Notebook

Calculus
102

الدكتورة

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الطالب

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calculus 2 (10)

4/3/2024

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الطالب عمرو خالد من
كلية الهندسة

7.1 Integration by parts

1] $\int x \sin x \, dx$

$u = x$ $\qquad dv = \sin x$

$\frac{du}{dx} = 1$ $\qquad \underline{-\int}$ $\quad \times$ $\qquad v = -\cos x$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

مسائل الـ ln والانفرس
دائمًا تُحل على
الأجزاء

2] $\int \ln x \, dx$

$u = \ln x$ $\qquad dv = (1)\, dx$

$du = \frac{1}{x} dx$ $\qquad v = x$

$= x\ln x - \int \frac{1}{x} \cdot x \, dx$

$= x\ln x - x + C$

u : نختارها الاسهل
للإشتقاق

dv : نختارها الاسهل
للتكامل

أجزاء للمرة الأولى

3] $\int t^2 e^t \, dt$

$u = t^2$ $\qquad dv = e^t \, dt$

$du = 2t \, dt$ $\qquad v = e^t$

$u = t$ $\qquad dv = e^t \, dt$

$du = dt$ $\qquad v = e^t$

أجزاء للمرة ②

$= e^t t^2 - 2\int t e^t \, dt$

~~$= t e^t - \int e^t \, dt$~~

$= e^t t^2 - 2\left(t e^t - \int e^t \, dt\right) = e^t t^2 - 2te^t + 2e^t + C$

طريقة 2 لحل سؤال 3

$\int t^2 e^t \, dt$

$= t^2 e^t - 2te^t + 2e^t + C$

نشتق u		نكامل dv
t^2	$+$	e^t
$2t$	$-$	e^t
2	$+$	e^t
0		e^t

قاعدة :

كثير حدود * اقتران أسي ($e^{\text{اقتران}}$ ، $\text{عدد}^{\text{اقتران}}$)
sin / cos
sinh / cosh
⇐ نستخدم طريقة الجدول

4] $\int t e^{-3t} \, dt$

$= -\frac{t}{3} e^{-3t} - \frac{e^{-3t}}{9} + C$

u		dv
t	$+$	e^{-3t}
1	$-$	$\frac{-e^{-3t}}{3}$
0		$\frac{e^{-3t}}{9}$

5] $\int t^2 \sin(Bt) \, dt$

$= -\frac{t^2}{B} \cos(Bt) + \frac{2t}{B^2} \sin(Bt)$

$+ \frac{2}{B^3} \cos(Bt) + C$

u		dv
t^2	$+$	$\sin(Bt)$
$2t$	$-$	$\frac{-\cos(Bt)}{B}$
2	$+$	$-\frac{\sin(Bt)}{B^2}$
0		$\frac{\cos(Bt)}{B^3}$

6] $\int x \cosh(ax) \, dx$

$= \frac{x}{a} \sinh(ax) - \frac{1}{a^2} \cosh(ax) + C$

u		dv
x	$+$	$\cosh(ax)$
1	$-$	$\frac{\sinh(ax)}{a}$
0		$\frac{\cosh(ax)}{a^2}$

7] $\int \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{1}{x^2+1} dx \qquad v = x$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{x^2+1} dx$

$= x\tan^{-1}x - \frac{1}{2} \ln|x^2+1|$ ← ما داخله دائما موجب فيمكن إزالة المطلق

$= x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + c$

8] $\int e^x \sin x \, dx$ (Periodic)

$e^x * \begin{matrix} \sin \\ \cos \end{matrix}$ } تكامل بالأجزاء مرتين

$u = e^x \qquad dv = \sin x$

$du = e^x dx \qquad v = -\cos x$ (الذي نفرضه في المرة الأولى u نفرضه نفس الشيء المرة 2)

$= -e^x \cos x + \int e^x \cos x$

$u = e^x \qquad dv = \cos x$

$du = e^x dx \qquad v = \sin x$

$\int e^x \sin x \, dx = -e^x \cos x + (e^x \sin x - \int e^x \sin x \, dx)$

[illegible]

[illegible]

$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} + c$

$= \frac{e^x}{2}(\sin x - \cos x) + c$

ضرب مشتقة ما داخله

$(\tan^{-1}(ax))' = \frac{1}{(ax)^2+1} * a$

$(\sin^{-1}(ax))' = \frac{1}{\sqrt{1-(ax)^2}} * a$

$\int \frac{1}{x^2+a^2} dx = \frac{\tan^{-1}(\frac{x}{a})}{a} + c$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a}) + c$

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طريقة 2 لحل سؤال 8 { عن طريق الجدول }

$$\int e^x \sin x \, dx$$

u		dv
e^x		$\sin x$
e^x	$+$	$-\cos x$
e^x	$-$	$-\sin x$
	$+\int$	

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} + c$$

نشتق مرتين
ونكامل مرتين

9] $\int e^{-\theta} \cos(2\theta) \, d\theta$

u		dv
$e^{-\theta}$		$\cos 2\theta$
$-e^{-\theta}$	$+$	$\frac{\sin 2\theta}{2}$
$+e^{-\theta}$	$-$	$-\frac{\cos 2\theta}{4}$
	$+\int$	

$$\int e^{-\theta} \cos(2\theta) \, d\theta = \frac{e^{-\theta}}{2} \sin 2\theta + \frac{e^{-\theta}}{4} \cos 2\theta - \frac{1}{4} \int e^{-\theta} \cos 2\theta$$

$$\int e^{-\theta} \cos 2\theta \, d\theta = \frac{4}{5} \left(\frac{e^{-\theta}}{2} \sin 2\theta - \frac{e^{-\theta}}{4} \cos 2\theta \right) + C$$

10] $\int_0^1 \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{dx}{1+x^2} \qquad v = x$

$$= x \tan^{-1} x \Big]_0^1 - \frac{1}{2} \int_0^1 \frac{2x}{x^2+1} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) \Big]_0^1 = \frac{\pi}{4} - \frac{\ln 2}{2}$$

$* \sin 2x = 2\sin x\cos x$ -1-1-

11] $\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{2}\int_0^{\pi} x\sin 2x\,dx$

u	dv
x	$+\sin 2x$
1	$-\cos 2x/2$
0	$-\sin 2x/4$

$= \frac{1}{2}\left(-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right)\Big]_0^{\pi}$

$= \frac{1}{2}\left(-\frac{\pi}{2}\cos 2\pi + \frac{1}{4}\cancel{\sin 2\pi}\right) - (0)$

$= -\frac{\pi}{4}$

Reduction Formulas.

(يوجد لدينا 5 أثباتات ويجب معرفة طريقتها)

II Prove

$\int \sin^n x\,dx$

{يجب أن نكتبها أولاً قبل حل السؤال الذي يأتي بهذه الصيغ الخمس}

sol: $\int \sin^{n-1}x\,\sin x\,dx$ | $u = \sin^{n-1}x \quad dv = \sin x$

$= -\cos x\sin^{n-1}x + \int (n-1)\sin^{n-2}x\,\underset{(1-\sin^2 x)}{\cos^2 x}\,dx$ | $du = (n-1)\sin^{n-2}x\cos x\,dx \quad v = -\cos x$

~~$= [illegible] + n\int \sin^{n-2}x\cos^2 x\,dx + \int \sin^{n-1}x\cos^2 x\,dx$ [illegible]~~

$\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

$+n\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$

$\int \sin^n x\,dx = -\frac{\cos x\sin^{n-1}x}{n} + \left(\frac{n-1}{n}\right)\int \sin^{n-2}x\,dx$

DOMS

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$$\boxed{\int \sin^2 x\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx}$$

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* في كل المسائل التي يطلب فيها تكامل $\sin^{n} x$ $\underline{3}$ أو أكثر نكتب الإثبات أولا ثم نطبق عليه

1] $\int \sin^3 x\,dx = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x\,dx$

$= -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\cos x + C$

2] $\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\,dx$

$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$

على الإثبات

3] $\int \sin^9 x\,dx = -\frac{1}{9}\cos x \sin^8 x + \underbrace{\frac{8}{9}\int \sin^7 x\,dx}$

$= -\frac{1}{9}\cos x \sin^8 x + \frac{8}{9}\left(-\frac{1}{7}\cos x \sin^6 x + \frac{6}{7}\int \sin^5 x\,dx\right)$

$= -\frac{1}{9}\cos x \sin^8 x - \frac{8}{63}\cos x \sin^6 x + \frac{48}{67}\left(-\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x\,dx\right)$

$= -\frac{1}{9}\cos x \sin^8 x - \frac{8}{63}\cos x \sin^6 x - \frac{6}{35}\cos x \sin^4 x + \frac{24}{35}\left(-\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x\,dx\right)$

$= -\frac{1}{9}\cos x \sin^8 x - \frac{8}{63}\cos x \sin^6 x - \frac{6}{35}\cos x \sin^4 x - \frac{24}{105} - \frac{48}{105}\cos x + C$

يوجد خطأ حسابي

$\boxed{2}$ Prove { لحل الأسئلة التي فيها $\cos^3 x$ او قوة أكثر من 3 }

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

$u = \cos^{n-1} x \qquad dv = \cos x$

$du = (n-1)\cos^{n-2} x(-\sin x) \qquad v = \sin x$

$$= \sin x \cos^{n-1} x + (n-1)\int \underbrace{\sin^2 x}_{\hookrightarrow (1-\cos^2 x)} \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x - \cos^n x \, dx$$

$$\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x \, dx$$

$$n\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \left(\frac{n-1}{n}\right)\int \cos^{n-2} x \, dx$$

$\boxed{3}$ Prove { لحل الأسئلة التي فيها $(\ln x)^{\overset{\text{عدد}}{\wedge}}$ }

$$\int (\ln x)^n dx$$

$u = (\ln x)^n \qquad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x} \qquad v = x$

$$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

$$\left\{ \begin{array}{l} \sec^2 x - \tan^2 x = 1 \\ \csc^2 x - \cot^2 x = 1 \end{array} \right\}$$

$\boxed{4}$ Prove

{ لحل الأسئلة التي فيها $\tan^3 x$ أو قوة أكبر من 3 }

$$\int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x \tan^{n-2} x\, dx - \int \tan^{n-2} x\, dx$$

let $u = \tan x \Rightarrow du = \sec^2 x\, dx$

$$\int u^{n-2}\, du - \int \tan^{n-2} x\, dx$$

$$\int \tan^n x\, dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\, dx$$

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$$

$\boxed{5}$ Prove

{ لحل الأسئلة التي فيها $\sec^3 x$ أو قوة أكبر من 3 }

$$\int \sec^n x\, dx = \int \sec^{n-2} x\, \sec^2 x\, dx =$$

$u = \sec^{n-2} x \qquad dv = \sec^2 x\, dx$

$du = (n-2)\sec^{n-3} x \sec x \tan x$

$v = \tan x$

$$= \tan x \sec^{n-2} x - \int (n-2) \sec^{n-2} x\, \underbrace{\tan^2 x}_{\hookrightarrow (\sec^2 x - 1)}\, dx$$

$$= \tan x \sec^{n-2} x - (n-2) \int \sec^n x - \sec^{n-2} dx$$

$$\int \sec^n x\, dx = \tan x \sec^{n-2} x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$$

ex: Find $\int \sec^5(3x)\,dx$

$u = 3x \qquad du = 3\,dx$

$$\int \sec^5(3x)\,dx = \frac{1}{3}\int \sec^5(u)\,du$$

$$= \frac{1}{3}\left[\frac{\tan u\sec^3 u}{4} + \frac{3}{4}\int \sec^3 u\,du\right]$$

$$= \frac{\tan u\sec^3 u}{12} + \frac{1}{4}\left(\frac{\tan u\sec u}{2} + \frac{1}{2}\int \sec u\,du\right)$$

$$= \frac{\tan u\sec^3 u}{12} + \frac{\tan u\sec u}{8} + \frac{1}{8}\ln|\sec u+\tan u| + C$$

$$= \frac{\tan 3x\sec^3 3x}{12} + \frac{\tan 3x\sec 3x}{8} + \frac{1}{8}\ln|\sec 3x+\tan 3x| + C$$

7.2 | Trigonometric Integrals

$\int \sin^m x \cos^n x \, dx$

If n is odd, ↙ نخرج cosx

If m is odd, ↙ نخرج sinx

(if n and m are even ↙ نستخدم متطابقات)

{ دائما نخرج الاس الفردي الاصغر }

والذي لم أخرج منه نفرضه u

1] $\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x (\sin x) \, dx$

$= \int (1-\cos^2 x)^2 \cos^2 x (\sin x) dx$

$u = \cos x$

$du = -\sin x \, dx$

$dx = -\frac{du}{\sin x}$

$= \int -(1-u^2)^2 (u^2) \frac{\cancel{\sin x} \, du}{\cancel{\sin x}} = \int -u^2(1-2u^2+u^4) \, du$

$= \int -u^2+2u^4-u^6 \, du = -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$

$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

2] $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x (\cos x) \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$ $\qquad u = \sin x$

$\qquad du = \cos x \, dx$

$= \int u^2 (1 - u^2) \, du = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

3] $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x \, dx = \int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x (\cos x) \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^7 x (1 - \sin^2 x)^2 (\cos x) dx$ $\qquad u = \sin x$

$\qquad du = \cos x \, dx$

$= \int_0^1 u^7 (1 - u^2)^2 \, du$ $\qquad x = \frac{\pi}{2} \Rightarrow u = 1$

$\qquad x = 0 \Rightarrow u = 0$

$= \int_0^1 u^7 (1 - 2u^2 + u^4) \, du$

$= \int_0^1 u^7 - 2u^9 + u^{11} \, du = \frac{u^8}{8} - \frac{u^{10}}{5} + \frac{u^{12}}{12} \Big]_0^1$

$= \frac{1}{8} - \frac{1}{5} + \frac{1}{12} - 0 = \frac{1}{120}$

$\int \tan^m x \sec^n x \, dx$

If n is even ⇐ نخرج - $\sec^2 x$ ثم نفرض $u = \tan x$

If m is odd ⇐ نخرج $\sec x \tan x$ ثم نفرض $u = \sec x$

If n is even and m is odd ⇐ إخراج $\sec^2 x$ / $\sec x \tan x$ ✓

If n is odd and m is even ⇐ نستخدم ال (reduction)

1] $\int \tan^6 x \sec^4 x \, dx$

$= \int \tan^6 x \sec^2 x \, (\sec^2 x) \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int \tan^6 x (\tan^2 x + 1)(\sec^2 x) \, dx$

$= \int u^6 (u^2 + 1) \, du = \int u^8 + u^6 \, du$

$= \frac{u^9}{9} + \frac{u^7}{7} + C = \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C$

2] $\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \sec^6 x (\sec x \tan x) \, dx$

$= \int (\sec^2 x - 1)^2 \sec^6 x (\sec x \tan x) \, dx$ | $u = \sec x$ $du = \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 u^6 \, du = \int u^6 (u^4 - 2u^2 + 1) \, du$

$= \int u^{10} - 2u^8 + u^6 \, du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$

$= \frac{\sec^{11} x}{11} - 2\frac{\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$

3] $\int \tan^4 x \sec^3 x \, dx$ { نستخدم متطابقة $\sec^2 x - \tan^2 x = 1$ وجعل المسألة تحوي $\sec x$ فقط }

$= \int (\sec^2 x - 1)^2 \sec^3 x \, dx = \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \, dx$

$= \int \sec^7 x - 2\sec^5 x + \sec^3 x \, dx$

$= \int \sec^7 x \, dx - 2 \int \sec^5 x \, dx + \int \sec^3 x \, dx$

{ نكتب إثبات $\sec^n x$ ثم نطبق هذا الإثبات على كل تكامل }

$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$

$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$

الترتيب مهم

1] $\int \sin(4x) \cos(5x) \, dx = \frac{1}{2} \int \sin(-x) + \sin(9x) \, dx$

$= -\frac{1}{2} \int \sin x \, dx + \frac{1}{2} \int \sin 9x \, dx = +\frac{1}{2} \cos x - \frac{1}{18} \cos 9x + C$

2] $\int_0^{\frac{\pi}{2}} \cos(5x) \cos(10x) \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(-5x) + \cos(15x) \, dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(5x) \, dx + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(15x) \, dx = \frac{\sin(5x)}{10} + \frac{\sin(15x)}{30} \Big]_0^{\frac{\pi}{2}}$

$= \frac{\sin \frac{5\pi}{2}}{10} + \frac{\sin \frac{15\pi}{2}}{30} - 0 = \frac{1}{10} - \frac{1}{30} = \frac{3-1}{30} = \frac{2}{30} = \frac{1}{15}$

لايمكن حلها بالتعويض لذلك نفكر مباشرة بالاجزاء

$$1] \int x \sec x \tan x \, dx$$

الاولوية لكثيرات الحدود بالنسبة لـ u

$u = x \qquad dv = \sec x \tan x \, dx$

$du = dx \qquad v = \sec x$

$$= x\sec x - \int \sec x \, dx$$

$$= x\sec x - \ln|\sec x + \tan x| + C$$

مهم

$$2] \int_{0}^{\frac{\pi}{4}} \sqrt{1 - \cos 4\theta} \, d\theta$$

لا يمكن حله بالتعويض او بالاجزاء لذلك نفكر بالمتطابقات بما أنه يوجد cos بالسؤال

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\rightarrow \cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$$

$$\cos 4\theta = 1 - \sin^2 2\theta - \sin^2 2\theta$$

$$1 - \cos 4\theta = 2\sin^2 2\theta$$

$$\int_{0}^{\frac{\pi}{4}} \sqrt{1 - \cos 4\theta} \, d\theta = \int_{0}^{\frac{\pi}{4}} \sqrt{2\sin^2(2\theta)} \, d\theta = \sqrt{2}\int_{0}^{\frac{\pi}{4}} |\sin 2\theta| \, d\theta$$

$$\theta \in [0, \tfrac{\pi}{4}]$$

$$2\theta \in [0, \tfrac{\pi}{2}]$$

ال sin دائماً (+) في الربع الاول

$$= \sqrt{2}\int_{0}^{\frac{\pi}{4}} \sin 2\theta \, d\theta$$

$$\int \sqrt{(\text{اقتران})^2} = \int \text{اقتران}$$

$$= \frac{-\sqrt{2}\cos 2\theta}{2} \Big]_{0}^{\frac{\pi}{4}}$$

$$\int_{a}^{b} \sqrt{(\text{اقتران})^2} = \int_{a}^{b} |\text{اقتران}|$$

$$= \frac{-1}{\sqrt{2}} \cos\frac{\pi}{2} + \frac{1}{\sqrt{2}} \cos 0 = \frac{1}{\sqrt{2}}$$

DOMS

3] $\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int \cos^2 x - \frac{\cancel{\cos^2 x}\,\sin^2 x}{\cancel{\cos^2 x}}\,dx$

$= \int \cos 2x\,dx = \frac{\sin 2x}{2} + C$

* * *

7.3 Trigonometric substitution

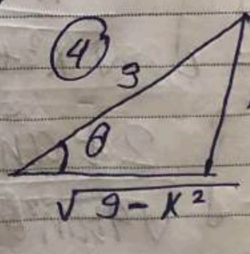
expression	substitution	
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2+x}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

عند وجود هذه الصيغ في التكامل إحيانًا نحل فقط عن طريق فرض x وأحيانًا نحل بهذه الطريقة وبالتعويض.

1] $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ ① $x = 3\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

② $dx = 3\cos\theta\,d\theta$

$= \int \frac{\sqrt{9-9\sin^2\theta}\,.\,3\cos\theta\,d\theta}{9\sin^2\theta}$ ③ $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

④


$= \int \frac{\sqrt{1-\sin^2\theta}\,\cos\theta}{\sin^2\theta}\,d\theta$

$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$

$= \int \csc^2\theta - 1\,d\theta = -\cot\theta - \theta + C$

$= \frac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

1) نفرض
2) نشتق
3) نجعل θ موضوعًا للقانون
4) نرسم المثلث القائم
5) نعوض قيمة x و dx
6) عندما نصل للجواب النهائي نجعله بصيغة x

2] $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

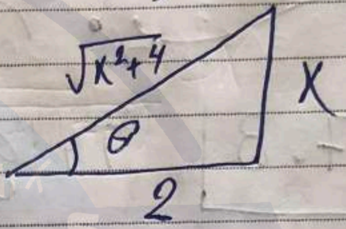
$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,dx$

$= \frac{1}{8}\int \frac{1}{\tan^2\theta\sec\theta}\,dx$



$= \frac{1}{8}\int \frac{\cos^2\theta\cos\theta}{\sin^2\theta}\,dx$

$= \frac{1}{8}\int \frac{\cos^3\theta}{\sin^2\theta}\cdot 2\sec^2\theta\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \frac{1}{4}\int \cot\theta\csc\theta\,d\theta = \frac{-1}{4}\csc\theta + C$

$= -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C = \frac{-\sqrt{x^2+4}}{4x} + C$

3] $\int \frac{x}{\sqrt{x^2+4}}\,dx$

$\boxed{\text{method1}}$

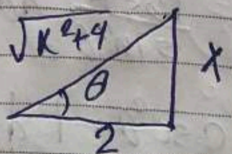
$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$= \int \frac{2\tan\theta\cdot 2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}}$

$= \frac{4}{2}\int \frac{\sec^{\cancel{2}}\theta\tan\theta}{\cancel{\sec\theta}}\,d\theta$



$= 2\sec\theta + C$

$= 2\,\frac{\sqrt{x^2+4}}{2} + c = \sqrt{x^2+4} + c$

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

method 2

$u = x^2+4 \qquad du = 2x\,dx \qquad dx = \frac{du}{2x}$

$$= \int \frac{x}{\sqrt{u}}\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= u^{\frac{1}{2}} + C = \sqrt{x^2+4} + C$$

4] $$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$$

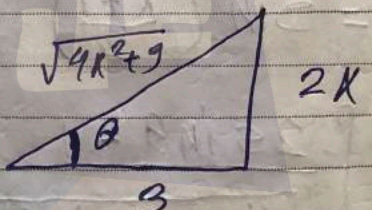
في السابق كنا نفرض = x أي
أننا كنا نأخذ الجذر لـ x^2
وهنا سنأخذ الجذر لـ $4x^2$ سينتج 2x
ولذلك سنفرض = 2x

$2x = 3\tan\theta$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

$\theta = \tan^{-1}\left(\frac{2x}{3}\right)$



$$= \int_0^{\frac{3\sqrt{3}}{2}} \frac{\frac{27}{8}\tan^3\theta}{\left(4\left(\frac{9}{4}\right)\tan^2\theta+9\right)^{\frac{3}{2}}}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$x = \frac{3\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$x = 0 \Rightarrow \theta = 0$

$\theta \in [0, \frac{\pi}{3}]$

$\Rightarrow \sec(+)$

لذلك نأخذ المطلق (+)

$$= \frac{\cancel{27}}{8}\cdot\frac{3}{2}\cdot\frac{1}{\cancel{27}}\int_0^{\frac{3\sqrt{3}}{2}} \frac{\tan^3\theta\sec^2\theta}{|\sec^3\theta|}\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$\theta = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$

$\theta = 0 \quad u = 1$

$$= \frac{3}{16}\int_1^{\frac{1}{2}} \frac{(1-u^2)}{u^2}\,\frac{\cancel{\sin\theta}\,du}{-\cancel{\sin\theta}} = \frac{3}{16}\int_1^{\frac{1}{2}} \frac{u^2-1}{u^2}\,du$$

$$= \left(u + \frac{1}{u}\right)\frac{3}{16}\Big]_1^{\frac{1}{2}} = \frac{3}{32}$$

$$5]\ \int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

نعمل إكمال مربع

1) نجعل معامل x^2 واحد

2) نقسم معامل x على 2 ونربع الناتج

3) نجمع ونطرح الناتج

$$= \int \frac{x}{\sqrt{-(x^2+2x-3)}}\,dx$$

$$= \int \frac{x}{\sqrt{-(x^2+2x+1-1-3)}}\,dx = \int \frac{x}{\sqrt{-((x+1)^2-4)}}\,dx$$

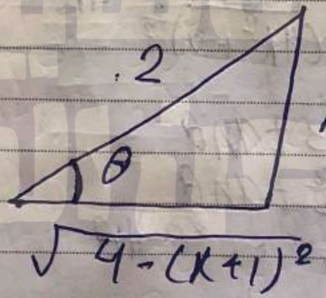
$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$x+1 = 2\sin\theta$

$x = 2\sin\theta - 1$

$dx = 2\cos\theta\,d\theta$

$\theta = \sin^{-1}\left(\frac{x+1}{2}\right)$



$$= \int \frac{2\sin\theta-1}{\sqrt{4-(2\sin\theta)^2}}\,dx = \int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta$$

$$= \frac{2}{2}\int \frac{\cancel{\cos\theta}(2\sin\theta-1)}{\sqrt{\cancel{\cos^2\theta}}}\,d\theta = \int 2\sin\theta - 1\ d\theta$$

$$= -2\cos\theta - \theta + C = -\frac{2\sqrt{4-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

6] $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\, dx$

$3x = 2\sin\theta$

$x = \frac{2}{3}\sin\theta$

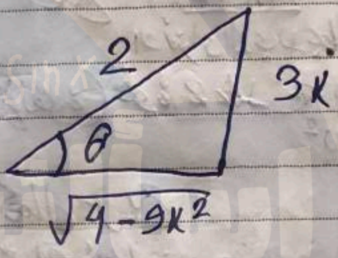
$= \int_0^{\frac{\pi}{2}} \sqrt{4-9(\frac{2}{3}\sin\theta)^2} \cdot \frac{2}{3}\cos\theta\, d\theta$

$dx = \frac{2}{3}\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} \cdot \frac{2}{3}\cos\theta\, d\theta$

$\theta = \sin^{-1}(\frac{3x}{2})$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$



$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$

$= \frac{2}{3}x + \frac{\cancel{2}}{3}\frac{\sin(2\theta)}{\cancel{2}} \Big]_0^{\frac{\pi}{2}}$

$x = \frac{2}{3} \Rightarrow \theta = \frac{\pi}{2}$

$= \frac{2}{3}(\frac{\pi}{2}) + \frac{1}{3}\sin(\pi) - (0)$

$x = 0 \Rightarrow \theta = 0$

$= \frac{\pi}{3}$

$\frac{\text{كثير حدود}}{\text{كثير حدود}}$ ↑

7.4] Integration of Rational function by partial fractions ع الكسور الجزئية 3

$\frac{\text{Polynomial}}{\text{Polynomial}}$ ⇒ درجة البسط أكبر أو تساوي درجة المقام نستخدم القسمة الطويلة

⇒ درجة البسط أقل من درجة المقام نستخدم الكسور الجزئية

{مكرر اي مرفوع لقوة}

/ /

* خاص بالكسور الجزئية *

1) نحلل المقام أكثر ما يمكن بحيث تكون الحدود مضروبة ببعضها

2) في حال كان ما في المقام (خطي) نضع $\frac{A}{\text{خطي}}$

3) 〃 〃 〃 〃 〃 〃 〃 (خطي مكرر) $\longrightarrow (x+1)^2 / (x)^3 / (5x+4)^6$

قوة 3 مثلاً ← $\frac{A}{\text{خطي}} + \frac{B}{(\text{خطي})^2} + \frac{C}{(\text{خطي})^3}$

4) 〃 〃 〃 〃 〃 〃 〃 (لا يحلل) نضع $\frac{Ax+B}{\text{الاقتران الذي لا يحلل}}$

5) 〃 〃 〃 〃 〃 〃 〃 (لا يحلل مرفوع لقوة) قوة 2 مثلاً ← $\frac{Ax+B}{\text{لا يحلل}} + \frac{Cx+D}{(\text{لا يحلل})^2}$

Ex: write out the form of the partial decomposition of the following [Dont find the coefficents]

{المطلوب صيغة الكسور الجزئية دون إيجاد الثوابت}

1] $f(x) = \frac{x-6}{x^2+x-6} = \frac{x-6}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

2] $\frac{1-x}{x^3+x^4} = \frac{1-x}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$

3] $\frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

4] $\frac{x^3+1}{x^3-3x^2+2x}$

$$\begin{array}{r|l} & 1 \\ x^3-3x^2+2x & x^3+1 \\ & \cancel{-x^3}+3x^2\mp 2x \\ \hline & 3x^2-2x+1 \end{array}$$

الباقي درجته أكبر او تساوي درجة المقسوم عليه نكمل القسمة أما إذا اصبحت درجته أقل من درجة المقسوم عليه نتوقف

$= 1 + \frac{3x^2-2x+1}{x^3-3x^2+2x}$

$= 1 + \frac{3x^2-2x+1}{x(x-2)(x-1)}$

$= 1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$

5] $f(x) = \frac{x^2-1}{x^3+x^2+x} = \frac{x^2-1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$

قسمة طويلة

6] $f(t) = \frac{t^6+1}{t^6+t^3} = \frac{-t^3+1}{t^3(t^3+1)} + 1 = \frac{-t^3+1}{t^3(t+1)\overbrace{(t^2-t+1)}} + 1$

دائما لا يحلل

$= \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{Dx+E}{t^2-t+1} + \frac{F}{t+1} + 1$

نفس عكس دائما (+)

7] $f(x) = \frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$

Ex: Find the following integrals:

1] $\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x-1 & x^3+x \\ & \underline{-x^3+x^2} \\ & x^2+x \\ & \underline{-x^2+x} \\ & 2x \\ & \underline{-2x+2} \\ & 2 \end{array}$$

$= \int x^2+x+2 + \frac{2}{x-1}\,dx$

$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$

2] $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx$

$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

نضربها بالذي لايوجد بالمقام بالنسبة للمقام الاساسي

$A(2x-1)(x+2) + B(x)(x+2) + Cx(2x-1)$

$= x^2+2x-1$

$x = 0 \Rightarrow -2A = -1 \quad A = \frac{1}{2}$

$x = -2 \Rightarrow C(-2)(-5) = 4-4-1$

$C = -\frac{1}{10}$

$x = \frac{1}{2} \Rightarrow \frac{B}{2}\left(\frac{5}{2}\right) = \frac{1}{4} + 1 - 1$

$B = \frac{1}{5}$

$= \int \frac{1}{2x} + \frac{1}{10x-5} + -\frac{1}{10x+20}$

$= \int \frac{1}{2}\cdot\frac{1}{x} + \frac{1}{5}\cdot\frac{2}{2x-1}\cdot\frac{1}{2} - \frac{1}{10}\cdot\frac{1}{x+2}$

$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2|$

$+ C$

3] $\int \frac{dx}{x^2-9} = \int \frac{dx}{(x-3)(x+3)}$

$\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$A(x+3) + B(x-3) = 1$

$x = 3 \Rightarrow A = \frac{1}{6}$

$x = -3 \Rightarrow B = -\frac{1}{6}$

$= \int \frac{1}{6} \cdot \frac{1}{x-3} + -\frac{1}{6} \cdot \frac{1}{x+3} \, dx$

$= \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + C$

$= \frac{1}{6} \left(\ln|x-3| - \ln|x+3| \right) + C = \frac{1}{6} \ln\left| \frac{x-3}{x+3} \right| + C$

4] $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx$

$$\begin{array}{r|l} & x + 1 \\ \hline x^3 - x^2 - x + 1 & x^4 - 2x^2 + 4x + 1 \\ & -\cancel{x^4} + x^3 + x^2 \mp x \\ \hline & +\cancel{x^3} - \cancel{x^2} + 3x + 1 \\ & -\cancel{x^3} + \cancel{x^2} + x \mp 1 \\ \hline & + 4x \end{array}$$

$= \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \, dx$

ضرب

$$\begin{array}{r|l} & x^2 - 1 \\ \hline x - 1 & x^3 - x^2 - x + 1 \\ & -x^3 + x^2 \\ \hline & -x + 1 \\ & +x \mp 1 \\ \hline & 0 \end{array}$$

كيف نحلل المقام؟ عن طريق القسمة الطويلة

* ننظر إلى الثابت => 1 ونحدد مضاعفاته (الأرقام التي يقسم عليها ويكون الناتج عدد صحيح) ثم نعوض بالمقام بحيث يكون ناتج التعويض يساوي صفر لذلك. فإن $x = 1$ تجعل المقام صفر فنقسم على $x - 1$ ويجب أن يكون الباقي صفر

$= \int x + 1 + \frac{4x}{(x-1)^2(x+1)} \, dx$

- $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$A(x-1)(x+1) + B(x+1) + C(x-1)^2 = 4x$

$x = 1 \Rightarrow B = 2 \quad x = -1 \Rightarrow C = -1$

$x = 0 \Rightarrow A = 1$

الناتج النهائي للتكامل: $\frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right|$

5] $$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$A(x^2+4) + (Bx + C)x = 2x^2 - x + 4$$

$$Ax^2 + 4A + Bx^2 + Cx = 2x^2 - x + 4$$

طريقة أخرى لإيجاد قيم الثوابت:

3 نقوم بعملية مساواة المعاملات بالثوابت المضروبة بالمتغير المحدد ع

المعاملات

$X^2 : 2 \rightarrow 2 = A + B \Rightarrow B = 1$

$X^1 : -1 \rightarrow -1 = C \Rightarrow C = -1$

$X^0 : 4 \rightarrow 4 = 4A \Rightarrow A = 1$

نضع المعاملات من اكبر متغير الى اصغر متغير من حيث الأسس

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{1}{x} + \frac{x - 1}{x^2+4}\,dx = \int \frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{\tan^{-1}\left(\frac{x}{2}\right)}{2} + C$$

6] $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$

$$\begin{array}{r} 1 \\ 4x^2-4x+3 \,\big)\, \overline{\cancel{4x^2}-3x+2} \\ \underline{-\cancel{4x^2}+4x\mp 3} \\ x-1 \end{array}$$

$= \int 1 + \frac{x-1}{4x^2-4x+3}\,dx$

$\hookrightarrow \Delta = b^2-4ac = (-4)^2-4(4)(3)$

$= 16-48 = -32$ { لا تتحلل

نكمل لكامل مربع

$= \int 1 + \frac{x-1}{4\left((x-\frac{1}{2})^2+\frac{1}{2}\right)}\,dx$

$$\left\{\begin{array}{l} 4x^2-4x+3 = 4(x^2-x+\frac{3}{4}) \\ = 4(x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{3}{4}) = 4\left((x-\frac{1}{2})^2+\frac{1}{2}\right) \end{array}\right.$$

$= x + \int \frac{\frac{1}{2}(u+1)-1}{4\left((\frac{u+1}{2}-\frac{1}{2})^2+\frac{1}{2}\right)}\cdot\frac{du}{2}$

$$\left\{\begin{array}{l} u = 2(x-\frac{1}{2}) = 2x-1 \\ du = 2\,dx \qquad x = \boxed{\frac{u+1}{2}} \end{array}\right.$$

$= x + \frac{1}{16}\int \frac{u+1-2}{\frac{u^2}{4}+\frac{1}{2}}\,du = x + \underset{④}{\cancel{\frac{1}{16}}}\int \frac{u-1}{\cancel{\frac{1}{4}}(u^2+2)}\,du$

$= x + \frac{1}{4}\left[\frac{1}{2}\int\frac{2u}{u^2+2}\,du - \int\frac{1}{u^2+2}\,du\right]$

$= x + \frac{1}{8}\ln(u^2+2) - \frac{1}{4}\frac{\tan^{-1}(\frac{u}{\sqrt{2}})}{\sqrt{2}} + c$

$= x + \frac{1}{8}\ln\left((2x-1)^2+2\right) - \frac{\tan^{-1}(\frac{2x-1}{\sqrt{2}})}{4\sqrt{2}} + c$

7] $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$A(x^2+1)^2 + (Bx+C)(x(x^2+1)) + (Dx+E)(x) = 1 - x + 2x^2 - x^3$$

x^4: $A + B = 0 \Rightarrow \boxed{B = -1}$

x^3: $\boxed{C = -1}$

x^2: $2A + B + D = 2 \Rightarrow \boxed{D = 1}$

x^1: $C + E = -1 \Rightarrow \boxed{E = 0}$

x^0: $\boxed{A = 1}$

$$= \int \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2} dx$$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$= \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+1} dx - \int \frac{1}{x^2+1} dx + \int \frac{\cancel{x}}{u^2} \frac{du}{2\cancel{x}}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) - \frac{1}{2u} + C$$

$$= \ln|x| - \ln\sqrt{x^2+1} - \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C$$

* مسائل خاصة *

1] $\int \frac{\sqrt{x+4}}{x}\,dx$

{ يمكننا أن نفرض الجذر كاملاً }

$u = \sqrt{x+4}$

$u^2 = x+4$

$2u\,du = dx$

$= \int \frac{u}{u^2-4} \cdot 2u\,du$

$= \int \frac{2u^2}{u^2-4}\,du$

$$\begin{array}{r|l} & 2 \\ \hline u^2-4 & 2u^2 \\ & -2u^2+8 \\ \hline & 8 \end{array}$$

$= \int 2 + \frac{8}{u^2-4}\,du$

$\frac{8}{u^2-4} = \frac{A}{u+2} + \frac{B}{u-2}$

$A(u-2) + B(u+2) = 8$

$u = 2 \Rightarrow B = 2$

$u = -2 \Rightarrow A = -2$

$= \int 2 - \frac{2}{u+2} + \frac{2}{u-2}\,du$

$= 2u - 2\ln|u+2| + 2\ln|u-2| + C$

$= 2\sqrt{x+4} - 2\ln|\sqrt{x+4}+2| + 2\ln|\sqrt{x+4}-2| + C$

$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|$

2] $\int \frac{dx}{x^2 + x\sqrt{x}}$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$= \int \frac{2u\,du}{u^4+u^3}$

$= \int \frac{2}{u^3+u^2}\,du = \int \frac{2}{u^2(u+1)}\,du$

$\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$

$A(u)(u+1) + B(u+1) + Cu^2 = 2$

$u = 0 \Rightarrow B = 2$

$u = -1 \Rightarrow C = 2$

$u = 1 \Rightarrow 2A + 2B + C = 2$

$2A + 4 + \cancel{2} = \cancel{2}$

$A = -2$

$= \int -\frac{2}{u} + \frac{2}{u^2} + \frac{2}{u+1}\,du = -2\ln|u| - \frac{2}{u} + 2\ln|u+1|$

$+ C$

3] $\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx = \int \frac{x^2 \cdot x}{\sqrt[3]{x^2+1}}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

$= \int \frac{(u-1)\cancel{x}}{\sqrt[3]{u}} \cdot \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int u^{\frac{2}{3}} - u^{-\frac{1}{3}}\,du = \frac{3u^{\frac{5}{3}}}{10} - \frac{3u^{\frac{2}{3}}}{4} + C$

$= \frac{3}{10}\sqrt[3]{(x^2+1)^5} - \frac{3}{4}\sqrt[3]{(x^2+1)^2} + C_1$

4] $\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

عند وجود أكثر من جذر نفرض
u تساوي جذر رتبته المضاعف
المشترك الأصغر

نأخذ مضاعفات العدد

$\sqrt[2]{x}$: 2, 4, $\boxed{6}$, 8
$\sqrt[3]{x}$: 3, $\boxed{6}$, 9

$= \int \frac{6u^5\,du}{u^3 - u^2}$

$u = \sqrt[6]{x}$
$x = u^6$
$dx = 6u^5\,du$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ u^3 - u^2 & 6u^5 \\ & -6u^5+6u^4 \\ \hline & 6u^4 \\ & -6u^4+6u^3 \\ \hline & 6u^3 \\ & -6u^3+6u^2 \\ \hline & 6u^2 \end{array}$$

$= \int 6u^2+6u+6 + \frac{6\cancel{u^2}}{\cancel{u^2}(u-1)}\,du$

$= 2u^3 + 3u^2 + 6u + \ln|u-1| + C$

$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + \ln|\sqrt[6]{x} - 1| + C$

5] $\int \frac{\sqrt{1+\sqrt{x}}}{x}\, dx$

$u = \sqrt{1+\sqrt{x}}$

$u^2 = 1+\sqrt{x}$

$(u^2-1)^2 = x$

$2(u^2-1)\cdot 2u = dx/du$

$dx = 4u(u^2-1)\,du$

$= \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du$

$= \int \frac{4u^2}{(u^2-1)}\, du$

$$\begin{array}{r|l} & 4 \\ \hline u^2-1 & 4u^2 \\ & -4u^2+4 \\ \hline & 4 \end{array}$$

$= \int 4 + \frac{4}{u^2-1}\, du$

$\frac{4}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$A(u+1) + B(u-1) = 4$

$u = -1 \Rightarrow B = -2$

$u = 1 \Rightarrow A = 2$

$= \int 4 + \frac{2}{u-1} - \frac{2}{u+1}\, du$

$= 4u + 2\ln|u-1| - 2\ln|u+1| + C$

$= 4\sqrt{1+\sqrt{x}} + 2\ln\left|\sqrt{1+\sqrt{x}} - 1\right| - 2\ln\left|\sqrt{1+\sqrt{x}}+1\right| + C$

$= 4\sqrt{1+\sqrt{x}} + 2\ln\left|\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right| + C$

6] $\int \frac{e^{2x}}{e^{2x}+3e^x+2}\, dx$

$u = e^x$

$du = e^x\, dx$

$dx = du/e^x$

$= \int \frac{u\,\cancel{e^x}}{u^2+3u+2} \cdot \frac{du}{\cancel{e^x}}$

$\frac{u}{u^2+3u+2} = \frac{A}{u+2} + \frac{B}{u+1}$

$A(u+1) + B(u+2) = u$

$u = -1 \Rightarrow B = -1$

$u = -2 \Rightarrow A = 2$

$= \int \frac{2}{u+2} - \frac{1}{u+1}\, du$

$= 2\ln|u+2| - \ln|u+1| + C$

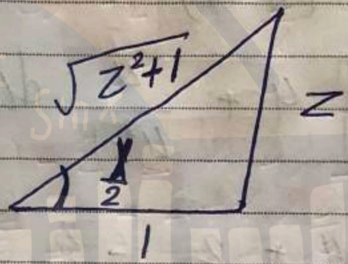
$= 2\ln|e^x+2| - \ln|e^x+1| + C$

⁂ Half Angle ⁂ 14/3/2024

نستخدم هذا الفرض في بعض الأسئلة التي تحوي cos / sin .

$$* \quad z = \tan\left(\frac{x}{2}\right) \quad * \qquad -\pi < x < \pi$$

1) $\sin\left(\frac{x}{2}\right) = \dfrac{z}{\sqrt{z^2+1}}$



2) $\cos\left(\frac{x}{2}\right) = \dfrac{1}{\sqrt{z^2+1}}$

3) $\sin(x) = \dfrac{2z}{z^2+1} \Rightarrow \sin(x) = 2\sin\frac{x}{2}\cos\frac{x}{2}$

4) $\cos(x) = \dfrac{1-z^2}{1+z^2} \Rightarrow \cos(x) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

وإما نحفظ الخمسة أو نشتقهم

5) $dx = \dfrac{2}{1+z^2}\, dz$

ex:

1] $\displaystyle\int \frac{dx}{1-\cos x} \qquad z = \tan\left(\frac{x}{2}\right)$ { فقط نقوم بالتعويض }

$$= \int \frac{\frac{2}{z^2+1}}{1-\left(\frac{1-z^2}{z^2+1}\right)}\, dz = \int \frac{\frac{2}{z^2+1}}{\frac{z^2+1-1+z^2}{z^2+1}}\, dz$$

$$= \int \frac{\cancel{2}}{\cancel{z^2+1}} * \frac{\cancel{z^2+1}}{\cancel{2}z^2}\, dz = \int z^{-2}\, dz = -z^{-1} + C$$

$$= \frac{-1}{\tan\frac{x}{2}} + C = -\cot\frac{x}{2} + C$$

2] $\int \frac{1}{3\sin x - 4\cos x}\,dx \qquad z = \tan(\frac{x}{2})$

$$= \int \frac{\frac{2}{z^2+1}}{3\left(\frac{2z}{z^2+1}\right) - 4\left(\frac{1-z^2}{z^2+1}\right)}\,dz = \int \frac{\frac{2}{z^2+1}}{\frac{6z-4+4z^2}{z^2+1}}\,dz$$

$$= \int \frac{\cancel{2}}{\cancel{2}(2z^2+3z-2)}\,dz = \int \frac{1}{(2z-1)(z+2)}\,dz$$

$$\frac{1}{(2z-1)(z+2)} = \frac{A}{2z-1} + \frac{B}{z+2} \qquad A(z+2) + B(2z-1) = 1$$

$z = -2 \Rightarrow B = \frac{-1}{5}$

$$= \int \frac{2}{5(2z-1)} + -\frac{1}{5(z+2)} \qquad z = \frac{1}{2} \Rightarrow A(\frac{5}{2}) = 1 \quad A = \frac{2}{5}$$

$$= \frac{1}{5}\ln|2z-1| - \frac{1}{5}\ln|z+2| = \frac{1}{5}\ln\left|\frac{2z-1}{z+2}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{2\tan(\frac{x}{2})-1}{\tan(\frac{x}{2})+2}\right| + C$$

$x = \frac{\pi}{2} \Rightarrow z = 1$

$x = 0 \Rightarrow z = 0$

$z = \tan(\frac{x}{2})$

3] $$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{2+\cos x}\,dx = \int_0^1 \frac{2\left(\frac{2z}{z^2+1}\right)\left(\frac{1-z^2}{z^2+1}\right)}{2+\left(\frac{1-z^2}{z^2+1}\right)} \cdot \frac{2}{z^2+1}$$

$$= \int_0^1 \frac{\frac{4z-4z^3}{(z^2+1)^2}}{\frac{2z^2+2+1-z^2}{z^2+1}} \cdot \frac{2}{z^2+1}\,dz = 8\int_0^1 \frac{z-z^3}{(z^2+1)^3} \cdot \frac{(z^2+1)}{(z^2+3)}$$

الطريقة هذه ستكون معقدة

لذلك نلجأ للتكامل بالتعويض

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{2+\cos x}\,dx$$

$u = 2+\cos x \qquad du = -\sin x\,dx \qquad dx = \frac{-du}{\sin x}$

$x = \frac{\pi}{2} \Rightarrow u = 2 \qquad x = 0 \Rightarrow u = 3$

$$2\int_3^2 \frac{\cancel{\sin x}(u-2)}{u}\cdot\frac{-du}{\cancel{\sin x}} = 2\int_2^3 \frac{u-2}{u}\,du = 2\int_2^3 1-\frac{2}{u}\,du$$

$$= 2\Big[u-2\ln|u|\Big]_2^3 = 2\big[3-2\ln(3)-2+2\ln(2)\big]$$

$$= 2\Big[1+2\ln\frac{2}{3}\Big] = 2+4\ln\frac{2}{3}$$

7.5 | Strategy of Integration

{ هذا الدرس عبارة عن مراجعة لما سبقه }

1] $\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x\sec^3 x\,dx$

$= \int \tan^2 x\sec^2 x.(\sec x\tan x)\,dx$

$u = \sec x$

$du = \sec x\tan x\,dx$

$\Big| = \int(\sec^2 x-1)\sec^2 x\,(\sec x\tan x)\,dx$

$= \int(u^2-1)u^2\,du = \int u^4-u^2\,du = \frac{u^5}{5}-\frac{u^3}{3}+C$

$= \frac{\sec^5 x}{5}-\frac{\sec^3 x}{3}+C$

__/__/__

2] $\int e^{\sqrt{x}}\,dx \qquad u=\sqrt{x} \qquad \begin{matrix} u^2 = x \\ 2u\,du = dx \end{matrix}$

$= \int e^u . 2u\,du = 2\int u e^u\,du$

$$\begin{array}{c|c} z & dv \\ \hline u & e^u \\ 1 & {}^{+}e^4 \\ 0 & {}^{-}e^u \end{array}$$

$= 2[ue^u - e^u] + C = 2ue^u - 2e^4 + C$

$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

3] $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$

$$\begin{array}{r} x^2 + 3x + 19 \\ x^3-3x^2-10x \overline{\big)\; x^5 + 1 \qquad\qquad} \\ \underline{-\cancel{x^5} + 3x^4 + 10x^3} \\ 3\cancel{x^4} + 10x^3 + 1 \\ \underline{-3\cancel{x^4} + 9x^3 + 10x^2} \\ 19\cancel{x^3} + 10x^2 + 1 \\ \underline{-19\cancel{x^3} + 57x^2 + 190x} \\ 67x^2 + 190x + 1 \end{array}$$

$= \int x^2+3x+19 + \frac{67x^2+190x+1}{x^3-3x^2-10x}$

ونكمل هنا عن طريق الكسور الجزئية فيكون الناتج النهائي

$19x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{3126}{35}\ln|x-5| - \frac{31}{14}\ln|x+2| + C$

4] $\int \sqrt{\frac{1-x}{1+x}}\,dx \qquad u = \sqrt{\frac{1-x}{1+x}} \Rightarrow$ هكذا ستكون معقدة جداً

(نضرب بالبسط)

$= \int \sqrt{\frac{1-x}{1+x}} * \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx$

$= \int \frac{1-x}{\sqrt{(1-x^2)}}dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad \begin{matrix} u = 1-x^2 \\ du = -2x\,dx \end{matrix}$

$= \sin^{-1}(x) + \sqrt{u} + C = \sin^{-1}(x) + \sqrt{1-x^2} + C$

5] $\int \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x$ $du = \frac{1}{x}\,dx$

$= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\ln x} + C$

تكاملات لا نستطيع أن نكاملها أبداً

$\int \sin(x^2)\,dx$ / $\int \sqrt{x^3+1}\,dx$

* Review of Limits *

1) $\frac{0}{a} = 0$ 2) $\frac{a}{0} = +\infty$ or $-\infty$ 3) $\frac{a}{\infty} = \frac{a}{-\infty} = 0$

4) $\infty + \infty = \infty$ 5) $-\infty - \infty = -\infty$ 6) $\infty \cdot \infty = \infty$

7) $\infty + a = \infty$ 8) $\infty - a = \infty$ 9) $\infty^r = \infty$ $r > 0$

10) $\sqrt[r]{\infty} = \infty$ 11) $a \cdot \infty = \infty$ 12) $\infty^{-r} = 0$ 13) $\ln \infty = \infty$

14) $\frac{0}{\infty} = 0$

$\frac{\infty}{\infty}$ or $\frac{0}{0}$ $\Rightarrow$ use L'Hospital Rule $\Rightarrow \frac{f'}{g'}$

$\boxed{\infty - \infty,\ 0 \cdot \infty}$ → $\boxed{L.R}$ نطبقها

$\frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$

$\boxed{1^\infty,\ 0^0,\ \infty^0}$ → ناخذ ln لما داخل النهاية ثم نرفع الجواب النهائي لـ e

$\Rightarrow y = \ln(\lim$ ما داخل ال$)$

1] $\lim\limits_{x \to 0} \frac{\cos x - 1}{x} = \frac{0}{0} \Rightarrow L.R$

$= \lim\limits_{x \to 0} \frac{-\sin x}{1} = 0$

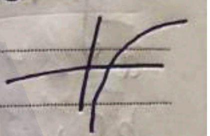
2] $\lim\limits_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \Rightarrow \boxed{L.R} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

3] $\lim\limits_{x\to-\infty} \frac{x^2-x+1}{1-3x^2} = \frac{\infty}{\infty} \Rightarrow \boxed{L.R} = \lim\limits_{x\to-\infty} \frac{2x-1}{-6x}$

$= \frac{\infty}{\infty} \Rightarrow \boxed{L.R} = \lim\limits_{x\to-\infty} \frac{2}{-6} = \frac{-1}{3}$

4] $\lim\limits_{x\to\infty} \sqrt{x^2+x} - x = \infty - \infty$ {Prob}

$\ln\infty = \infty$
$\ln 0^+ = -\infty$



$= \lim\limits_{x\to\infty} \sqrt{x^2+x} - x * \frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x}$

$= \lim\limits_{x\to\infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2+x} + x} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x})} + x} = \lim\limits_{x\to\infty} \frac{x}{\underbrace{|x|}_{\text{موجبة}}\sqrt{1+\frac{1}{x}} + x}$

$= \lim\limits_{x\to\infty} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+\frac{1}{x}} + 1)} = \frac{1}{2}$

5] $\lim\limits_{x\to\infty} x e^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty} \Rightarrow \boxed{L.R}$

$= \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

$y = x^{e^{-x}}$

6] $\lim\limits_{x\to\infty} x^{e^{-x}} = \infty^0 \Rightarrow \ln y = \ln x^{e^{-x}}\ \ln y = e^{-x}\ln x$

ثم نأخذ $\lim\limits_{x\to\infty}$ للطرفين (او يمكن أن نأخذ الـln داخل الحل الـ lim مباشرةً)

$= \lim\limits_{x\to\infty} e^{-x}\ln x = \lim\limits_{x\to\infty} \frac{\ln x}{e^x} \Rightarrow \boxed{L.R} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \frac{0}{\infty} = 0$

$\supset \lim\limits_{x\to\infty} x^{e^{-x}} \subset = e^0 = 1$

$$\boxed{\begin{array}{l}\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}\\ \lim_{x\to 0}\left(1+ax\right)^{\frac{b}{x}}=e^{ab}\end{array}}$$

7] $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{3x}$

$= e^{-6}$

7.8 Improper Integrals.

Type 1: infinite intervals. f is continous

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

في حال كان أحد حدي
التكامل $(\infty/-\infty)$ وكان
الإقتران متصلاً
على الفترة

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

$$= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{t\to\infty}\int_a^u f(x)\,dx$$

وجود الـ ∞ في حدود
التكامل يعتبر مشكلة
و $^{\infty}_{-\infty}$ فهنا يوجد مشكلتين
فيجب أن يحتوي
كل تكامل على مشكلة
واحدة فقط

1] $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left.\frac{-1}{x}\right]_1^t$

$= \lim_{b\to\infty}\left(1-\frac{1}{t}\right) = 1$ (converge)

يجب كتابتها في نهاية الحل
ونكتب في حال كان الجواب رقم
أما إذا كان infinite نكتب (Diverge)

$$2]\ \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln|t| - 0) = \infty$$ (Divarge)

$$3]\ \int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

$$\begin{array}{c|c} u & dv \\ \hline x & +e^x \\ 1 & -e^x \\ 0 & e^x \end{array}$$

$$= \lim_{t\to-\infty}(xe^x - e^x)\Big]_t^0$$

$$= \lim_{t\to-\infty}(-1 - te^t + e^t) = -1 - \lim_{t\to-\infty} te^t$$

$$= -1 - \lim_{t\to-\infty}\frac{t}{e^{-t}} = -1 - \lim_{t\to-\infty}\frac{-1}{e^{-t}} = -1$$ (conv.)

Type 2: Discontinous Integrals

1] If f continous on $[a,b)$ and discont. at b then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

2] If f continous on $(a,b]$ and discont. at a then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

3] If f continous on $[a,b]$ except at $c \Rightarrow a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{u\to c^+}\int_u^b f(x)\,dx$$

1] $$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+}\left(2\sqrt{x-2}\right)\Big]_t^5 = \lim_{t\to 2^+}\left(2\sqrt{3} - 2\sqrt{t-2}\right)$$

$= 2\sqrt{3}$ (conv.)

2] $$\int_0^{\frac{\pi}{2}} \sec x\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx = \lim_{t\to\frac{\pi}{2}^-}\int_0^t \frac{1}{\cos x}\,dx$$

$$= \lim_{t\to\frac{\pi}{2}^-} \ln|\sec x+\tan x|\Big]_0^t = \lim_{t\to\frac{\pi}{2}^-}\left(\ln|\sec t+\tan t| - 0\right)$$

$= \ln\left(\sec\frac{\pi}{2}+\tan\frac{\pi}{2}\right) = \ln\infty = \infty$ (diverge)

3] $$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

$$= \lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} + \lim_{u\to 1^+}\int_u^3 \frac{dx}{x-1} = \lim_{t\to 1^-}\ln|x-1|\Big]_0^t + \lim_{u\to 1^+}\ln|x-1|\Big]_u^3$$

$$= \lim_{t\to 1^-}\left(\ln|t-1| - 0\right) + \lim_{u\to 1^+}\left(\ln(2) - \ln|u-1|\right) = -\infty + \ln 2 - \infty$$

$= -\infty$ (Div.)

4] $\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$

$u = \ln x \quad dv = dx$
$du = \frac{dx}{x} \quad v = x$

$= \lim_{t \to 0^+} \left(x \ln x - \int dx \right)$

$= \lim_{t \to 0^+} \left(x \ln x - x \Big]_t^1 \right) = \lim_{t \to 0^+} \left(-1 - t \ln t + t \right)$

$= -1 - \lim_{t \to 0^+} t \ln t = -1 - \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} \rightarrow \frac{\infty}{\infty} \Rightarrow \boxed{L.R}$

$= -1 - \lim_{t \to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = -1 - \lim_{t \to 0^+} -t = -1$ (converge)

Ex: write the following integrals as sum of limits:

1] $\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)} \, dx$

المشكلة 1 وجود ∞
المشكلة 2 الإقتران غير متصل عند $x=0$

تذكر: يجب أي يحوي
كل تكامل على مشكلة
واحدة فقط إما infinite
او عدم إتصال

$= \int_0^1 \frac{1}{\sqrt{x}\,(x+1)} \, dx + \int_1^\infty \frac{1}{\sqrt{x}\,(x+1)} \, dx$

$= \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}\,(x+1)} \, dx + \lim_{u \to \infty} \int_1^u \frac{1}{\sqrt{x}\,(x+1)} \, dx$

2] $\int_{-\infty}^{\infty} \frac{1}{\sqrt[3]{x^2 - 1}} \, dx = \int_{-\infty}^{\infty} f(x) \, dx$

المشاكل $(-\infty / \infty / -1 / 1)$

$= \int_{-\infty}^{-2} f(x) \, dx + \int_{-2}^{-1} f(x) \, dx + \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx + \int_{1}^{2} f(x) \, dx + \int_{2}^{\infty} F(x) \, dx$

ثم تكمل الحل

/ /

ليس ضمن المجال فهو مستثنى

1] $\int_{-\infty}^{0} \frac{1}{3-\frac{4}{x}}\, dx$ المجال كل $(0 / -\infty / \frac{4}{3})$

$$= \int_{-\infty}^{-1} \frac{1}{3-\frac{4}{x}}\, dx + \int_{-1}^{0} \frac{1}{3-\frac{4}{x}}\, dx$$

$$= \lim_{t \to -\infty} \int_{t}^{-1} \frac{1}{3-\frac{4}{x}}\, dx + \lim_{u \to 0^-} \int_{-1}^{u} \frac{1}{3-\frac{4}{x}}\, dx$$

$$= \lim_{t \to -\infty} \int_{t}^{-1} \frac{1}{3} + \frac{4}{9} \cdot \frac{3}{(3x-4)}\, dx + \lim_{u \to 0^-} \int_{-1}^{u} \frac{1}{3} + \frac{4}{9} \cdot \frac{3}{(3x-4)}\, dx$$

$$= \lim_{t \to -\infty} \left(\frac{x}{3} + \frac{4}{9} \ln|3x-4|\right)\Big]_{t}^{-1} + \lim_{u \to 0^-} \left(\frac{x}{3} + \frac{4}{9} \ln|3x-4|\right)\Big]_{-1}^{u}$$

ثم نكمل السؤال لمعرفة فيما إذا كان (conv.) و (Dive.)

2] $\int_{-\infty}^{0} 2^r\, dr$ method 1 $= \int_{-\infty}^{0} 2^r\, dr = \frac{2^r}{\ln 2}\Big]_{-\infty}^{0}$

$$= \frac{2^0}{\ln 2} - \frac{\overset{\to 0}{2^{-\infty}}}{\ln 2} = \frac{1}{\ln 2} \text{ (conv.)}$$

استخدم الطريقة 2 دائماً في الإمتحان

method 2: $\int_{-\infty}^{0} 2^r\, dr = \lim_{t \to -\infty} \int_{t}^{0} 2^r\, dr$

$$= \lim_{t \to -\infty} \frac{2^r}{\ln 2}\Big]_{t}^{0} = \lim_{t \to -\infty} \left(\frac{1}{\ln 2} - \frac{2^r}{\ln 2}\right) = \frac{1}{\ln 2} - 0$$

(conv.)

3] $\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$

$= \lim_{t \to -\infty} \int_{t}^{0} x e^{-x^2} dx + \lim_{u \to +\infty} \int_{0}^{u} x e^{-x^2} dx$

$u = -x^2$

$du = -2x\,dx$

$x\,dx = \frac{du}{-2}$

$= \lim_{t \to -\infty} \int_{t}^{0} \frac{e^u}{-2} du + \lim_{u \to \infty} \int_{0}^{u} \frac{e^u}{-2} du$

مختلفة عن ال u في $\int_{0}^{u}$

$= \lim_{t \to -\infty} \frac{e^u}{-2} \Big]_{t}^{0} + \lim_{u \to \infty} \frac{e^u}{-2} \Big]_{0}^{u}$

$= \lim_{t \to -\infty} \frac{-e^{-x^2}}{2} \Big]_{t}^{0} + \lim_{u \to \infty} \frac{-e^{-x^2}}{2} \Big]_{0}^{u}$

$= \lim_{t \to -\infty} \left(\frac{-1}{2} + \frac{e^{-t^2}}{2} \right) + \lim_{u \to \infty} \left(\frac{-e^{-u^2}}{2} + \frac{1}{2} \right)$

$= \frac{-1}{2} + \frac{1}{2} = 0$ (conv.)

4] $\int_{-2}^{3} \frac{1}{x^4} dx = \int_{-2}^{0} \frac{1}{x^4} dx + \int_{0}^{3} \frac{1}{x^4} dx$

$= \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^4} dx + \lim_{u \to 0^+} \int_{u}^{3} \frac{1}{x^4} dx$

$= \lim_{t \to 0^-} \frac{-x^{-3}}{3} \Big]_{-2}^{t} + \lim_{u \to 0^+} \frac{-x^{-3}}{3} \Big]_{u}^{3} = \infty$ (div.)

[نحتاج أن نراجع موضوع
الرسم من ≤ 1 من أجل
الوحدة ال 6]

lecture 3

20/10/2023
الجمعة

section 1.2

1*) polynomials $\Rightarrow f(x) = ax^n$

n: عدد صحيح موجب
أو الصفر

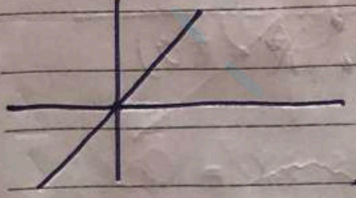
$f(x) = 3 \; / \; g(x) = 2 + 4x \; / \; P(x) = x^{17} + 3x^9 + x \Rightarrow$ polynomials

$h(x) = 3 + x^{\frac{1}{2}} \; / \; Q(x) = 3x^{-2} \; / \; l(x) = 5\sqrt[4]{x^3} + 3 \Rightarrow$ non polynomials

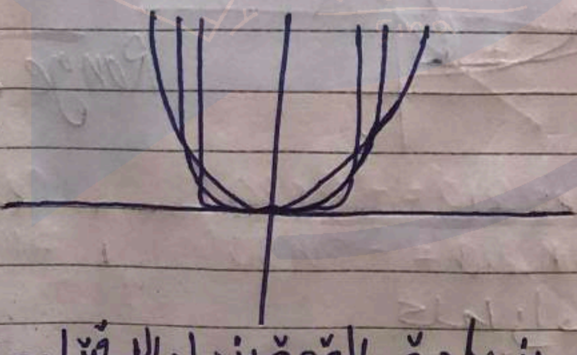
The domain of any polynomial is $\mathbb{R}$, but the range depends on the function

2) power function $\Rightarrow f(x) = ax^n \Rightarrow n \in R$

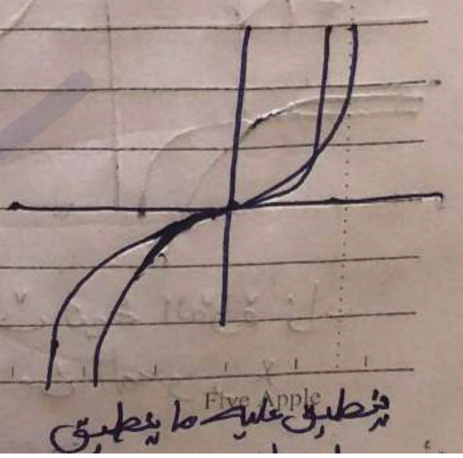
$f(x) = x$



$f(x) = x^2, x^4, x^6, \dots$



$f(x) = x^3, x^5, x^7, \dots$



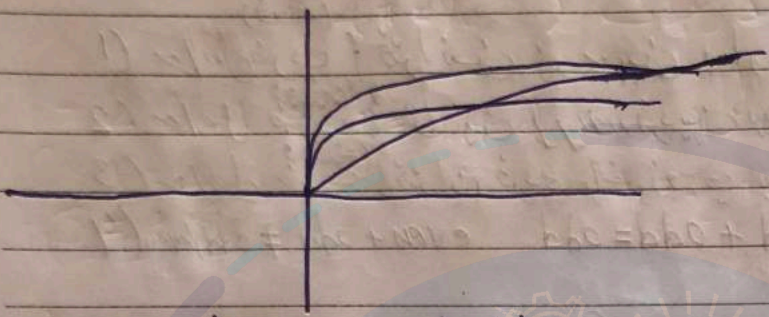
بزيادة القوة يزداد الاقتراب نحو المحور x

$\Leftarrow x^6$ أقرب من x^2, x^4 من المحور x

تنطبق عليه ما ينطبق

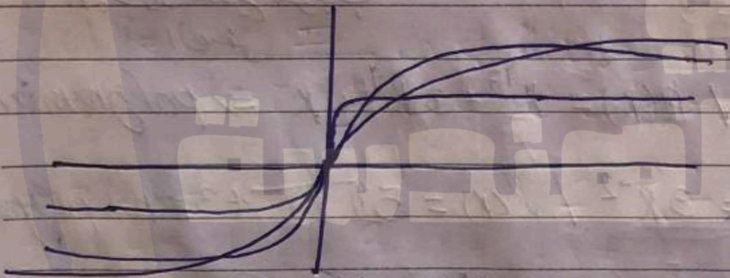
$f(x) = \sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \ldots$

Domain: $[0, \infty)$
Range: $[0, \infty)$



كلما زادت رتبة الجذر يزيد
الإقتراب من المحور y

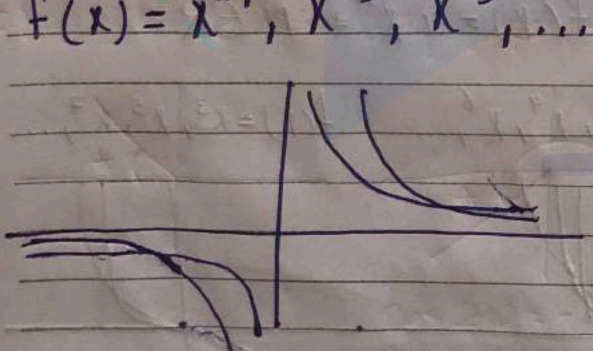
$f(x) = \sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \ldots$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

كلما زادت رتبة الجذر يزيد
الإقتراب من المحور y

$f(x) = x^{-1}, x^{-3}, x^{-5}, \ldots$



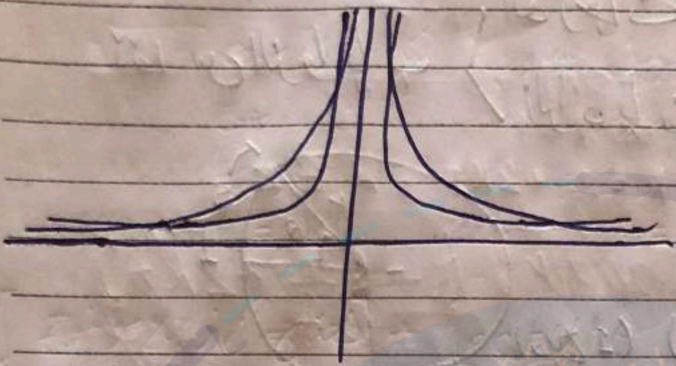
Domain: $R - \{0\}$

Range: $R - (0)$

كلما زادت قيمة القوة زاد
الإقتراب من المحور x

$f(x) = x^{-2}, x^{-4}, x^{-6}, \ldots$

Domain: $R - \{0\}$
Range: $(0, \infty)$



كلما زادت قيمة القوة زاد
الإقتراب من المحور x

3) Rational functions {الإقترانات العكسية}

↳ $\dfrac{\text{polynomial}}{\text{polynomial}}$ Domain: $R - \{$أصفار المقام إن وجدت$\}$

4) Algebraic functions {الإقترانات الجبرية}

↳ اقترانات تستخدم العمليات الجبرية

↓

(الجمع / الطرح / الضرب / القسمة / أخذ الجذور)

$f(x) = \dfrac{x\sqrt{x} - 1}{x^{\frac{5}{3}} + x\sqrt{x}}$ $g(x) = \sqrt{x^2+1}$

5) Transcendental functions

↳ الإقترانات غير الجبرية

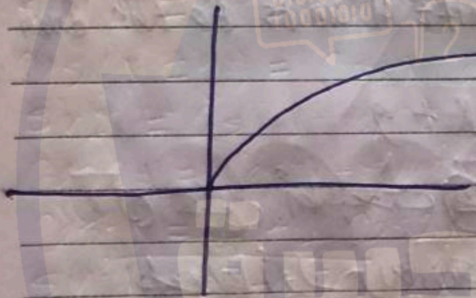
$f(x) = \cos x$ $g(x) = \sin x$ $h(x) = \log x$

section 1.3

vertical and horizontal shifts:

1) $y = f(x) + c$ c is a distance to the upward (للأعل
2) $y = f(x) - c$ c is a distance to the downward (للأسفل)
3) $y = f(x - c)$ c is a distance to the right
4) $y = f(x + c)$ c is a distance to the left

$f(x) = \sqrt{x}$



* نرسم الرسمة الأساسية وعلى أساسها نرسم البقية

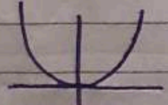
sketch $f(x) = x^2 - 4x + 5$

* نحول لإكمال مربع

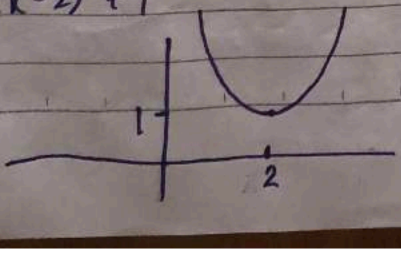
$x^2 - 4x + 5 = x^2 - 4x + 4 - 4 + 5$

$= (x-2)^2 + 1$

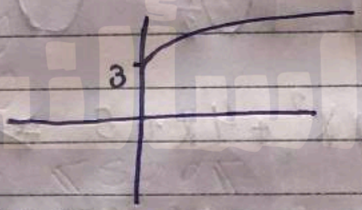
x^2



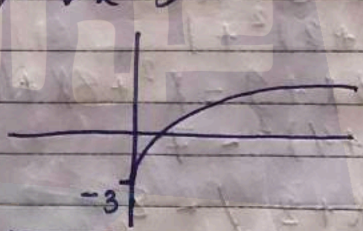
$f(x) = (x-2)^2 + 1$



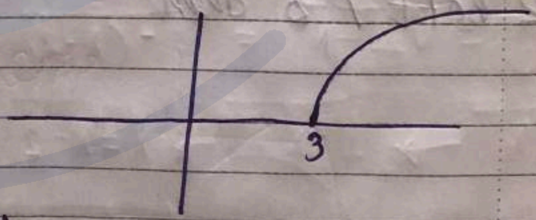
1) $f(x) = \sqrt{x} + 3$



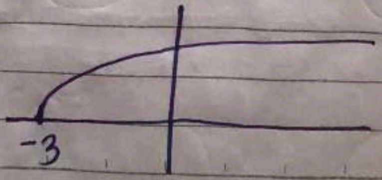
2) $f(x) = \sqrt{x} - 3$



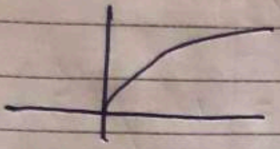
3) $f(x) = \sqrt{x-3}$



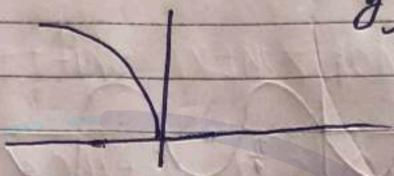
4) $f(x) = \sqrt{x+3}$



$\sqrt{x}$

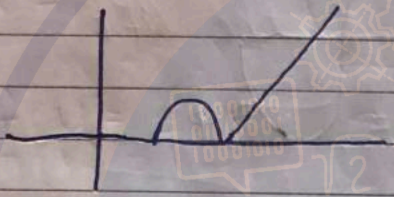


$\sqrt{-x}$

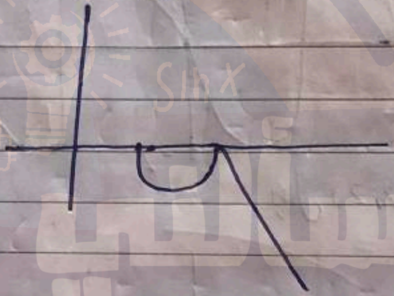


$\Rightarrow f(-x)$
بانعكاس حول محور y

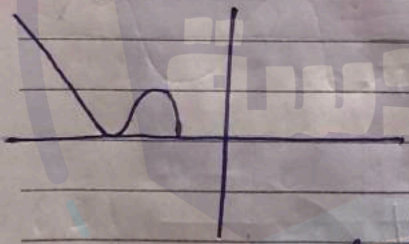
$f(x)$



$-f(x)$



$f(-x)$



$f(x) = \sqrt{1+2x}$ from $g(x) = \sqrt{x}$

(1) $f(x) = \sqrt{2(\frac{1}{2}+x)}$

(2) $\sqrt{x} \longrightarrow \sqrt{2x}$

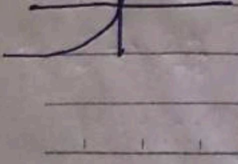
(3) $x = \frac{1}{2}+x \rightarrow \sqrt{2(\frac{1}{2}+x)}$

(1) horizontal ~~shift~~ shrinking of $\sqrt{x}$ by a factor of 2.
(2) shifting $\frac{1}{2}$ unit to the left.

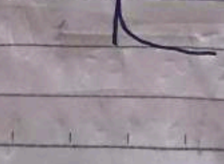
$y = \sqrt[3]{2-x} = \sqrt[3]{-(x-2)}$

$\sqrt[3]{x} \rightarrow \sqrt[3]{-x} \rightarrow x = x-2 \rightarrow \sqrt[3]{-(x-2)}$

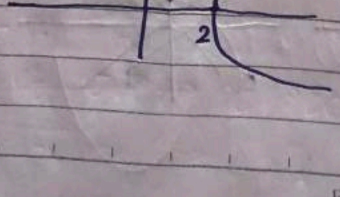
①



②



③



3.11 | Hyperbolic functions lecture 15

① $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ Dom f(x): R

Range of f(x): $[1, \infty)$

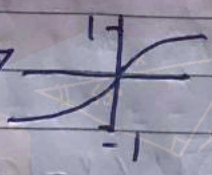
$* \cosh^2 x - \sinh^2 x = 1$

② $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$ Dom f(x): R

Range of f(x): R

R
(-1,1) مدى

$\tanh x = \frac{\sinh x}{\cosh x} \Rightarrow$



$\operatorname{sech} x = \frac{1}{\cosh x}$

$\operatorname{csch} x = \frac{1}{\sinh x}$ $\coth x = \frac{\cosh x}{\sinh x}$

$\sinh(-x) = -\sinh x$

$\cosh(-x) = \cosh x$

$\tanh(-x) = -\tanh x$

$* \operatorname{sech}^2 x + \tanh^2 x = 1$

chapter 6: Applications of Integrati

6.1] Areas Between curves:

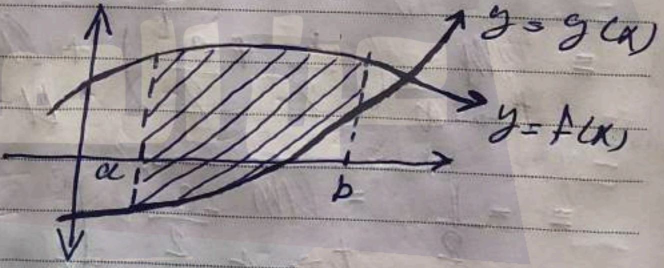
1] في حال طلب المساحة المحصورة بين الإقتران $y = f(x)$ والإقتران $y = g(x)$ وخطين يمران بالمحور x ($x = a$ / $x = b$) فإن المساحة تكون:

$$A = \int_a^b (f(x) - g(x))\,dx$$

الإقتراه العلوي − الإقتران السفلي

[بإعتبار أن $g(x) \leq f(x)$ لكل قيم $x \in [a,b]$
وكلا هما متصلان على الفترة ↑]

{وفي هذه الحالة لا نضع مطلق لأننا نعرف الإقتران العلوي من السفلي}

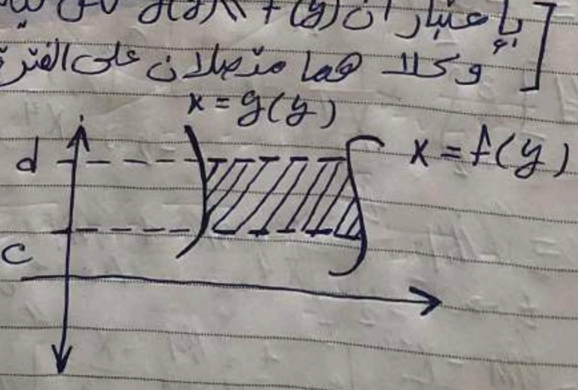


2] في حال طلب المساحة المحصورة بين الإقتران $x = f(y)$ والإقتران $x = g(y)$ وخطين يمران بالمحور y ($y = c$ / $y = d$) فإن المساحة تكون:

$$A = \int_c^d (f(y) - g(y))\,dy$$

الإقتران الاكبر − الإقتران الأصغر

[بإعتبار أن $g(y) \leq f(y)$ لكل قيم $y \in [c,d]$
وكلا هما متصلان على الفترة ↑]



3] المساحة المحصورة بين $y = f(x)$ و $y = g(x)$ والخطين $(x=a / x=b)$
في حال عدم معرفة الإقتران العلوي من السفلي

$$A = \int_a^b |f(x) - g(x)|\, dx$$

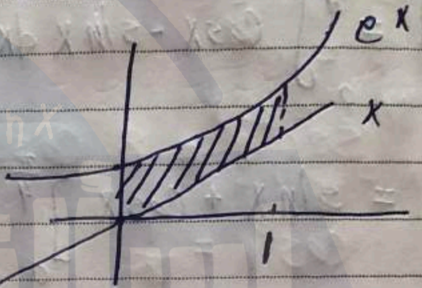
{نأخذ قيمة مطلقة للجواب النهائي}

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & f \geq g \\ g(x) - f(x) & f < g \end{cases}$$

Ex: 1 اوجد المساحة المحصورة بين $y = e^x$ و $y = x$ والخطين $x=1 / x=0$

$$A = \int_0^1 (e^x - x)\, dx = e^x - \frac{x^2}{2} \Big]_0^1$$

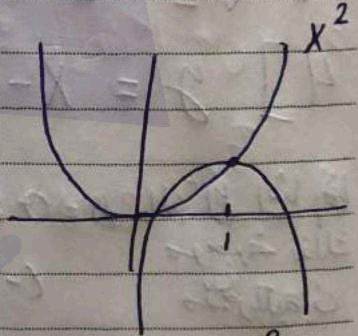
$$= \left(e - \frac{1}{2}\right) - (1) = e - \frac{3}{2}$$



2] $y = x^2$ / $y = 2x - x^2$ اوجد المساحة المحصورة بين الإقترانين

الرأس $= -\frac{b}{2a} = \frac{-2}{-2} = 1$

$(1, 1)$

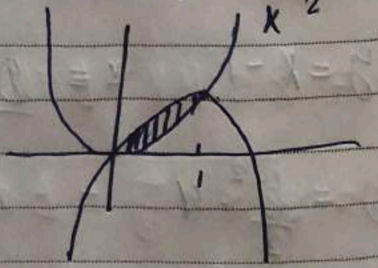


$x^2 = 2x - x^2$ {لمعرفة نقاط التقاطع}

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 0 \quad x = 1$

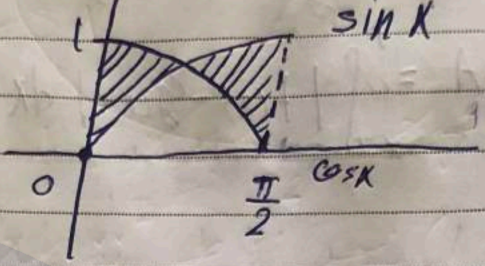


$$A = \int_0^1 (2x - x^2 - x^2)\, dx$$

$$= \int_0^1 (2x - 2x^2)\, dx = x^2 - \frac{2x^3}{3} \Big]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

3] $y = \sin x \;/\; y = \cos x$ اوجد المنطقة المحصورة

$x = 0 \qquad x = \frac{\pi}{2}$



$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4} \Rightarrow$ نقطة التقاطع

$$A = \int_{0}^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx$$

$$= \sin x + \cos x \Big]_{0}^{\frac{\pi}{4}} + -\cos x - \sin x \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (1) + (-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2} - 2 + \sqrt{2} = 2\sqrt{2} - 2$$

4] $y = x - 1 \;/\; y^2 = 2x + 6$ أوجد المنطقة المحصورة

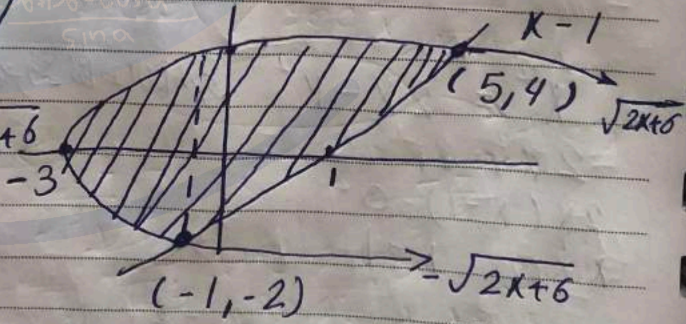
ال y^2 اشارتها موجبة لذلك مقعر لليمين

$y = 0 \Rightarrow 2x + 6 = 0$

$x = -3$

$(-3, 0)$

$y = \pm\sqrt{2x+6}$



$y = x - 1 \qquad x = \frac{y^2 - 6}{2}$

$y = \frac{y^2 - 4}{2}$

$2y = y^2 - 8$

$y^2 - 2y - 8 = 0$

$(y - 4)(y + 2) = 0$

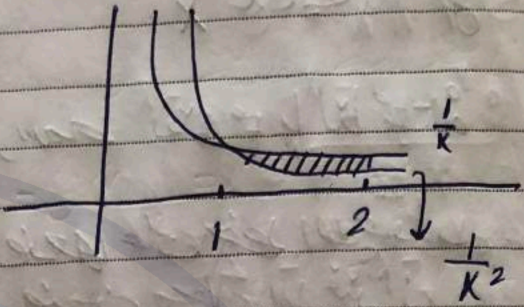
$y = 4 \quad y = -2$

$$A = \int_{-3}^{-1} \sqrt{2x+6} - (-\sqrt{2x+6}) \, dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1 \, dx$$

$$= 18$$

5] $y=\frac{1}{x}$ / $y=\frac{1}{x^2}$ اوجد المساحة المحصورة

$x=2$



$\frac{1}{x}=\frac{1}{x^2}$

$x^2=x$

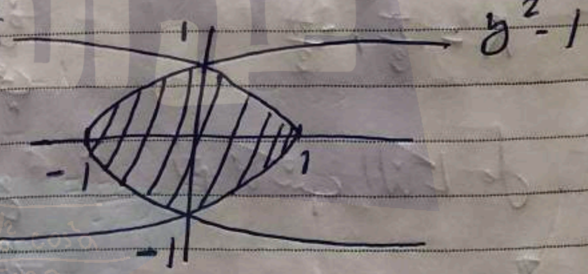
$x(x-1)=0$

$x=0 \quad x=1$

$$A=\int_1^2 \frac{1}{x}-\frac{1}{x^{+2}}\,dx = \ln|x|+\frac{1}{x}\Big]_1^2 = \ln 2+\frac{1}{2}-1$$

$$=\ln 2-\frac{1}{2}$$

6] $x=1-y^2 \qquad x=y^2-1$



$$A=\int_{-1}^{1} 1-y^2-(y^2-1)\,dy$$

$$=\int_{-1}^{1} 2-2y^2\,dy$$

$$=2y-\frac{2}{3}y^3\Big]_{-1}^{1} = 2-\frac{2}{3}-\left(-2+\frac{2}{3}\right)$$

$$=2-\frac{2}{3}+2-\frac{2}{3}=4-\frac{4}{3}=\frac{8}{3}$$

7] أوجد المنطقة المحصورة ضمن الفترة $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

$y = \sec^2 x \qquad y = 8\cos x$

في حال اعطاني إقتران لا أعرف كيف أرسمه نقوم بالتالي:

1) نجد نقاط التقاطع

2) نضعها على خط الاعداد ضمن مجال الإقتران

3) نعوض رقم بين كل رقمين لمعرفة العلوي من السفلي

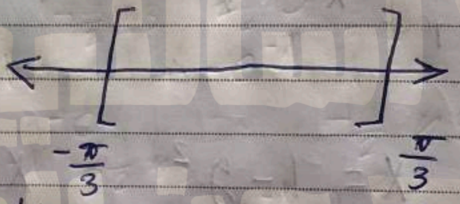
1) $\sec^2 x = 8\cos x$

$\cos^3 x = \frac{1}{8}$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, -\frac{\pi}{3}$

2)

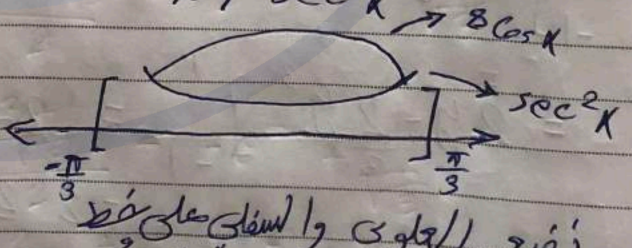


نقاط التقاطع هي نفسها اطراف الفترة نعوض رقم بينهم صفر مثلا في كلا الإقترانين

$y = 8\cos 0 \qquad y = \sec^2 0$

$y = 8 \qquad y = 1$

$8\cos x > \sec^2 x$



نضع العلوي والسفلي على خط الاعداد لتفادي الخطأ

3) ثم نكتب التكامل

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 8\cos x - \sec^2 x \, dx$$

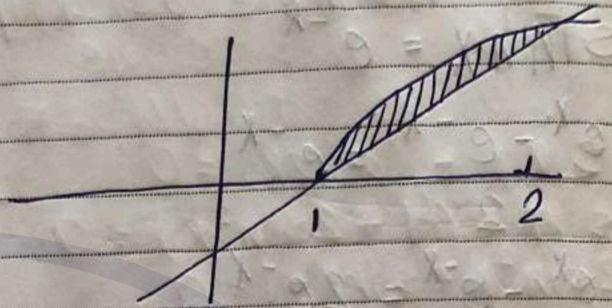
$$= 8\sin x - \tan x \Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 8\sin\frac{\pi}{3} - \tan\frac{\pi}{3} - \left(-8\sin\frac{\pi}{3} + \tan\frac{\pi}{3}\right)$$

$$= \frac{8\sqrt{3}}{2} - \sqrt{3} + \frac{8\sqrt{3}}{2} - \sqrt{3} = 8\sqrt{3} - 2\sqrt{3}$$

$$= 6\sqrt{3}$$

8] $y = \sqrt{x-1}$ $\quad x - y = 1 \Rightarrow y = x - 1$



$\sqrt{x-1} = x - 1$

$(x-1) - (x-1)^2 = 0$

$(x-1)(1 - (x-1)) = 0$

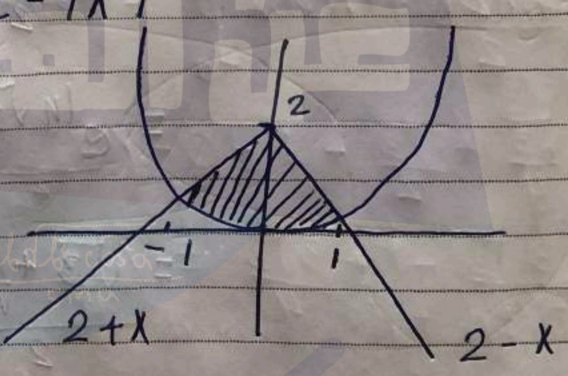
$x = 1 \qquad x = 2$

$$A = \int_1^2 \sqrt{x-1} - x + 1 \, dx = \frac{2(x-1)^{\frac{3}{2}}}{3} - \frac{x^2}{2} + x \Big]_1^2$$

$$= \frac{2}{3} - \frac{4}{2} + 2 - \left(0 - \frac{1}{2} + 1\right)$$

$$= \frac{2}{3} + -\frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

9] $y = x^4 \qquad y = 2 - |x|$



$2 - x = x^4 \Rightarrow$ نجرب التعويض

$2 - (1) \overset{?}{=} (1)^4$ بدلا من التحليل

$1 = 1$ ✓

$2 + x = x^4$

$2 + (-1) \overset{?}{=} (-1)^4$

$1 = 1$ ✓

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x \le 0 \end{cases}$$

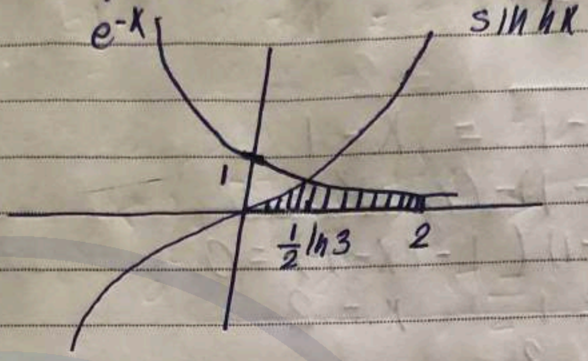
$$A = \int_{-1}^{0} 2 + x - x^4 \, dx + \int_0^1 2 - x - x^4 \, dx$$

$$y = \begin{cases} 2 - x, & x > 0 \\ 2 + x, & x \le 0 \end{cases}$$

$$= 2x + \frac{x^2}{2} - \frac{x^5}{5} \Big]_{-1}^{0} + 2x - \frac{x^2}{2} - \frac{x^5}{5} \Big]_0^1$$

$$= -\left(-2 + \frac{1}{2} + \frac{1}{5}\right) + 2 - \frac{1}{2} - \frac{1}{5} = 4 - 1 - \frac{2}{5} = 3 - \frac{2}{5} = \frac{13}{5}$$

10] $y = \sinh x$, $y = e^{-x}$, $x = 0$, $x = 2$, $y = 0$



$\sinh x = e^{-x}$

$$\frac{e^{x} - e^{-x}}{2} = e^{-x}$$

$e^{x} - e^{-x} = 2e^{-x}$

$3e^{-x} = e^{x}$

$3 = e^{2x}$

$\ln 3 = 2x$

$x = \frac{1}{2}\ln 3$

$$A = \int_{0}^{\frac{1}{2}\ln 3} \sinh x \, dx + \int_{\frac{1}{2}\ln 3}^{2} e^{-x} \, dx$$

$$= \cosh x \Big]_{0}^{\frac{1}{2}\ln 3} + -e^{-x} \Big]_{\frac{1}{2}\ln 3}^{2}$$

$$= \frac{e^{x} + e^{-x}}{2} \Big]_{0}^{\frac{1}{2}\ln 3} - e^{-x} \Big]_{\frac{1}{2}\ln 3}^{2}$$

$$= \frac{e^{\ln\sqrt{3}} + e^{\ln\frac{1}{\sqrt{3}}}}{2} - \frac{1+1}{2} - e^{-2} + e^{\ln\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} + \frac{1}{\sqrt{3}}}{2} - 1 - e^{-2} + \frac{1}{\sqrt{3}}$$

$$= \frac{\frac{3+1}{\sqrt{3}}}{2} - 1 - e^{-2} + \frac{1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - 1 - \frac{1}{e^{2}}$$

$$= \sqrt{3} - 1 - \frac{1}{e^{2}}$$

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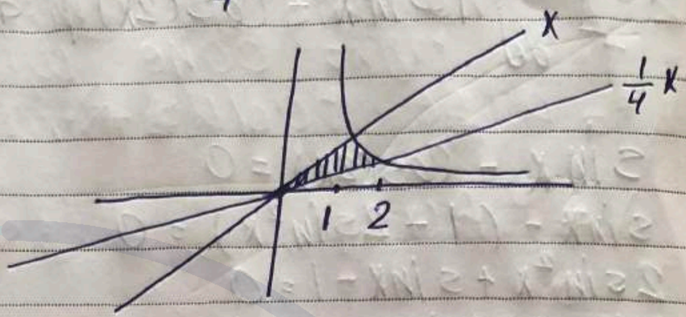
11] $y = \frac{1}{x}$, $y = x$ $y = \frac{1}{4}x$ $x > 0$



$\frac{1}{x} = x$
$x^2 = 1$
$x = 1$

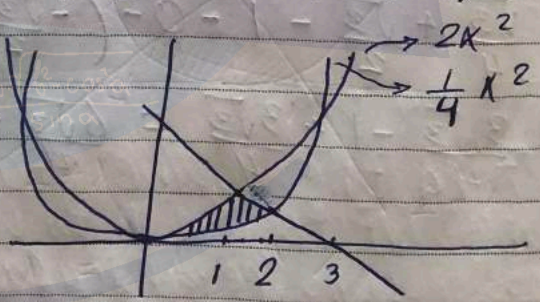
$\frac{1}{x} = \frac{1}{4}x$
$x^2 = 4$
$x = 2$

$$A = \int_0^1 x - \frac{1}{4}x \, dx + \int_1^2 \frac{1}{x} - \frac{1}{4}x \, dx$$

$$= \frac{x^2}{2} - \frac{x^2}{8} \Big]_0^1 + \ln|x| - \frac{x^2}{8} \Big]_1^2$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{8}} + \ln 2 - \cancel{\frac{1}{2}} - (0 - \cancel{\frac{1}{8}}) = \ln 2$$

12] $y = \frac{1}{4}x^2$, $y = 2x^2$, $y = 3 - x$, $x \geq 0$



$\cancel{\frac{1}{4}x^2 = 2x^2}$
$\cancel{x^2 = 8x^2}$

$\frac{1}{4}x^2 = 3 - x$
$x^2 = 12 - 4x$
$x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
$x = 2$

$2x^2 = 3 - x$
$2x^2 + x - 3 = 0$
$(2x + 3)(x - 1) = 0$
$x = 1$

$$A = \int_0^1 2x^2 - \frac{1}{4}x^2 \, dx + \int_1^2 3 - x - \frac{1}{4}x^2 \, dx = \frac{3}{2}$$

DOMS

13] $\int_{0}^{\frac{\pi}{2}} |\sin x - \cos(2x)| \, dx$ $\quad f(x) = \sin x - \cos 2x$

$\sin x - \cos(2x) = 0$

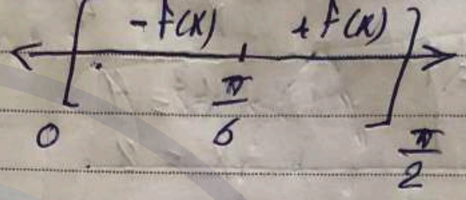
$\sin x - (1 - 2\sin^2 x) = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2} \qquad \sin x = -1$

$x = \frac{\pi}{6} \qquad x = \notin 3$



$$\int_{0}^{\frac{\pi}{2}} |\sin x - \cos(2x)| \, dx = \int_{0}^{\frac{\pi}{6}} \cos 2x - \sin x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x - \cos 2x \, dx$$

$$= \frac{\sin 2x}{2} + \cos x \Big]_{0}^{\frac{\pi}{6}} - \cos x - \frac{\sin 2x}{2} \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 - 0 - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{3\sqrt{3}}{4} - 1 + \frac{3\sqrt{3}}{4} = \frac{6\sqrt{3}}{4} - 1 = \frac{3\sqrt{3}}{2} - 1$$

6.2 Volumes :

* Cross section Method * → [Disc Method]

{ لإيجاد حجوم الأشكال غير المنتظمة }

$$V = \int_a^b A(x)\,dx$$

$A(x)$ → مساحة ال cross section

ناخذها $A(x)$ في حال كان ال cross section عامودي على ال x-axis

[a,b] بداية الشكل على ال x-axis ونهايته

$$V = \int_c^d A(y)\,dy$$

⇒ أخذناها $A(y)$ لأننا أخذنا ال cross section عامودي على ال y-axis

[c,d] بداية الشكل على ال y-axis ونهايته

** طريقة ال cross section تكون بدون دوران

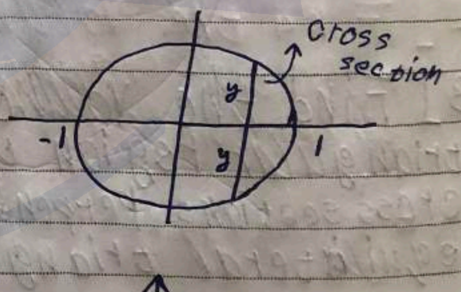
1] Find the volume of the solid with a circular base of radius 1. parallel cross sections perpendicular to the base are equilateral triangles.

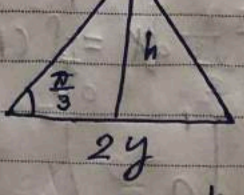
$$V = \int_{-1}^{1} A(x)\,dx \qquad A(x) = \frac{1}{2}bh$$

الدائرة نصف قطرها 1 فهي دائرة الوحدة

$$y^2 + x^2 = 1$$
$$y^2 = 1 - x^2$$

$$= \frac{1}{2}(2y)(\sqrt{3}y)$$
$$= \sqrt{3}y^2$$
$$= \sqrt{3}(1-x^2)$$





$$\tan\frac{\pi}{3} = \frac{h}{y}$$
$$\sqrt{3}y = h$$

$$V = \int_{-1}^{1} \sqrt{3} - \sqrt{3}x^2\,dx$$

$$= \sqrt{3}x - \frac{\sqrt{3}}{3}x^3\Big]_{-1}^{1}$$

$$= \sqrt{3} - \frac{\sqrt{3}}{3} - \left(-\sqrt{3} + \frac{\sqrt{3}}{3}\right)$$

$$= 2\sqrt{3} - \frac{2\sqrt{3}}{3} = \frac{4\sqrt{3}}{3} = \frac{4}{\sqrt{3}}$$

2] Find the volume of a frustum of a right cone with height h, lower base radius R, and top radius r.

$$V=\int_0^h A(x)\,dx$$

$$=\pi\int_0^h\left(\frac{R-r}{h}x+r\right)^2$$

$$=\pi\left.\frac{\left(\frac{R-r}{h}x+r\right)^3}{3\cdot\frac{R-r}{h}}\right]_0^h$$

$$=\frac{\pi}{3}h\left(R^2+rR+r^2\right)$$

$$A(x)=\pi R^2=\pi y^2$$

معادلة الخط المستقيم الواصل من (0,r) → (h,R)

$$m=\frac{R-r}{h}\qquad y-y_1=m(x-x_1)$$

$$y-r=\frac{R-r}{h}(x-0)$$

$$y=\frac{R-r}{h}x+r$$

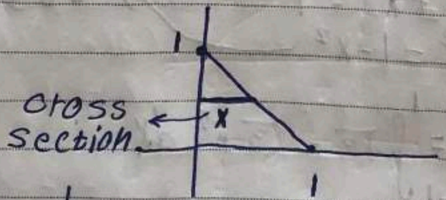
$$A(x)=\pi\left(\frac{R-r}{h}x+r\right)^2$$

3] Find the volume of a solid with base is triangular region with vertices (0,0),(0,1) and (1,0). cross sections perpendicular to the y-axis are equilateral triangles.

$$V=\frac{1}{2}\int_0^1 x^2\frac{\sqrt{3}}{2}\,dy=\frac{\sqrt{3}}{4}\int_0^1(1-y)^2\,dy$$

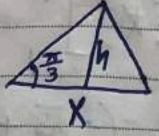
=

$$=\frac{\sqrt{3}}{12}$$



$$\tan\frac{\pi}{3}=\frac{h}{\frac{x}{2}}$$

$$\frac{\sqrt{3}}{2}x=h$$



$$y=1-x$$

$$x=1-y$$

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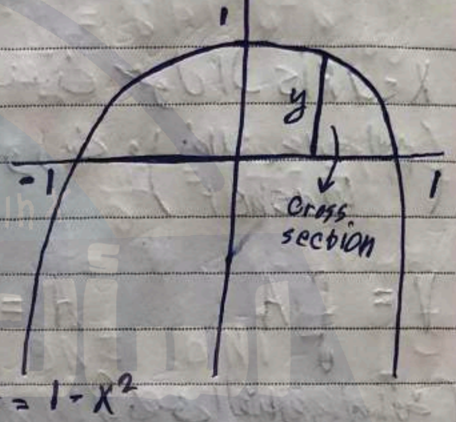
4] Find the volume of a solid with base is the region enclosed by the purabola $y=1-x^2$ and the x-axis cross sections perpendicular to the x-axis are isosceles tringles with heigh equal to the base.

$V = \int_{-1}^{1} A(x)\,dx$

$A(x) = \frac{1}{2} by$

$= \frac{1}{2} y^2$

$= \frac{1}{2} (1-x^2)^2$

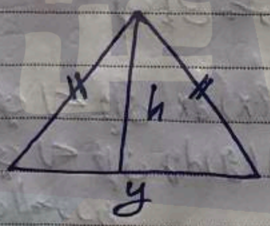


$= \int_{-1}^{1} \frac{1}{2} (1-x^2)^2 dx$

$= \frac{1}{2}\int_{-1}^{1} 1-2x^2+x^4 \, dx$

$= \frac{1}{2}\left(x-\frac{2}{3}x^3+\frac{x^5}{5}\right)\Big]_{-1}^{1}$

$= \frac{8}{15}$



$h = y$

$\Rightarrow \frac{1}{2}\int_{-1}^{1}(1-2x^2+x^4)dx = \frac{2}{2}\int_{0}^{1}(1-2x^2+x^4)\,dx$ even لأنّ

function

※ إيجاد الحجم الناتج عن دوران منحنى حول الـ x-axis او الـ y-axis او خط موازي لأحد المحورين .

$V = \int A \qquad A = \pi r^2$

(عند دوران المنحنى سينتج لدينا دوائر فالمساحة دائما تكون هي مساحة الدائرة) ⇐ {cross section is a disc}

※ إيجاد الحجم الناتج عن دوران منحنيين حول الـ x-axis او الـ y-axis او خط موازي لاحد المحورين . (نطلق عليه اسم washer)

⇐ {cross section is a washer}

$V = \int A \qquad A = \pi (R_1^2 - R_2^2)$

R_1: الدائرة ذات نصف القطر الاكبر

R_2: الدائرة ذات نصف القطر الاصغر

{نصف القطر هو المسافة العامودية من الإقتران إلى محور الدوران}

⇐ إذا كان الدوران حول x-axis او محور (خط) موازي له فإن $V = \int_b^a A(x)\,dx$

⇐ إذا كان الدوران حول y-axis او محور موازي له فإن $V = \int_d^c A(y)\,dy$

{a/b} قيم على المحور x   {c/d} قيم على المحور y

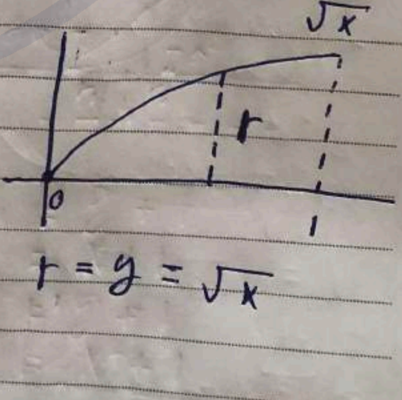
1] Find the volume of the solid obtaind by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1

$V = \int_0^1 A(x)\,dx \qquad A = \pi r^2$

$\qquad\qquad\qquad\qquad = \pi (\sqrt{x})^2$

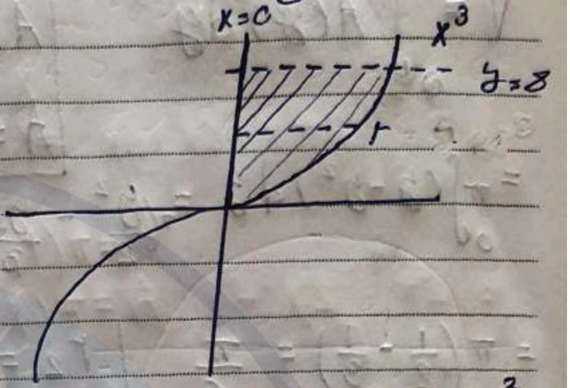
$= \int_0^1 \pi x\,dx \qquad = \pi x$

$= \frac{\pi}{2} x^2 \Big]_0^1 = \frac{\pi}{2}$



$r = y = \sqrt{x}$

2] Find the volume of the solid obtaind by rotating the region bounded by $y = x^3$, $y=8$ and $x=0$ about the y-axis.



$V = \int_0^8 A(y)\,dy$ $\quad A = \pi r^2$

$= \int_0^8 \pi y^{\frac{2}{3}} dy$ $\quad A = \pi y^{\frac{2}{3}}$

$= \frac{3\pi}{5} y^{\frac{5}{3}} \Big]_0^8$

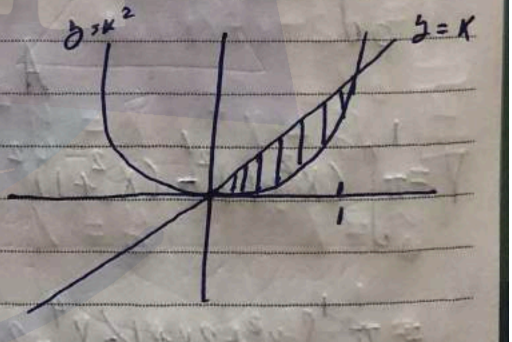
$= \frac{3\pi}{5} \sqrt[3]{(8)^5} = \frac{96\pi}{5}$

$r = x \quad y = x^3$

$\Downarrow \quad y^{\frac{1}{3}} = x$

$r = y^{\frac{1}{3}}$

3] Find the volume of region obtained by rotating $y=x$ and $y=x^2$ about the x-axis.



$V = \int_0^1 A(x)dx$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

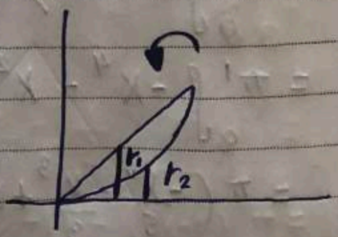
$= \pi \int_0^1 (x^2 - x^4)\,dx$

$A(x) = \pi(r_1^2 - r_2^2)$

$= \pi(x^2 - x^4)$

$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big]_0^1$

$= \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$



(يفضل أخذ ال r بالقرب من منحنى الإقتران)

$r_1 = y = x$

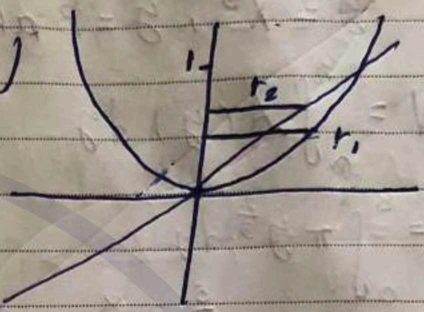
$r_2 = y = x^2$

4] Find the volume of the solid obtained by rotating $y=x$, $y=x^2$ about y-axis.

$V = \int_0^1 A(y)\,dy$

$A = \pi(r_1^2 - r_2^2)$

$A = \pi(y - y^2)$



$= \pi\int_0^1 (y-y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_0^1$

$= \pi\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi}{6}$

$r_2 = x$
$r_2 = y$

$r_1 = x$
$r_1 = \sqrt{y}$

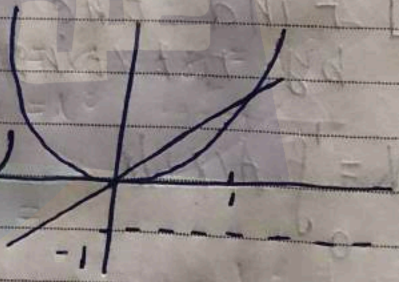
$y = x^2$
$x = \sqrt{y}$

5] Find the volume of the solid obtained by rotating the region enclosed by $y=x$, $y=x^2$ about the line $y=-1$

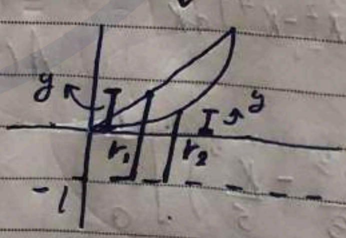
$V = \int_0^1 A(x)\,dx$

$A = \pi(r_1^2 - r_2^2)$

$A = \pi((x+1)^2 - (x^2+1)^2)$



$V = \pi\int_0^1 (x+1)^2 - (x^2+1)^2\,dx$



$= \pi\int_0^1 x^2+2x+\cancel{1}-x^4-2x^2-\cancel{1}\,dx$

$= \pi\int_0^1 -x^4 - x^2 + 2x\,dx$

$= \pi\left(-\frac{x^5}{5} - \frac{x^3}{3} + x^2\right)\Big]_0^1$

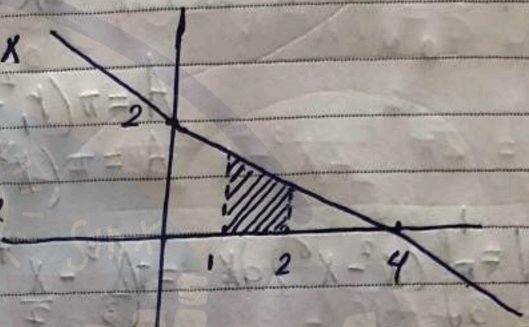
$= \pi\left(-\frac{1}{5} - \frac{1}{3} + 1\right) = \frac{7\pi}{15}$

$r_1 = y + 1$
$= x + 1$

$r_2 = y + 1$
$= x^2 + 1$

6] Find the volume of the solid obtained by rotating the region bounded by the given curves about the given lines.

$\boxed{1}$ $y = 2 - \frac{1}{2}x$, $y = 0$, $x = 1$, $x = 2$, about x-axis



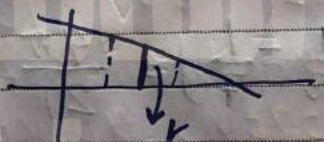
$V = \int_1^2 A(x)\,dx$ $\qquad A = \pi r^2$

$= \pi\left(2 - \frac{1}{2}x\right)^2$

$= \pi\int_1^2 \left(2 - \frac{1}{2}x\right)^2 dx = \pi\int_1^2 4 - 2x + \frac{1}{4}x^2\,dx$

$= \pi\left(4x - x^2 + \frac{x^3}{12}\right)\Big]_1^2$

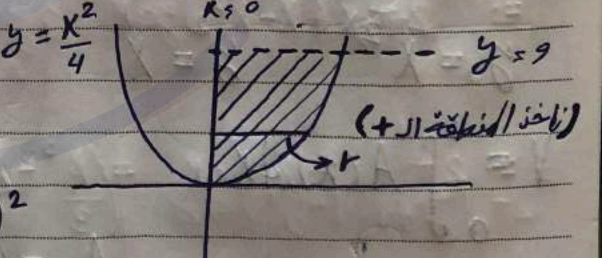
$= \frac{19\pi}{12}$



$r = y \qquad y = 2 - \frac{1}{2}x$

$r = 2 - \frac{1}{2}x$

$\boxed{2}$ $y = \frac{x^2}{4}$, $y = 9$, $x = 0$ about y-axis



$V = \int_0^9 A(y)\,dy$ $\qquad A = \pi r^2$

$A = \pi(2\sqrt{y})^2$

$A = 4\pi y$

$V = \int_0^9 4\pi y\,dy$

$= 2\pi y^2\Big]_0^9$

$= 162\pi$

$r = x \qquad y = \frac{x^2}{4}$

$r = 2\sqrt{y} \qquad x = 2\sqrt{y}$

$\boxed{3}$ $y = x^3 , y = x , x \geq 0$ about the x-axis

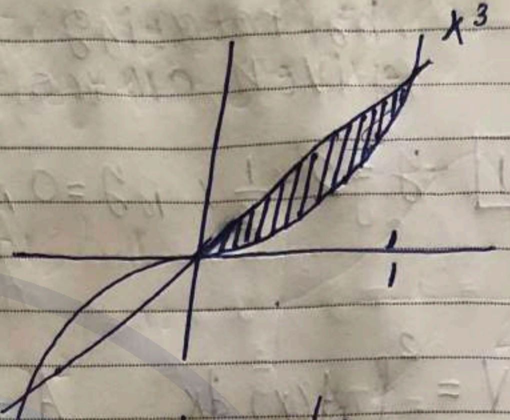
$V = \int_0^1 A(x)\, dx$

$x^3 = x$

$x^3 - x = 0$

$x(x^2-1) = 0$

$x = 0, -1, 1$

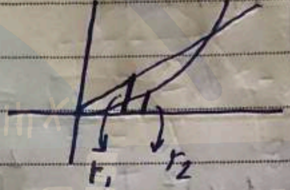


$A = \pi(r_1^2 - r_2^2)$

$A = \pi(x^2 - x^6)$

$V = \pi\int_0^1 (x^2 - x^6)\, dx = \pi\left(\frac{x^3}{3} - \frac{x^7}{7}\right)\Big]_0^1$

$= \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4\pi}{21}$



$r_1 = y$ $\quad y = x$

$r_1 = x$

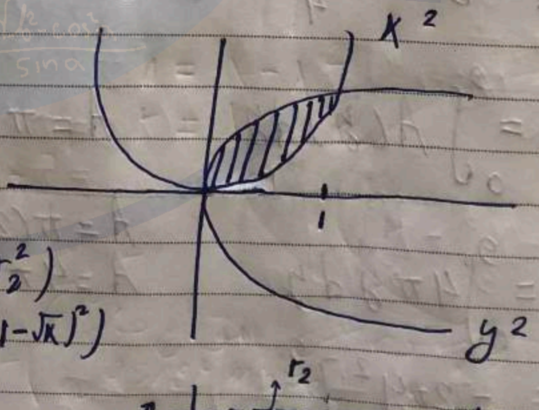
$r_2 = y$ $\quad y = x^3$

$r_2 = x^3$

$\boxed{4}$ $y = x^2 , x = y^2$ about $y = 1$

$y = x^2$ $\quad y^2 = x$

$V = \int_0^1 A(x)\, dx$

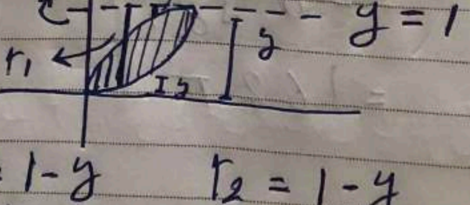


$A = \pi(r_1^2 - r_2^2)$

$= \pi\left((1-x^2)^2 - (1-\sqrt{x})^2\right)$

$V = \pi\int_0^1 (1-x^2)^2 - (1-\sqrt{x})^2\, dx$

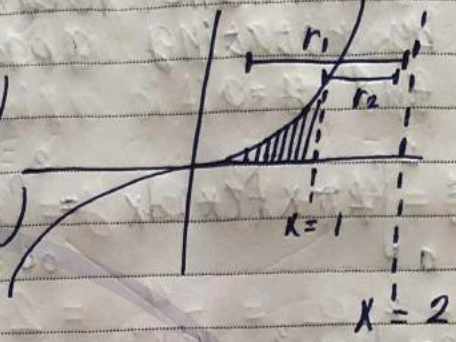
$= \frac{11\pi}{30}$



$r_1 = 1 - y$ $\quad r_2 = 1 - y$

$r_1 = 1 - x^2$ $\quad r_2 = 1 - \sqrt{x}$

$\boxed{5}$ $y = x^3, \; y = 0, \; x = 1$ about $x = 2$



$V = \int_0^1 A(y)\,dy \qquad A = \pi(r_1^2 - r_2^2)$

$A = \pi\left((2-\sqrt[3]{y})^2 - 1\right)$

$V = \pi\int_0^1 (2-\sqrt[3]{y})^2 - 1\,dy$

$= \pi\int_0^1 4 - 4\sqrt[3]{y} + y^{\frac{2}{3}} - 1\,dy \qquad r_1 = 2 - x \qquad y = x^3$

$r_1 = 2 - \sqrt[3]{y} \qquad x = \sqrt[3]{y}$

$= \pi\left(4y - \cancel{4}\,y^{\frac{4}{3}} \cdot \frac{3}{\cancel{4}} + \frac{3}{5}y^{\frac{5}{3}} - y\right)\Big]_0^1 \qquad r_2 = 1$

$= \pi\left(\cancel{4} - \cancel{3} + \frac{3}{5} \cancel{-1}\right)$

$= \frac{3}{5}\pi$

* Volumes by cylindrical shells *

Rotating about y-axis

$V = \int_a^b 2\pi x\, f(x)\, dx$ (او محور موازي لل y-axis)

$2\pi x \Rightarrow$ circumference

$x \Rightarrow$ radius

$f(x) \Rightarrow$ height

$dx \Rightarrow$ thickness

Rotating about x-axis

$V = \int_c^d 2\pi y\, f(y)\, dy$ (او محور موازي لل x-axis)

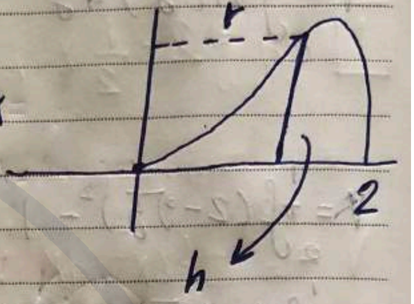
$y \Rightarrow$ radius

$f(y) \Rightarrow$ height

$dy \Rightarrow$ thickness

الإرتفاع يكون موازي لمحور الدوران أما
الـ R يكون من الخط الذي اعتبرته ارتفاع الى
محور الدوران حتى إن وجد اكثر من إقتران

1] Find the volume of the solid obtained by rotating about the y-axis $y = 2x^2 - x^3$ and $y = 0$



$$V = \int_0^2 2\pi x f(x)\,dx = \int_0^2 2\pi x(2x^2 - x^3)\,dx$$

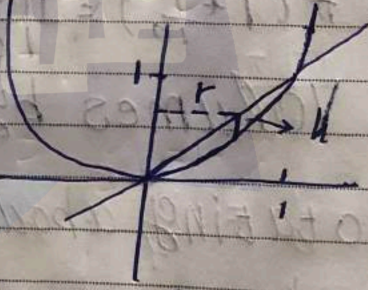
$$= 2\pi \int_0^2 2x^3 - x^4\,dx = 2\pi\left(\frac{x^4}{2} - \frac{x^5}{5}\right)\Big]_0^2$$

$h = 2x^2 - x^3$

$$= 2\pi\left(\frac{16}{2} - \frac{32}{5}\right) = \frac{32\pi}{10} = \frac{16\pi}{5}$$

$r = x$

2] Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$



$$V = \int_0^1 2\pi x f(x)\,dx$$

$$= \int_0^1 2\pi x(x - x^2)\,dx$$

$h = x - x^2$

$r = x$

$$= 2\pi \int_0^1 x^2 - x^3\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)2\pi\Big]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$

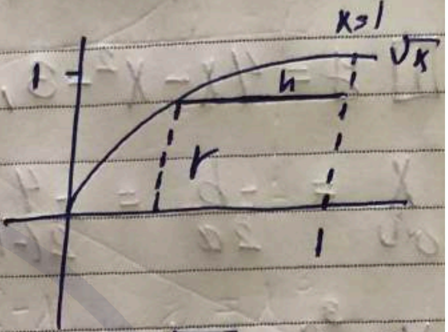
(ويمكن حل السؤال عن
طريق ال washer mehod)

3] Use cylindrical shells to find the volume of the solid obtained by rotating about x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$V = \int_0^1 2\pi y f(y)\,dy$$

$$= 2\pi\int_0^1 y(1-y)\,dy = 2\pi\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big]_0^1$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi}{3}$$



$h = 1 - \sqrt{x} = 1 - y$

$r = y$

4] Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$

$y = x - x^2$

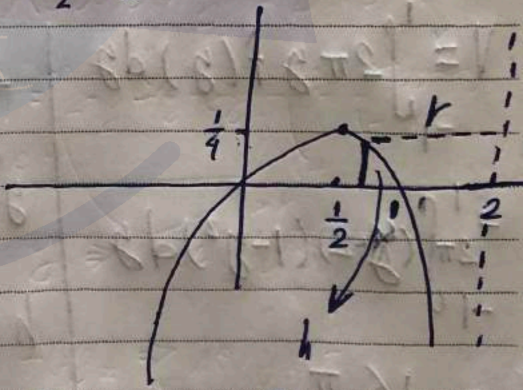
$x = \frac{-b}{2a} = \frac{-1}{-2} = \frac{1}{2}$

$$V = \int_0^1 2\pi x f(x)\,dx$$

$$= 2\pi\int_0^1 (2-x)(x-x^2)\,dx$$

$$= 2\pi\int_0^1 2x - 3x^2 + x^3\,dx$$

$$= 2\pi\left(x^2 - x^3 + \frac{x^4}{4}\right)\Big]_0^1 = \frac{\pi}{2}$$



$h = x - x^2$

$r = 2 - x$

5] Use the method of cylindrical shells to find the volume by rotating the region bounded by the given curves about the specific axis.

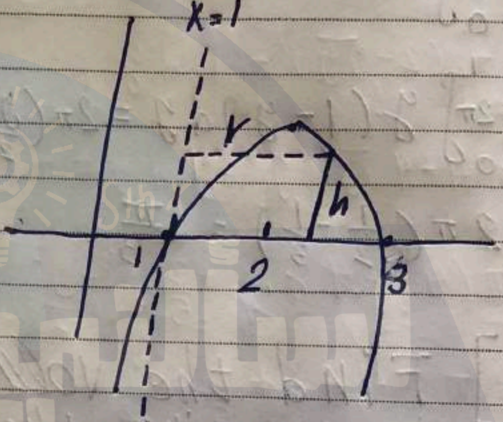
$\boxed{1}$ $y = 4x - x^2 - 3,\ y = 0$, about $x = 1$

$x_{\text{رأس}} = \frac{-b}{2a} = \frac{-4}{2(-1)} = 2$

$$V = \int_1^3 2\pi x f(x)\, dx$$

$$= 2\pi \int_1^3 (x-1)(4x-x^2-3)\, dx$$

$$= \frac{8\pi}{3}$$



$h = 4x - x^2 - 3$

$r = x - 1$

$\boxed{2}$ $x = y^2 + 1,\ x = 2y^2$, about $y = -2$

$$V = \int_{-1}^{1} 2\pi y f(y)\, dy$$

$$= 2\pi \int_{-1}^{1} (y+2)(1-y^2)\, dy$$

$$= \frac{16\pi}{8}$$

$2y^2 = y^2 + 1$

$y^2 = 1$

$y = \pm 1$

$h = y^2 + 1 - 2y^2$

$h = 1 - y^2$

$r = y + 2$