اسألني عن الهندسة
2018

التفاضل و التكامل (1)

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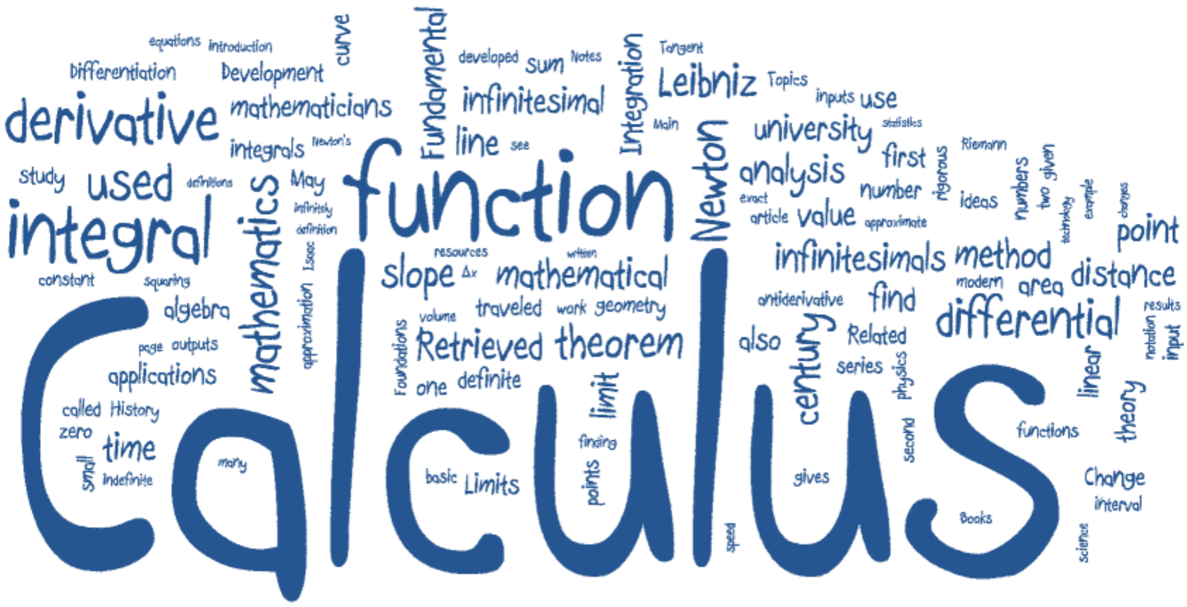
إعداد الطالب : قتيبة الروابدة



بسم الله الرحمن الرحيم

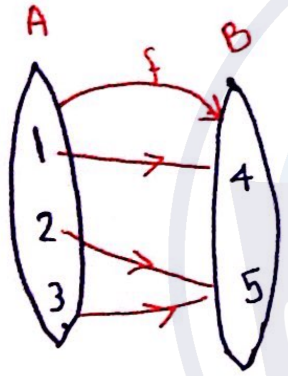
CALCULUS "101"

Chapter # 1

⇒ Functions 8

$\boxed{1.1}$:)

⇒ Definition (Def) :- a function is that assigns to every element in a set "A" a unique element in a set "B". $f: A \longrightarrow B$



$f(1) = 4$

$f(2) = 5$

$f(3) = 5$

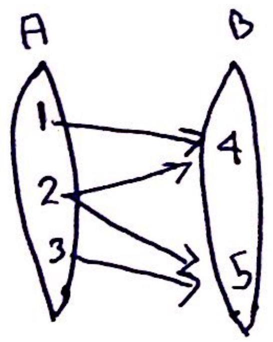
* → Domain of $(f) = D_f = \text{dom}(f)$ ⇒ A is the Domain of f "المجال"

$D_f = \{1, 2, 3\}$

→ Range of $(f) = R_f$ ⇒ B is the range of f "المدى"

$R_f = \{4, 5\}$

⇒



It is a relation

It isn't a function

Since ⇒ $f_{(2)} = 4$ and $f_{(3)} = 4$

⇑

because

⇒ Real numbers ≡ $\mathbb{R}$ $(-\infty, \infty)$

rationals	irrationals
2	$\sqrt{5}$
-3	π
0	
$\frac{4}{5}$	e ← نيبيري

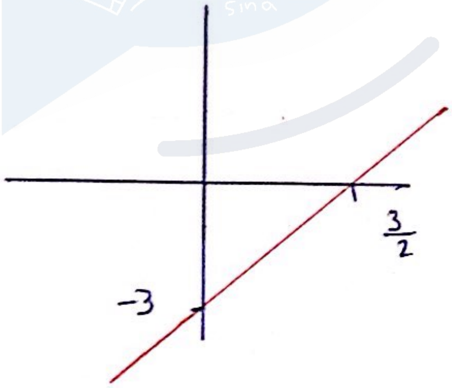
⇒ Set : $\{1, 2, 3\}$

Interval : $(2, 5)$ / $[2, 5)$

⇒ Basic functions :

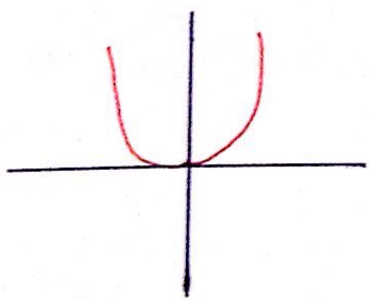
(1) $f(x) = 2x - 3$ (Linear function)

$D_f = \mathbb{R} = R_f$



(2) $f(x) = x^2$ (Parabola)

مكافئ



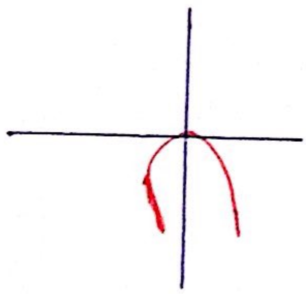
$D_f = \mathbb{R}$

$R_f = [0, \infty)$

(3) $f(x) = -x^2$

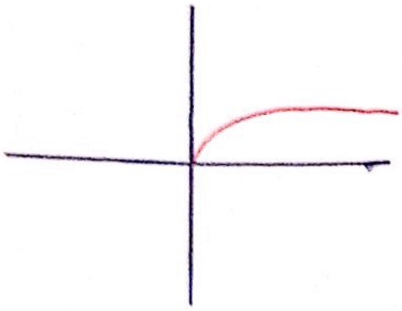
$D_f = \mathbb{R}$

$R_f = (-\infty, 0]$



(4) $f(x) = \sqrt{x}$

$D_f = [0, \infty) = R_f$



$f(-2) =$ Undefine

Ex ⇒ Find the domain: $f(x) = \sqrt{3 - x}$

$3 - x \geq 0$

$3 \geq x$

$D_f = (-\infty, 3]$

(5) $f(x) = \frac{1}{x}$

$f(1) = 1$

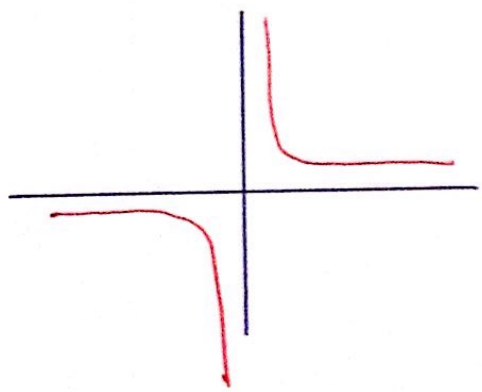
$f(2) = \frac{1}{2}$

$f(\frac{1}{100}) = 100$

$f(0) = \frac{1}{0}$ Undefine

$D_f = \mathbb{R}^* = (-\infty, 0) \cup (0, \infty) = (-\infty, \infty) - \{0\} = R_f$

⇒ $\mathbb{R}^*$ means : $\mathbb{R}$ except "0"



1) ⇒ $f(x) = \dfrac{x^2 - 4}{x - 2}$, Find the domain :

Don't simplify

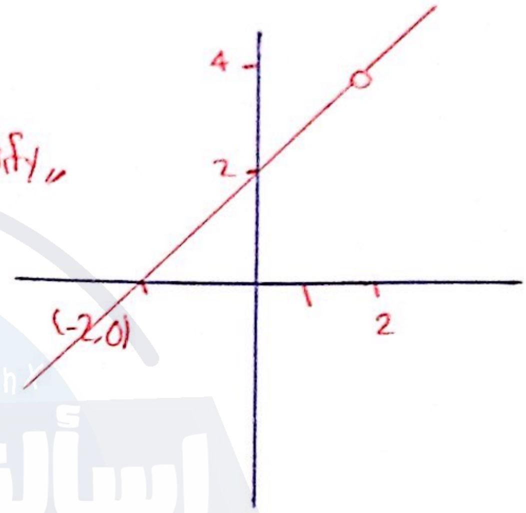
$$x - 2 \neq 0$$
$$x \neq 2$$

$\therefore\ D_f = \mathbb{R} - \{2\}$

⇒ Find the range and graph «simplify»

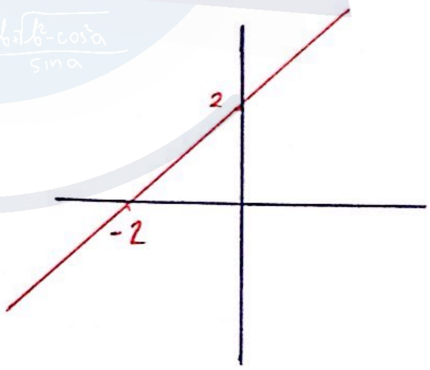
$$\frac{x^2 - 4}{x - 2} = \frac{(\cancel{x-2})(x+2)}{\cancel{x-2}}$$

$$R_f = \mathbb{R} - \{4\}$$



2) $g(x) = x + 2$

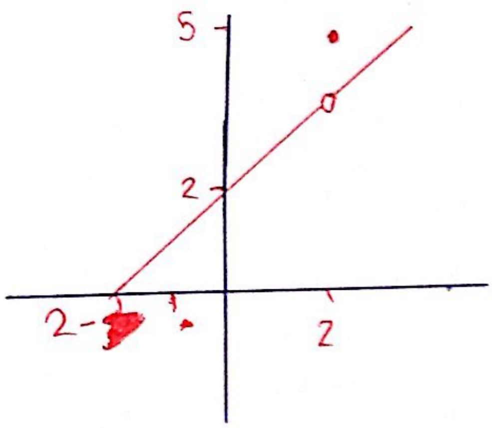
$D_g = \mathbb{R}$, $R_g = \mathbb{R}$



⇒ is $f \equiv g$??

No; Since $D_f \neq D_g$

3) $h(x) = \begin{cases} x+2 & , x \neq 2 \\ 5 & , x = 2 \end{cases}$



$D_h = \mathbb{R}$,

⇒ $h \neq g$; since $h(2) \neq g(2)$ (4)

⇒ The domain rules:

1) Square root: Under square root should be greater than or equal to zero.

2) Denominator is never zero.

3) Do not simplify

Ex ⇒ Find the domain:

1) $f(x) = \sqrt{3-x} - \frac{x}{\sqrt{2+x}}$

$3 - x \geq 0$ & $2 + x > 0$

$3 \geq x$ & $x > -2$

$D_f = (-2, 3]$

2) $h(x) = 3 - x + \frac{x}{2+x}$

$2 + x \neq 0$

$x \neq -2$

$D_h = \mathbb{R} - \{-2\}$

3) $g(x) = \frac{x^2}{x^2+4}$

$D_g = \mathbb{R}$

4) $f(t) = \frac{t^3 - 5}{t^2 + t - 6}$

$t^2 + t - 6 \neq 0$

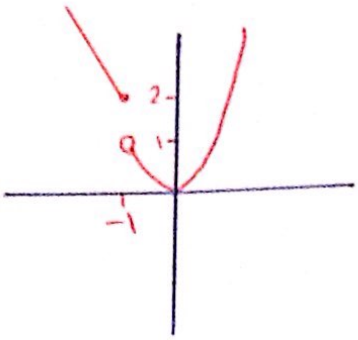
$(t+3)(t-2) \neq 0$

$D_f = \mathbb{R} - \{-3, 2\}$

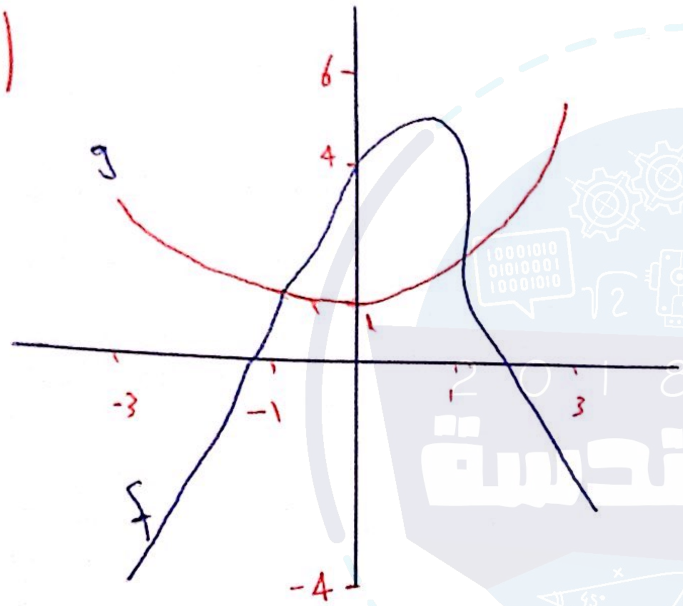
5) $f(x) = \begin{cases} 1-x & , x \le -1 \\ x^2 & , x > -1 \end{cases}$ → Piecewise function

$D_f = \mathbb{R}$

$R_f = [0, \infty)$



6)



⇒ Find:

A) $D_f = [-3, 3]$

B) $D_g = [-3, 3]$

C) $f(-3) = -4$

D) $g(-3) = 3$

E) For what values of (x) is $f(x) = g(x)$

$x = \{-1, 1\}$

F) $R_f = [-4, 4]$

G) $R_g = [1, 6]$

H) Solve Equation (Eqn) $f(x) = -1$

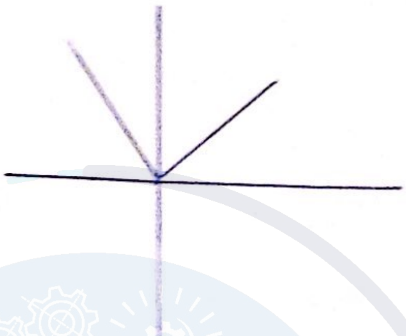
$x = \{3, -2\}$

* Absolute value function:

$$f(x) = |x| \Rightarrow \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

$D_f = \mathbb{R}$

$R_f = [0, \infty)$

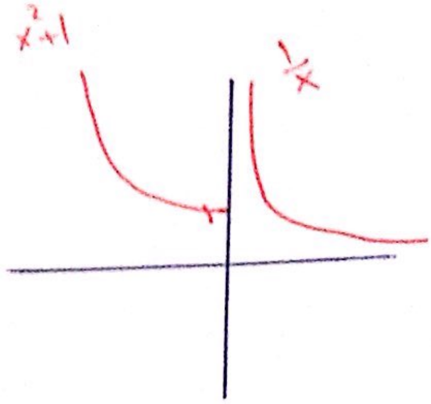


* $|ab| = |a|\,|b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$|a+b| \leq |a| + |b|$

Ex: 1) $f(x) = \begin{cases} x^2+1 & , x \leq 0 \\ \frac{1}{x} & , x > 0 \end{cases}$

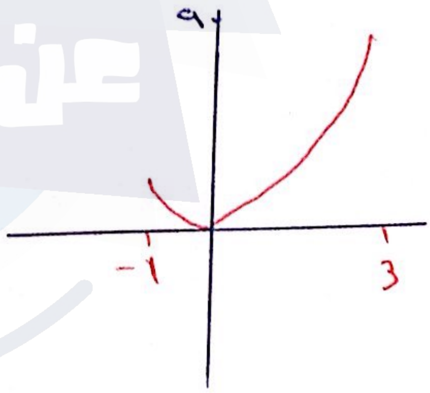


$\Rightarrow D_f = \mathbb{R}$

$R_f = (0, \infty)$

* بهذا المثال ← محور (y) هو خط تقارب عمودي.
← محور (x) هو خط تقارب أفقي.

2) $f(x) = x^2 \quad , x \in [-1, 3]$



$\Rightarrow D_f = [-1, 3]$

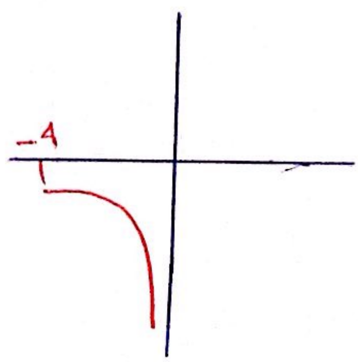
natural domain $= \mathbb{R}$

$R_f = [0, 9]$

3) $f(x) = \frac{1}{x} \quad , x \in [-4, 0)$

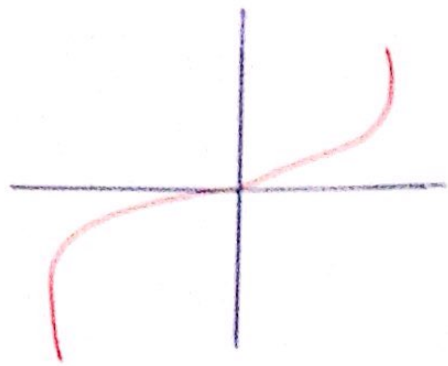
$D_f = [-4, 0)$, natural domain $= \mathbb{R}^*$

$R_f = (-\infty, -\frac{1}{4}]$



⇒ 4) $f(x) = x^3$

$D_f = R_f = \mathbb{R}$

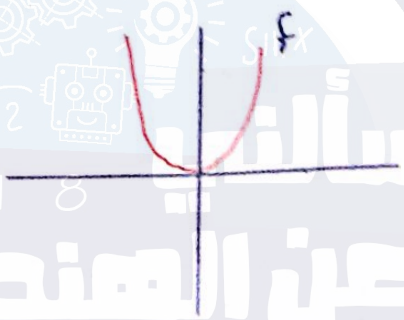


* odd function : $f(-x) = -f(x)$

↓

Symmetrical about the origin

⇒ $f(x) = x^2$



$f(2) = f(-2)$

f is even function

↓

Symmetrical about y - axis

$f(-x) = f(x)$

⇒ Even function ⇒ $f(-x) = f(x)$

↳ odd function ⇒ $f(-x) = -f(x)$

⇒ **Ex:** 1) $f(x) = x\,|x|$ Even, odd or ~~neither~~ neither??

$f(-x) = -x\,|-x| \Rightarrow |-x| = |x|$

$f(-x) = \underbrace{-x\,|x|}_{-f(x)}$

it's an odd function; since $f(-x) = -f(x)$

2) $f(x) = x^2|x|$

$f(-x) = (-x)^2|-x|$

$f(-x) = x^2|x| = f(x)$

it is an even function

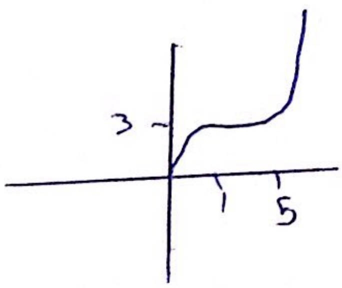
3) $f(x) = x^2 + x^3$ E, O, neither??

$f(-x) = (-x)^2 + (-x)^3$

$f(-x) = x^2 - x^3 \neq f(x)$
$\neq -f(x)$

it is neither

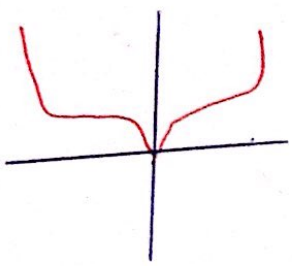
⇒ **Ex:** f is defined on $[-5, 5]$ ⇒



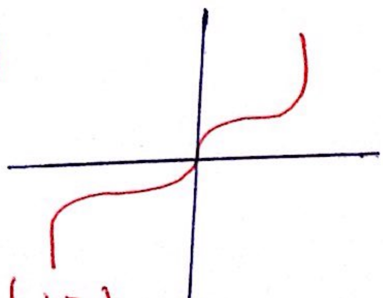
1) Complete to get an even function

2) complete to get an odd function

1)



2)



Ex: graph f:

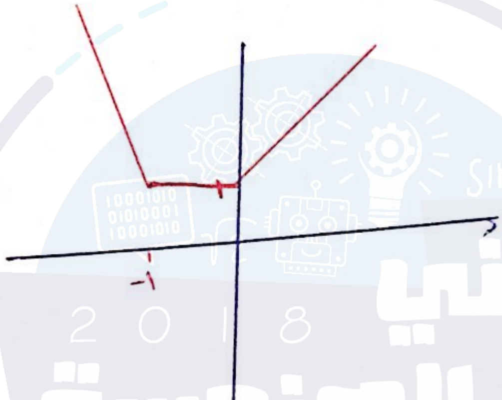
$$f(x) = |x| + |x+1|$$

$|x|$: $-x$ for $x<0$, x for $x>0$

$|x+1|$: $-x-1$ for $x<-1$, $x+1$ for $x>-1$

$-2x-1$ for $x<-1$, 1 for $-1<x<0$, $2x+1$ for $x>0$

$$f = \begin{cases} -2x-1 & , x \leq -1 \\ 1 & , -1<x<0 \\ 2x+1 & , x \geq 0 \end{cases}$$



$\boxed{1.2}$ Mathematical models:

1) polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a$$

⇒ n is a positive integer

⇒ n = degree of poly

⇒ $a \in \mathbb{R}$

⇒ $D_f = \mathbb{R}$

⇒ Ex: $f(x) = -3x^5 + \frac{1}{2}x^3 - \sqrt{3}\,x^2 + 100$

* poly of degree = 5

$a_5 = -3,\ a_4 = 0,\ a_3 = \frac{1}{2},\ a_2 = -\sqrt{3},\ a_1 = 0,\ a_0 = 100$

* rational = $\dfrac{\text{Polynomials}}{\text{Polynomials}}$

⇒ $f(x) = \dfrac{x^3 + 4x + 1}{x^2 - 4}$

$D_f = \mathbb{R} - \{-2, 2\}$

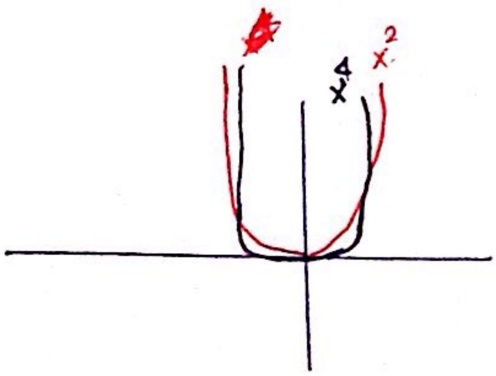
$f(x) = \dfrac{\sqrt{x} + 3}{x - 1}$

$D_f = [0, \infty) - \{1\}$

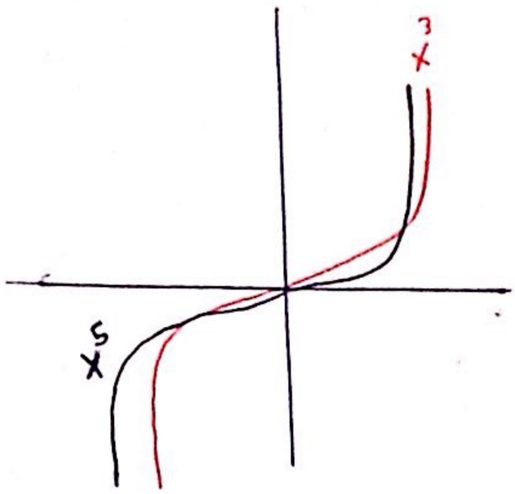
* Not rational since $x^{\frac{1}{2}} \notin \mathbb{Z}^+$

$f(x) = x^n$ ⇒ n is even

$D_f = \mathbb{R}$, $R_f = [0, \infty)$

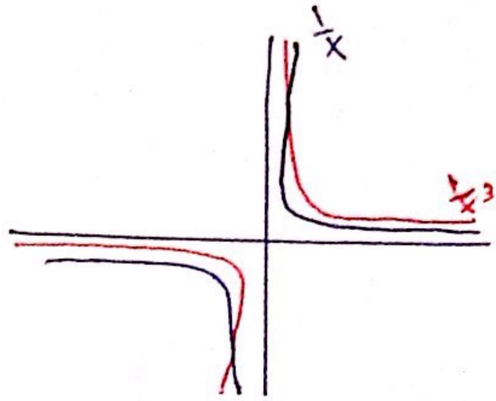


$f(x) = x^n \Rightarrow n$ is odd



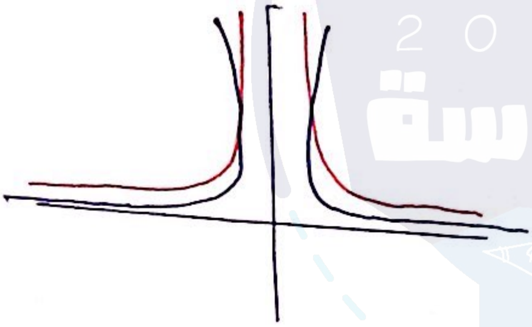
$f(x) = x^n \Rightarrow n$ is a negative integer

* n is odd



$f(x) = x^n$

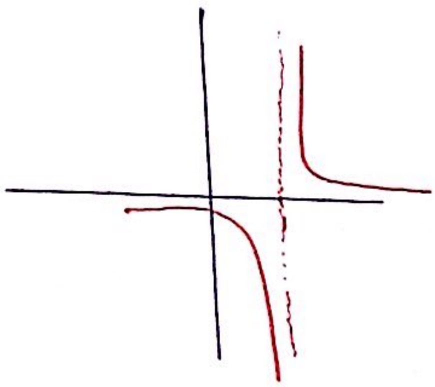
* n is a negative integer
* n is even



$\Rightarrow f(x) = \frac{1}{x-1}$

$D_f = \mathbb{R} - \{1\}$

$R_f = \mathbb{R} - \{0\}$



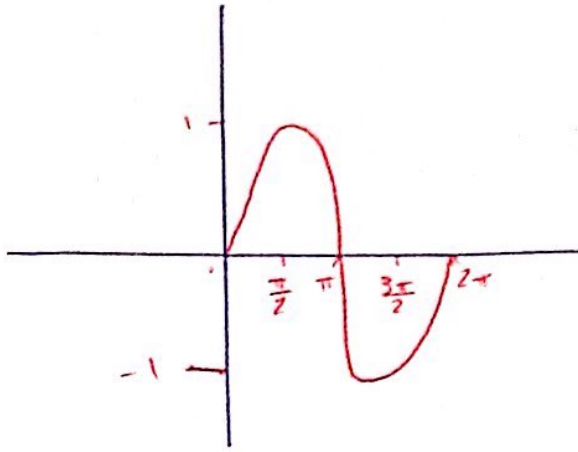
⇒ Trigonometric functions.

1) $\sin x$

$D_f = \mathbb{R}$

$R_f = [-1, 1]$

P. = Period $= 2\pi$



⇒ $\sin x$ is odd ⇒ $\sin -x = -\sin x$

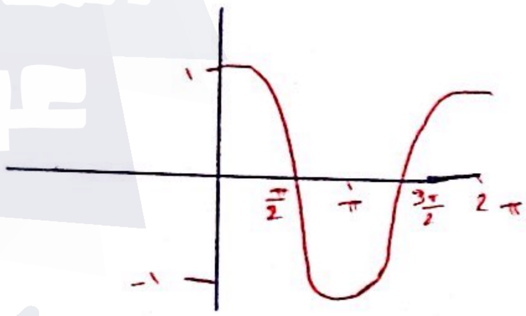
Ex: $\sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

2) $\cos x$

$D_f = \mathbb{R}$

$R_f = [-1, 1]$

P. = Period $= 2\pi$



$\cos x$ is even ⇒ $\cos -x = \cos x$

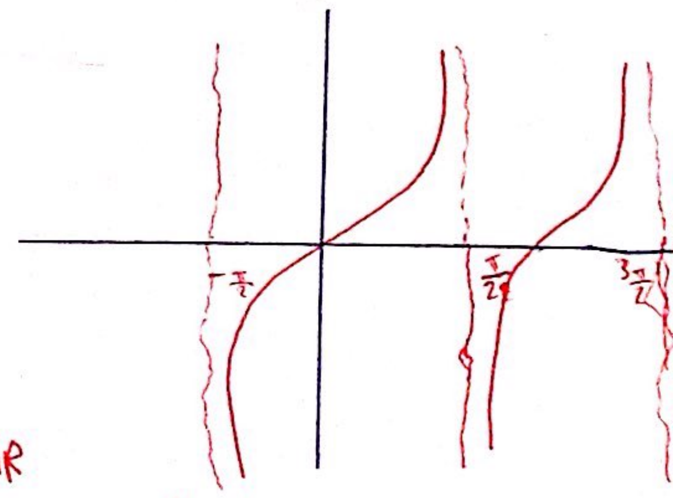
* $\cos(x + 2n\pi) = \cos x$

Ex: $\cos(\pi + 4\pi) = \cos \pi = -1$

3) $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$D_f = \mathbb{R} - \{ n \times \frac{\pi}{2} \}$

odd



$\tan x$ is an odd function

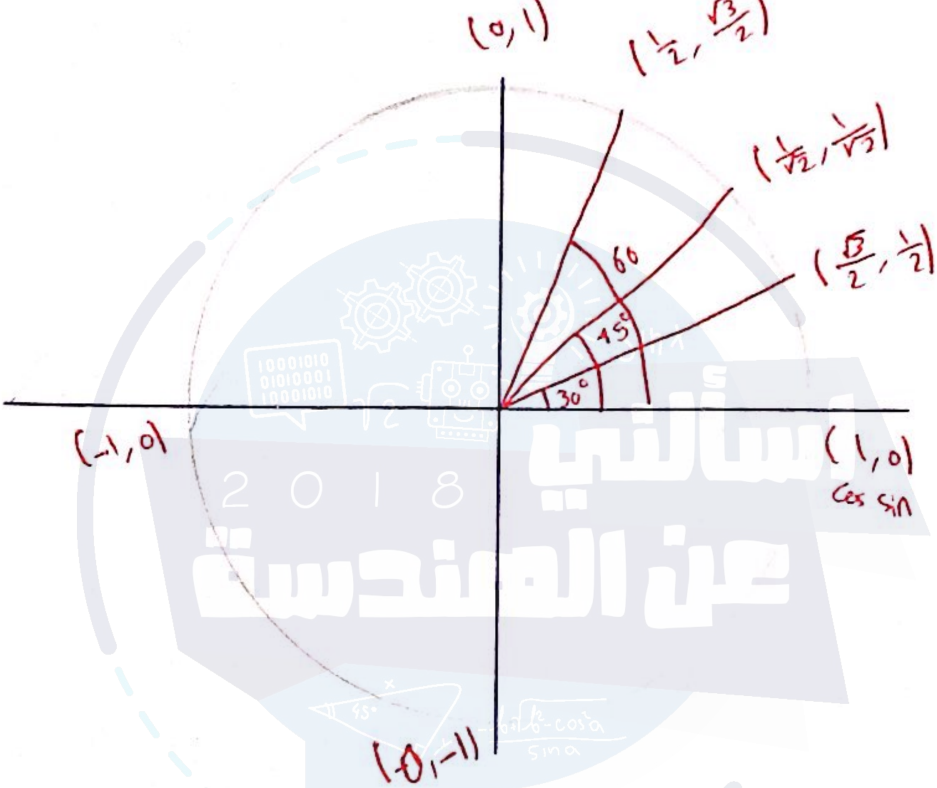
$\tan(-x) = -\tan x$, $R_f = \mathbb{R}$

P. = Period $= \pi$, * $\tan(x + n\pi) = \tan x$

4) $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

5) $\sec x = \frac{1}{\cos x}$

6) $\csc x = \frac{1}{\sin x}$



$\Rightarrow \cos \frac{4\pi}{3} = \cos\left(\left(1 + \frac{1}{3}\right)\pi\right)$

$= \cos\left(\pi + \frac{\pi}{3}\right)$

$= -\frac{1}{2}$

$\Rightarrow \sin\left(4\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$\Rightarrow \sin \frac{5\pi}{6} = \frac{1}{2}$

* Identities :-

1) $\cos^2 x + \sin^2 x = 1$

2) $1 + \tan^2 x = \sec^2 x$

3) $1 + \cot^2 x = \csc^2 x$

4) $\sin(x \mp y) = \sin x \cos y \mp \cos x \sin y$

5) $\cos(x \mp y) = \cos x \cos y \pm \sin x \sin y$

6) $\sin 2x = 2 \sin x \cos x$

7) $\cos 2x \to \cos^2 x - \sin^2 x$

$\to 2\cos^2 x - 1$

$\to 1 - 2\sin^2 x$

$\Rightarrow$ double angle formula

8) $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

9) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

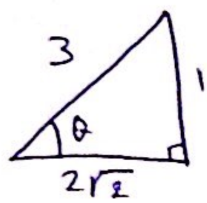
$\Rightarrow$ half angle formula

$\Rightarrow$ If $\sin\theta = \frac{1}{3}$, θ bet. $\frac{\pi}{2} < \theta < \pi$:

1) $\sin 2\theta = 2 \sin\theta \cos\theta$

$= 2 \cdot \frac{1}{3} \cdot -\frac{2\sqrt{2}}{3}$

$= \dfrac{-4\sqrt{2}}{9}$



2) $\csc\theta \Rightarrow$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{3}} = 3$

3) $\cos 2\theta$

$= \cos^2\theta - \sin^2\theta$

$= \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$

$\Rightarrow$ $\boxed{1.3}$ New functions from old

* Transformation

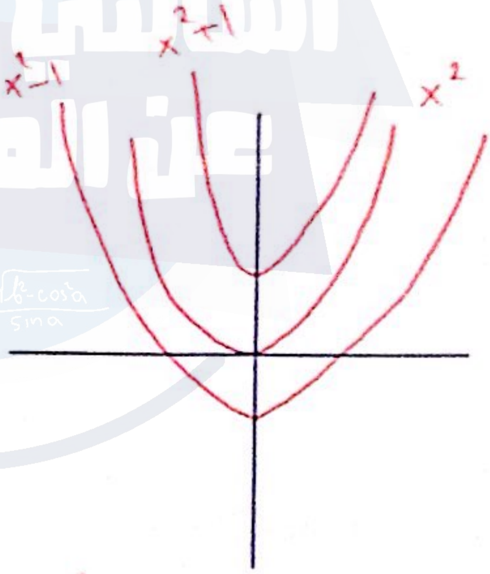
1) vertical and Horizontal shift.

Ex: $f(x) = x^2$

graph: ① $g(x) = x^2 + 1$

② $h(x) = x^2 - 1$

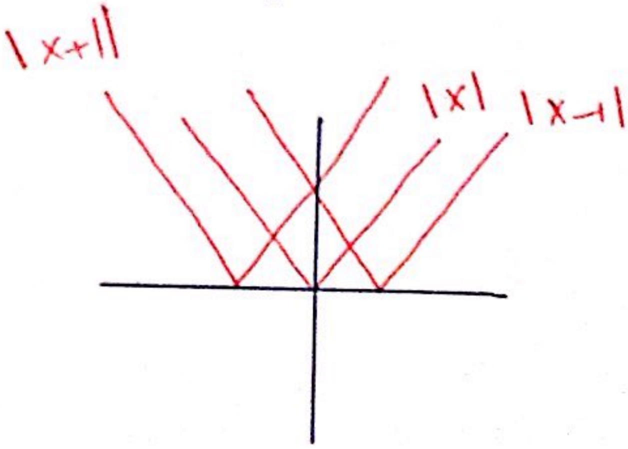
* Range may changes



$\Rightarrow f(x) = |x|$

graph: $g(x) = |x-1|$

$h(x) = |x+1|$



* Vertical and Horizontal shift.

$c > 0$, to obtain the graph of:

1) $y = f(x) + c$, shift f "c" units to the up
2) $y = f(x) - c$, shift f "c" units to the down

] R_f may changes

3) $y = f(x - c)$, shift f "c" units to the right
4) $y = f(x + c)$, shift f "c" units to the left

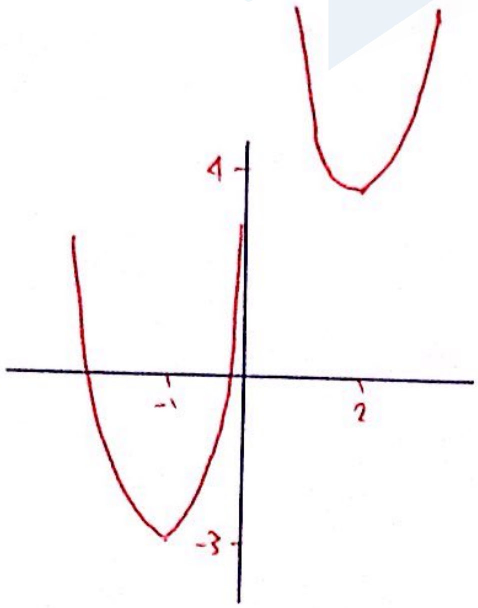
] Domain may changes

⇒ 1) $f(x) = x^2 + 2x$

write $g(x)$ obtain by shifting f 3 units to the left then 2 units to the down.

$g(x) = \left((x+3)^2 + 2(x+3) \right) - 2$

2)



g is obtained from f by shifting f 3 units to the right then 7 units to the up.

Ex: $f(x) = x^2 + 4x + 1$

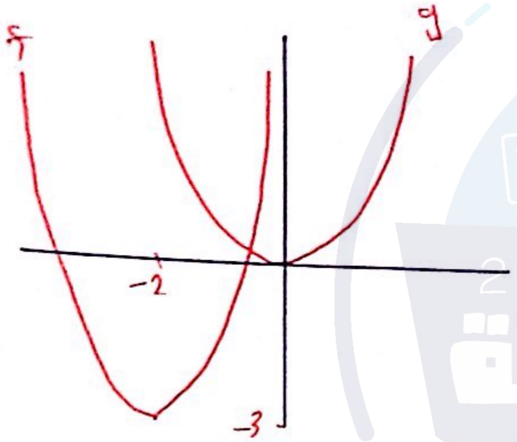
$g(x) = x^2$

$\Rightarrow$ The two shifts H. and V. that transforms g into f are??

$f = x^2 + 4x + 2^2 - 2^2 + 1$

$f(x) = (x+2)^2 - 3$ $\therefore$ 2 units left

3 units down



1.3 If you have two functions

$f(x)$ & $g(x)$

$\Rightarrow$ New functions $\to$ $f \mp g$

$\to$ $f \cdot g$

$\to$ $\frac{f}{g}$

* $D_{f \mp g} = D_f \cap D_g$

* $D_{f \cdot g} = D_f \cap D_g$

* $D_{\frac{f}{g}} = D_f \cap D_g$ & $g(x) \neq 0$

Ex: ① $f(x) = \sqrt{3-x}$ & $g(x) = \sqrt{x^2-1}$

Find: 1) $f+g$ 2) $\frac{f}{g}$

1) $\Rightarrow f+g = \sqrt{3-x} + \sqrt{x^2-1}$

f: $+$ $\quad 3 \quad$ $-$

g: $+$ $\quad -1 \quad$ $-$ $\quad 1 \quad$ $+$

Domain $= (-\infty, -1] \cup [1, 3]$

2) $\frac{f}{g} = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$

Domain $= (-\infty, -1) \cup (1, 3]$

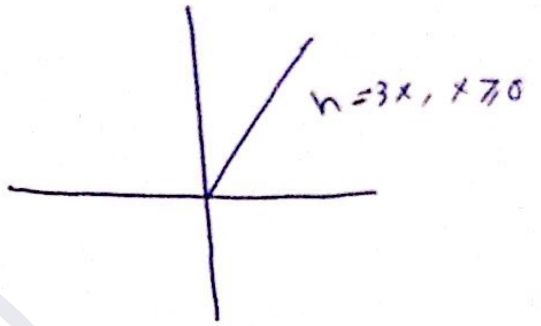
Ex: $f(x) = \sqrt{x}$ & $g(x) = 3\sqrt{x}$

find: 1) $h = f \cdot g$ 2) $h(-4)$ 3) $h(4)$

1) $h(x) = 3x$, $D_h = [0, \infty)$

2) $h(-4) =$ undefined

3) $h(4) = 12$



$\Rightarrow$ f & g are two functions

Composite of f & g

$(f \circ g) \Rightarrow$ New function

$* (f \circ g)(x) = f(g(x))$

$\Rightarrow x \in D_g$ & $g(x) \in D_f$

$\Rightarrow$

x	1	2	3	4	5	6
f(x)	3	1	4	2	2	5
g(x)	6	3	2	1	2	3

find:

1) $(f \circ g)(1)$

$= f(g(1)) = f(6) = 5$

2) $(g \circ f)(1) = g(f(1)) = 2$

3) $(f \circ f)(4) = f(f(4)) = 1$

$\Rightarrow$ **Ex:** $f(x) = x^2$ & $g(x) = \sqrt{4-x}$

Find: 1) $g(f(x))$

$$g(f(x)) = \sqrt{4-x^2}$$

2) $f(g(x))$

$$f(g(x)) = \left(\sqrt{4-x}\right)^2$$

$$= 4 - x$$

3) $f(g(1)) = 4 - 1 = 3$

4) $f(g(6))$

$$= f(\sqrt{4-6}) = f(\sqrt{-2}) = \text{undefined}$$

5) $(f \circ f)(x)$

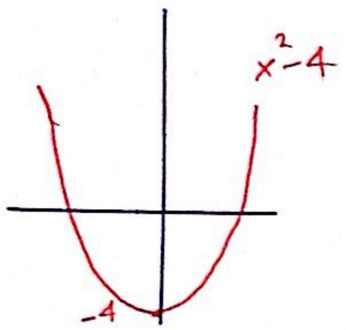
$$= f(x^2) = x^4$$

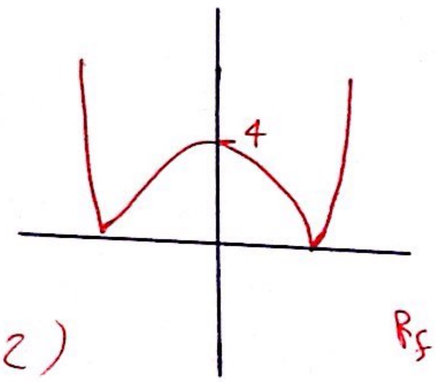
6) $(g \circ g)(x)$

$$= g(\sqrt{4-x})$$

$$= \sqrt{4-\sqrt{4-x}}$$

$\Rightarrow$ **Ex:** graph: $f(x) = |x^2 - 4|$

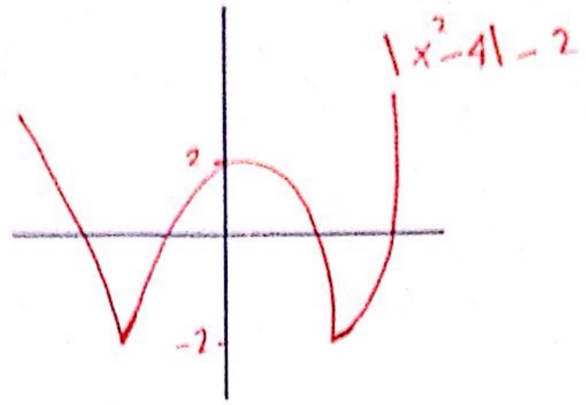




$R_f = [0, \infty)$

Ex: 1) graph: $g(x) = |x^2-4|-2$

$R_g = [-2, \infty)$



⇒ 2) if $f(2x+5) = x+1$, find: 1) $f(x)$

$$y = 2x+5$$

$$y-5 = 2x$$

$$\frac{y-5}{2} = x$$

$$\Rightarrow f(y) = \frac{y-5}{2} + 1$$

$$f(y) = \frac{y}{2} - \frac{3}{2}$$

2) $f(9)$

$$= \frac{9}{2} - \frac{3}{2} = 3$$

$\boxed{1.4}$ Exponential Functions

$f = x^2 \rightarrow$ Polynomial

$f = 2^x \rightarrow$ exp.

$\Rightarrow f(x) = a^x$, $a > 0$ & $a \neq 1$

$f(x) = -2^x$
$f(\frac{1}{2}) =$ undefined

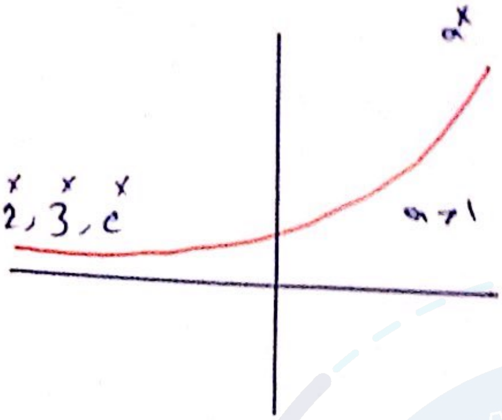
$f(x) = 2^x$

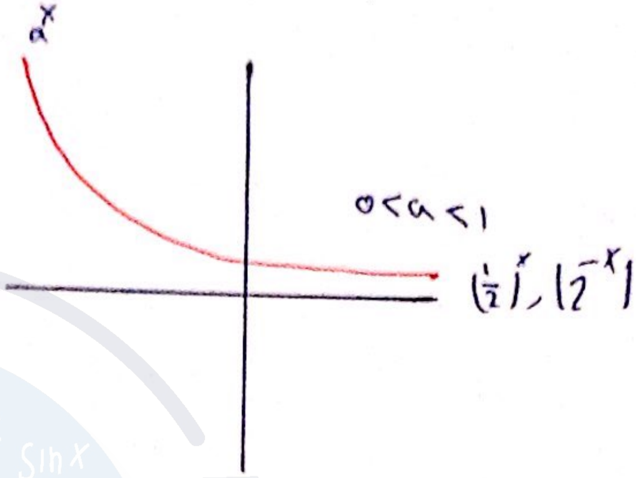
$f(\frac{1}{2}) = \sqrt{2}$

1.4 The exponential functions:

$$f(x) = a^x \ , \ a > 0 \ , \ a \neq 1$$

$a \equiv$ base





$D_f = \mathbb{R}$, $R_f = (0, \infty)$

$\Rightarrow$ solve: $2^x = 0$
$3^x = -1$ $\Big] \rightarrow$ no solutions

* Properties :

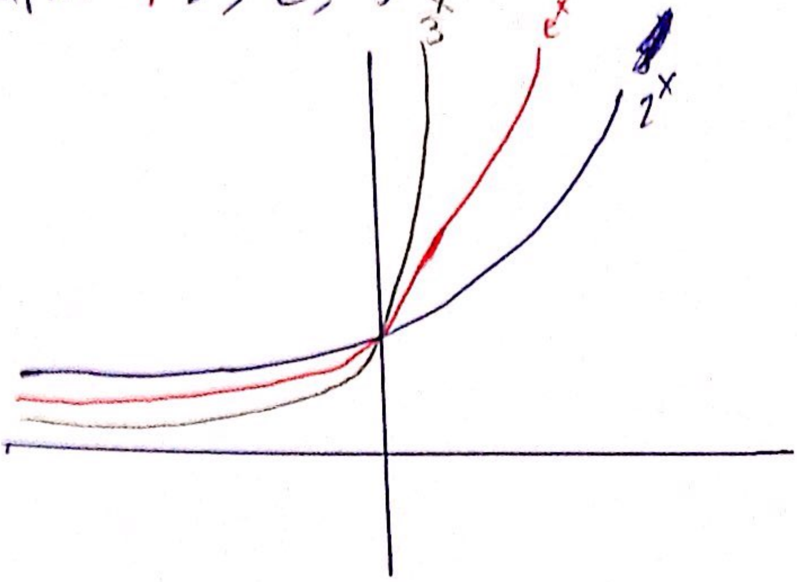
1) $a^x \cdot a^y = a^{x+y}$

2) $\frac{a^x}{a^y} = a^{x-y}$

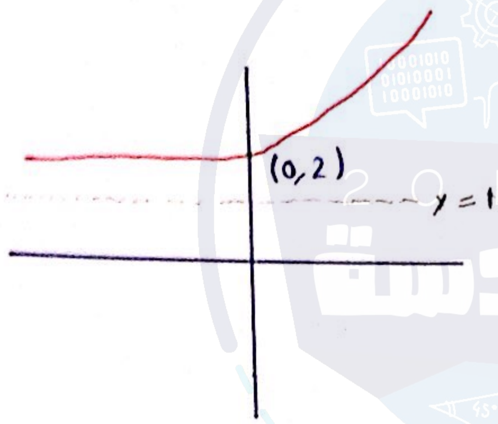
3) $a^0 = 1$

4) $a^x \cdot b^x = (ab)^x \Rightarrow 2^3 \cdot 5^3 = (10)^3 = 1000$

⇒ graph: 1) 2^x, e^x, 3^x



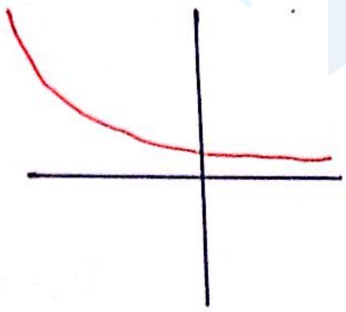
2) $2^x + 1$



$D_f = \mathbb{R}$

$R_f = (1, \infty)$

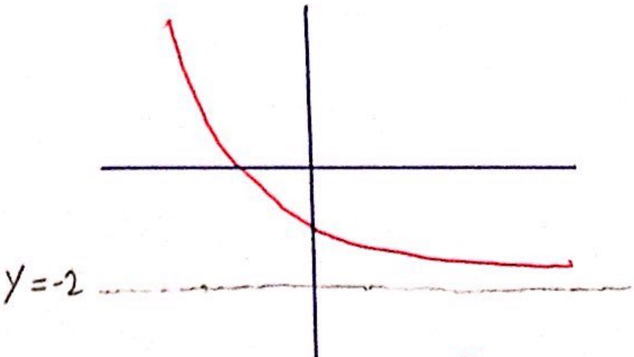
3) e^{-x}



$D_f = \mathbb{R}$

$R_f = (0, \infty)$

4) $e^{-x} - 2$



$D_f = \mathbb{R}$

$R_f = (-2, \infty)$

⇒ Find the Domain:

1) $f(x) = \dfrac{1-e^{x^2}}{1-e^{4-x^2}}$

$1-e^{4-x^2} \Rightarrow e^{4-x^2} = 1$

$4-x^2 = 0$

$\boxed{x = \mp 2}$

$D_f = \mathbb{R} - \{-2, 2\}$

2) $f(x) = \dfrac{1-e^{x^2}}{1-e^{x^2+4}}$

$e^{x^2+4} = 1$

$x^2+4 = 0 \Rightarrow$ no solution

$D_f = \mathbb{R}$

3) $f(x) = \dfrac{1-e^{x^2}}{1+e^{x^2-4}}$

$e^{x^2-4} = -1 \Rightarrow$ no solution

$D_f = \mathbb{R}$

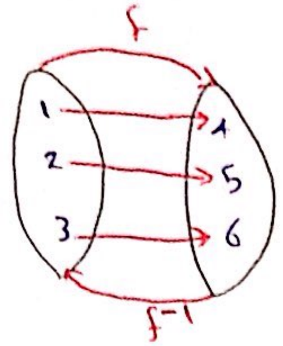
4) $f(x) = \dfrac{1-e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}$ $\quad x \neq 0$

$e^{\frac{1}{x}} = -1 \Rightarrow$ no solution

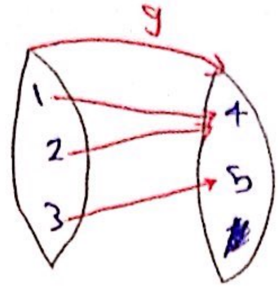
$D_f = \mathbb{R} - \{0\}$

1.5 Inverse Functions & Logarithms.

Ex:



1-1



not 1-1

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f$

$f(1) = 4$, $f(2) = 5$

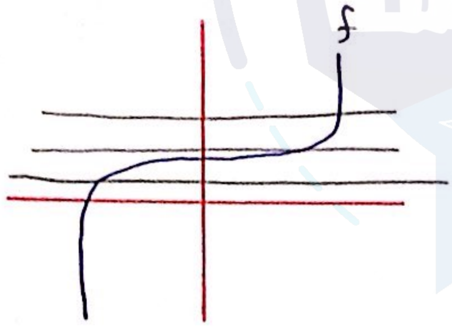
$f^{-1}(4) = 1$, $f^{-1}(5)\,2$

$\Rightarrow$ $f^{-1}(f(1)) = 1 \Rightarrow 1 \in D_f$

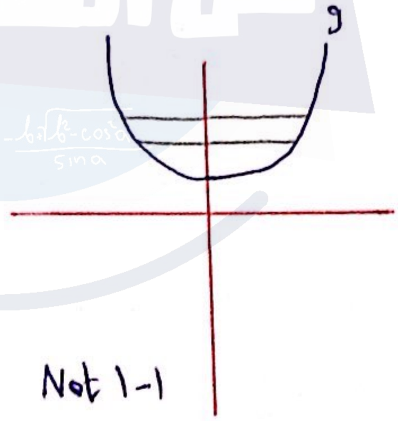
$f(f^{-1}(4)) = 4 \Rightarrow 4 \in D_{f^{-1}}$

$\Rightarrow (f \circ f^{-1})(x) = x$

$\Rightarrow$



1-1

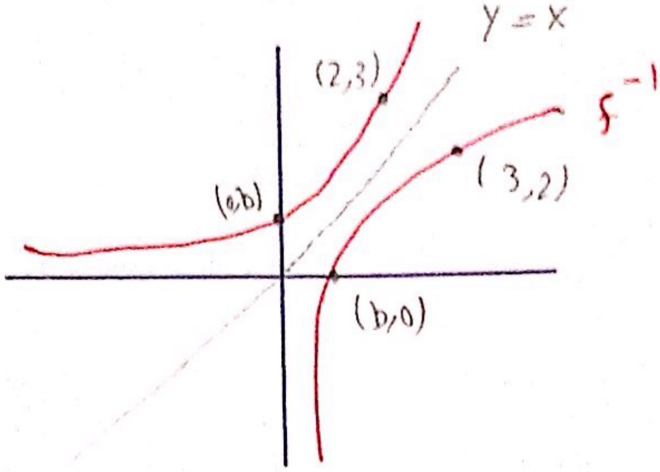


Not 1-1

* Horizontal line property to determine "f" is 1-1.

  If the horizontal line cuts the curve once, then "f" is 1-1.

* graph:



⇒ Inverse Function

1) $f : A \xrightarrow{1-1} B$

* If "f" is 1-1 ⟶ "f" has an inverse.

2) "f" is 1-1 ⟶ 1) No two different elements has same image.

2) Use horizontal line property.

* Is the function "f" is increasing ($f' > 0$) on its domain or "f" is is decreasing on its domain, then f is 1-1.

⇒ $f^{-1} : B \to A$

$D_{f^{-1}} = R_f$, $R_{f^{-1}} = D_f$

* Reflect the graph of "f" in line $y = x$ to get the graph of "f^{-1}".

⇒ $f(f^{-1})^{-1} = f$

⇒ - $f^{-1}(f(x)) = x$, $x \in D_f$

- $f(f^{-1}(x)) = x$, $x \in R_f$

* Rule of f^{-1} = ??

$\boxed{y = f(x)}$

to find "f^{-1}" write x in terms of y.

Ex: 1) $f(x) = x^3 - 4$

$f'(x) = 3x^2 \Rightarrow$ 1-1

find $f^{-1}(x)$

$y = x^3 - 4$ $\quad y \in R_f$

$y + 4 = x^3$ $\quad x \in D_f$

$x = \sqrt[3]{y+4}$

$f^{-1}(x) = (x+4)^{\frac{1}{3}}$, $x \in R_f$

2) $y = \dfrac{1+x}{2-3x}$, find $f^{-1}(x)$:

$y = \dfrac{1+x}{2-3x}$

$2y - 3yx = 1 + x$

$2y - 1 = 3yx + x$

$x = \dfrac{2y-1}{3y+1}$

$f^{-1}(x) = \dfrac{2x-1}{3x+1}$

3) $f(x) = e^x + 1 \Rightarrow$ 1-1

find: $f(2)$, $f^{-1}(2)$

$f(2) = e^2 + 1$

$f^{-1}(2) =$

$2 = e^x + 1$

$e^x = 1$

$\boxed{x = 0}$ $\quad \boxed{f^{-1}(2) = 0}$

4) $f(1) = 7$

"f" is 1-1

$f^{-1}(7) = 1$

5) $f(x) = x^5 + x^3 + x \Rightarrow$ 1-1

$f^{-1}(3) =$

$3 = x^5 + x^3 + x$

$\boxed{x = 1}$

$\boxed{f^{-1}(3) = 1}$

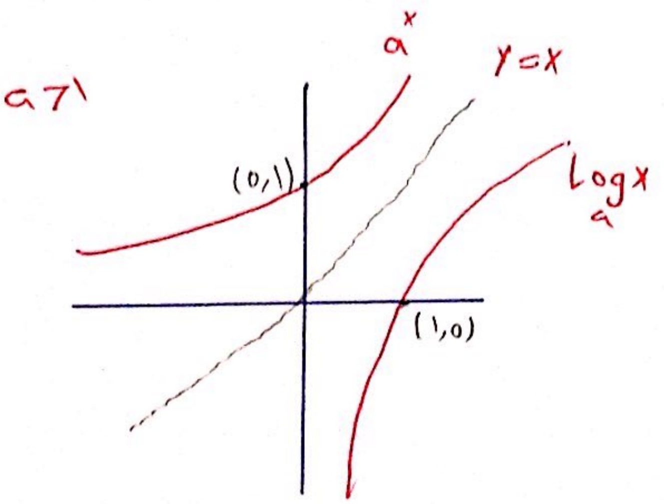
* $f(f^{-1}(2)) = 2$

* Logarithmic functions:

exp. a^x, $a > 0$, $a \neq 1 \Rightarrow f$ is 1-1

f has an Inverse call it Logarithm

$\Rightarrow$ natural Logarithm $= \ln x \Rightarrow e^x$



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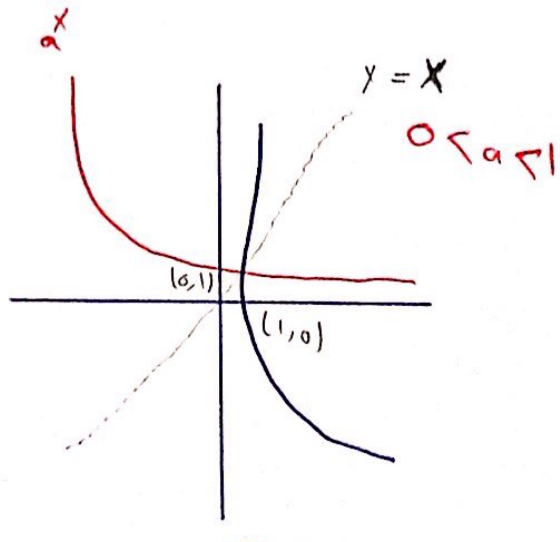


متناقص

1) Domain of $\log_a^x = (0,\infty)$

2) Range of $\log_a^x = \mathbb{R}$

3) $\log_a^1 = 0$

4) ~~x~~ $\log_a^0 =$ undefined

5) $\log^{-1} =$ undefined

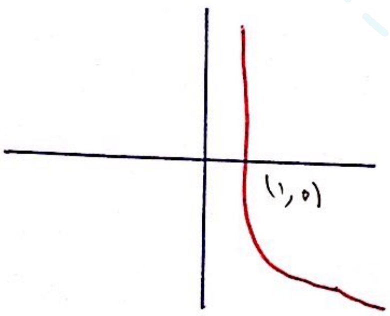
Ex: $y = \log_3^{x-2}$, find the domain

$x - 2 > 0$

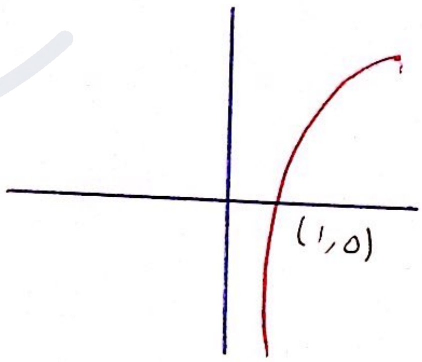
$x > 2$

$\boxed{D_f = (2,\infty)}$

Ex: graph: 1) $\log_{\frac{1}{3}}^x$ 2) $\log_3^x$



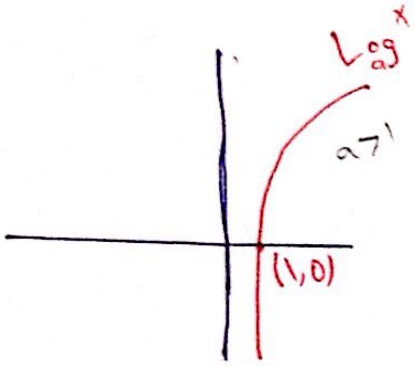
$\log_{\frac{1}{3}}^x$

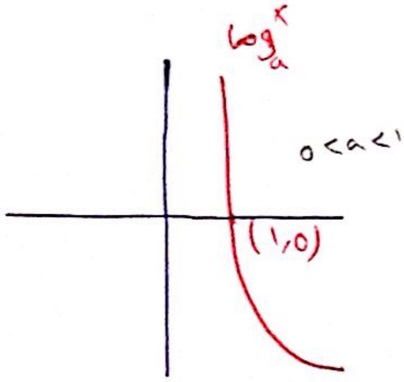


$D_f = (0,\infty)$

$R_f = \mathbb{R}$

* Logarithmic functions.





⇒ properties:

1) $\log_a x \Rightarrow D_{\log} = (0,\infty) \Rightarrow R_{\log} = \mathbb{R}$

2) $\log_a 1 = 0$

3) $\log_a a = 1$

4) $\log_a a^x = x$

↳ $a^{\log_a x} = x$

⇒ Since log & exponential are inverse.

5) $\log_e x = \ln x$ ⇒ natural logarithm.

6) $\log_a (xy) = \log_a x + \log_a y$

7) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

8) $\log_a x^r = r \log_a x$

9) $\log_a b = \frac{\ln b}{\ln a}$

⇒ Ex: $\log_2^3 = \frac{\ln 3}{\ln 2}$

10) $\log_a^b = \frac{1}{\log_b^a}$ $\quad a > 0,\ a \neq 1$
$b > 0$

⇒ Prove "10"

$\log_a^b = \frac{\ln b}{\ln a} = \frac{1}{\frac{\ln a}{\ln b}} = \frac{1}{\log_b^a}$

11) $\log_a^b = k \Rightarrow a^k = b$

Prove "11"

$a^{\log_a^b} = a^k$

$b = a^k$

Ex: Solve the equation:

$\log_3^x = 2$

$x = 3^2 = 9$

12) $a^x = e^{x \ln a} \Rightarrow 2^x = e^{x \ln 2}$

~~Prove "12"~~

~~$e^{x \ln a} = a^x$~~

Ex: $f(x) = \ln(e^x - 3)$ ⇒ ~~solve the equation~~
find the domain

$e^x - 3 > 0$
$e^x > 3$

$x > \ln 3$

$D_f = (\ln 3, \infty)$

Ex: Solve:

① $\log_2^{x} + \log_2^{(x+2)} = 3$

$\log_2^{x^2+2x} = 3$

$x^2 + 2x = 8$

$x^2 + 2x - 8 = 0$

$x = -4, \; x = 2$

Solution $\Rightarrow \boxed{x = 2}$

② $\log_2^{x} - 2\log_2^{6} = \log_2^{\frac{1}{6}}$

$\log_2 \frac{x}{36} = \log_2 \frac{1}{6}$

$\frac{x}{36} = \frac{1}{6}$

$\boxed{x = 6}$

③ $2^{x-5} = 3$

$\log_2^{3} = x - 5$

$x = 5 + \log_2^{3}$

$x = 5 + \frac{\ln 3}{\ln 2}$

④ $e^{2x} - 3e^{x} + 2 = 0$

$(e^{x} - 2)(e^{x} - 1) = 0$

$e^{x} = 2$, $x = \ln 2$

$e^{x} = 1$, $\boxed{x = 0}$

Ex: $\log_3 10 + \log_9 16 = \log_3 (\ )$

$$\frac{\ln 10}{\ln 3} + \frac{\ln 16}{\ln 9} = \log_3 (\ )$$

$$\frac{\ln 10}{\ln 3} + \frac{\cancel{2}\ln 4}{\cancel{2}\ln 3} = \log_3 (\ )$$

$$\frac{\ln 10 + \ln 4}{\ln 3} = \log_3 (\ )$$

$$\frac{\ln 40}{\ln 3} = \log_3 (40)$$

Ex: $f(x) = \frac{e^x}{1-5e^x}$

Find: $f^{-1}(x)$

$$y = \frac{e^x}{1-5e^x} \Rightarrow y - 5y\,e^x = e^x$$

$$y = e^x(1+5y)$$

$$e^x = \frac{y}{1+5y}$$

$$x = \ln\left(\frac{y}{1+5y}\right)$$

$$f^{-1}(x) = \ln\left(\frac{x}{1+5x}\right)$$

Ex: $f(x) = x^2 - 4x \rightarrow$ not (1-1)

if $x \geq 3 \Rightarrow$ it is (1-1)

$$y = x^2 - 4x \ , \ x \geq 3$$

$$x^2 - 4x - y = 0$$

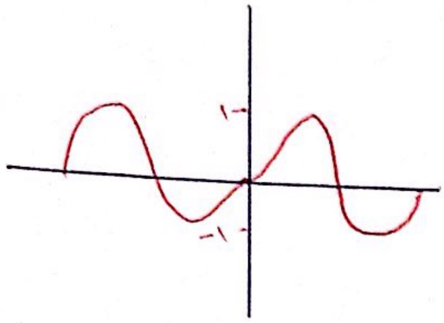
$$x = \frac{4 \mp \sqrt{16 + 4y}}{2}$$ يهمل

$$= 2 + \sqrt{4+y}$$

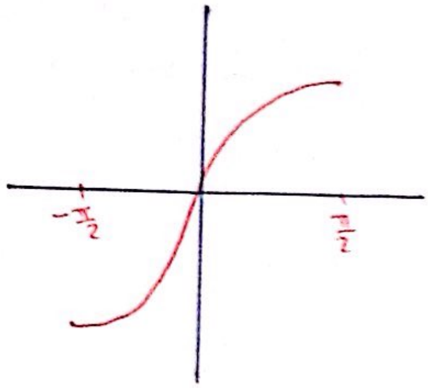
$$f^{-1}(x) = 2 + \sqrt{4+x}$$

* Inverse trigonometric functions

First 13/11
9–10



not 1-1



$\sin x : \underset{D}{[-\frac{\pi}{2}, \frac{\pi}{2}]} \xrightarrow{1-1} \underset{R}{[-1, 1]}$

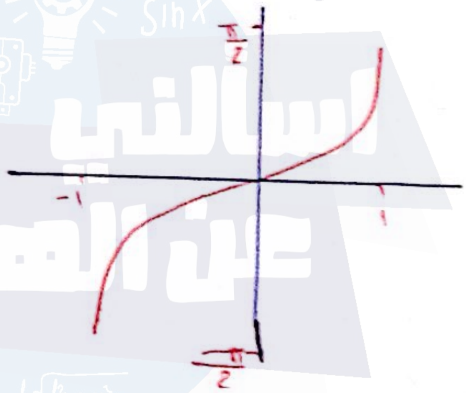
$$\left.\begin{aligned} * \ \sin(-x) &= -\sin x \\ \sin^{-1}(-x) &= -\sin^{-1} x \end{aligned}\right] \text{odd}$$

$\sin^{-1}(0) = 0°$

$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

$\sin^{-1}(1) = \frac{\pi}{2}$

$\sin^{-1}(\frac{-1}{2}) = -\sin^{-1}(\frac{1}{2}) = \frac{-\pi}{6}$



$\sin^{-1} x : \underset{D}{[-1, 1]} \longrightarrow \underset{R}{[-\frac{\pi}{2}, \frac{\pi}{2}]}$

$\Rightarrow \sin(\sin^{-1}(x)) = x \quad , x \in [-1, 1]$

$\sin^{-1}(\sin(x)) = x \quad , x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

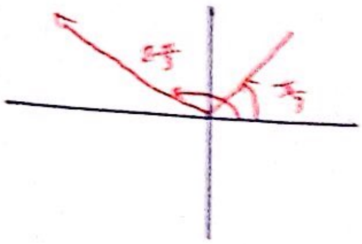
$\Rightarrow$ Ex: 1) $\sin(\sin^{-1}(\frac{1}{2})) = \frac{1}{2}$

2) $\sin(\sin^{-1}(\frac{1}{4})) = \frac{1}{4}$

3) $\sin(\sin^{-1}(\frac{-1}{3})) = \frac{-1}{3}$

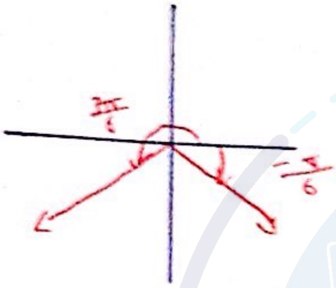
4) $\sin^{-1}(\sin(\frac{\pi}{3})) = \frac{\pi}{3}$

5) $\sin^{-1}(\sin(\frac{2\pi}{3})) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$



$(\pi - \frac{2\pi}{3}) = \frac{\pi}{3}$

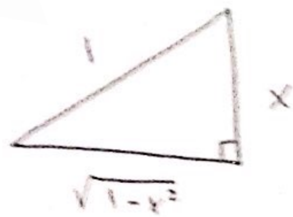
6) $\sin^{-1}(\sin(\frac{7\pi}{6})) = \sin^{-1}(\frac{-1}{2}) = \frac{-\pi}{6}$



$\Rightarrow$ Ex : $\sin(2\underbrace{\sin^{-1}(x)}_{\theta})$ $\Rightarrow$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\theta = \sin^{-1}(x)$

$= 2x\sqrt{1-x^2}$



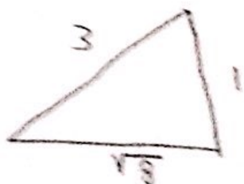
$\Rightarrow$ Find: $\cos(\sin^{-1}(\frac{1}{3}) + \sin^{-1}(\frac{-1}{4}))$

* يرافق كل تسمية مثلث قائم الزاوية

$= \cos(\sin^{-1}\frac{1}{3} - \sin^{-1}\frac{1}{4})$

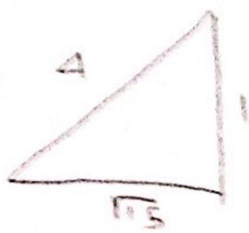
$\alpha = \sin^{-1}\frac{1}{3}$, $\beta = \sin^{-1}\frac{1}{4}$

$\sin\alpha = \frac{1}{3}$



$= \cos(\alpha - \beta)$

$\sin\beta = \frac{1}{4}$

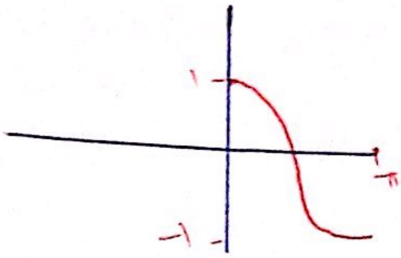


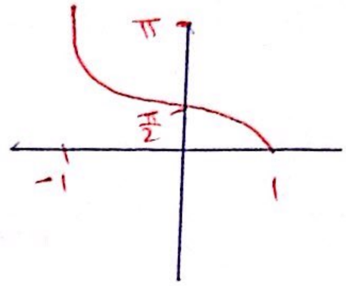
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{\sqrt{8}}{3}\cdot\frac{\sqrt{15}}{4} + \frac{1}{3}\cdot\frac{1}{4}$

$= \frac{121}{12}$

$\Rightarrow \cos(x) : \underset{D}{[0, \pi]} \to \underset{R}{[-1, 1]}$ | $\cos^{-1}(x) : \underset{D}{[-1, 1]} \to \underset{R}{[0, \pi]}$





Ex: 1) $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. 2) $\cos^{-1}\left(\frac{-1}{2}\right) = 2\frac{\pi}{3}$. 3) $\cos^{-1}(0) = \frac{\pi}{2}$

4) $\cos^{-1}(1) = 0^{\circ}$ 5) $\cos^{-1}(-1) = \pi$

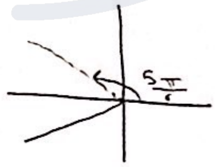
* $\cos(\cos^{-1}(x)) = x$, $x \in [-1, 1]$

$\cos^{-1}(\cos(x)) = x$, $x \in [0, \pi]$

Ex: 1) $\cos\left(\cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{3}$ 2) $\cos\left(\cos^{-1}\left(\frac{-1}{4}\right)\right) = \frac{-1}{4}$

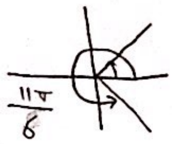
3) $\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = 5\frac{\pi}{6}$ 4) $\cos^{-1}\left(\cos\left(\frac{\pi}{10}\right)\right) = \frac{\pi}{10}$

5) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \frac{5\pi}{6}$



6) $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}$

7) $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$



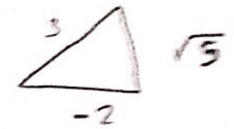
1) $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ 2) $\sin^{-1} -\frac{\sqrt{3}}{2} = -\sin^{-1}\frac{\sqrt{3}}{2} = -\frac{\pi}{3}$

3) $\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ 4) $\cos^{-1} -\frac{\sqrt{3}}{2} = 5\frac{\pi}{6}$

* الجواب (زاوية سالبة) فقط في $\sin^{-1}$, $\tan^{-1}$

Ex: $\sin(\underbrace{\cos^{-1}\frac{-2}{3}}_{\alpha} + \underbrace{\sin^{-1}\frac{1}{4}}_{\beta}) = \sin(\alpha+\beta)$

$\cos\alpha = \frac{-2}{3}$

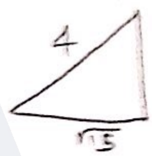


$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

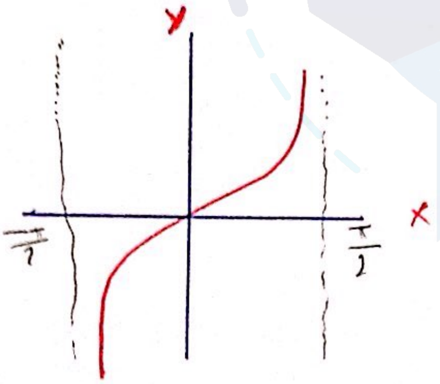
$= \frac{\sqrt{5}}{3} \cdot \frac{\sqrt{15}}{4} + \frac{-2}{3} \cdot \frac{1}{4}$

$= \frac{5\sqrt{3} - 2}{12}$

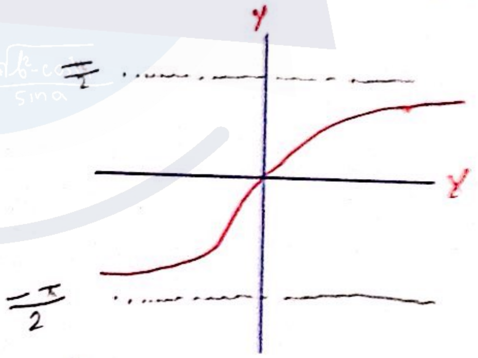
$\sin\beta = \frac{1}{4}$



* $\tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

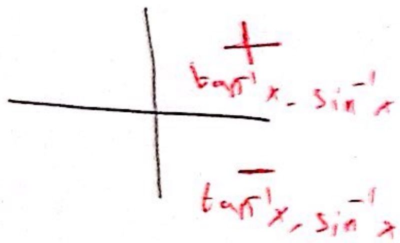


$\tan^{-1}x : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$



odd

$\tan^{-1}(-x) = -\tan^{-1}(x)$



Ex: 1) $\tan^{-1}0 = 0°$ 2) $\tan^{-1}1 = \frac{\pi}{4}$

3) $\tan^{-1}-1 = -\tan^{-1}1 = -\frac{\pi}{4}$

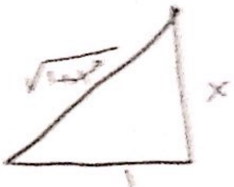
* $\arctan = \tan^{-1}$

4) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ 5) $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Ex: 1) $\sin(2\underbrace{\tan^{-1}x}_{\theta}) = \sin 2\theta = 2\sin\theta\cos\theta$

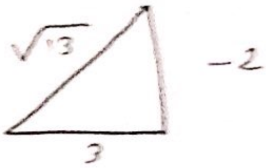
$$= \frac{2x}{1+x^2}$$

$\tan\theta = x$



2) $\cos(\underbrace{\tan^{-1}\frac{-2}{3}}_{\theta}) = \cos\theta$

$$= \frac{3}{\sqrt{13}}$$



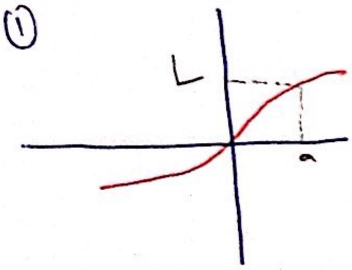
3) $\sin(\tan^{-1}\frac{-2}{3}) = \frac{-3}{\sqrt{13}}$

Chapter "2" Limits and derivatives

$$\lim_{\underbrace{x \to a}} f(x) = L$$

x goes to a

①

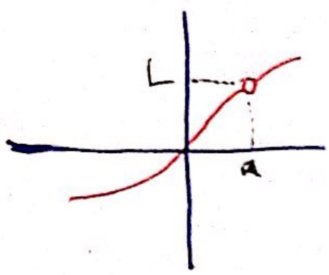


$\lim_{x\to a} f = L$

$f(x) = L$

$\therefore a \in D_f$

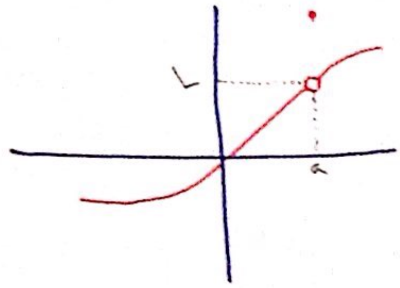
②



$\lim_{x\to a} f = L$

$a \notin D_f$

③

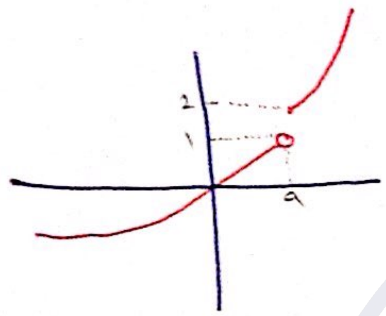


$\lim_{x\to a} f = L$

$f_a = 4 \neq L$

* 1, 2, 3 } → Limit exists

④

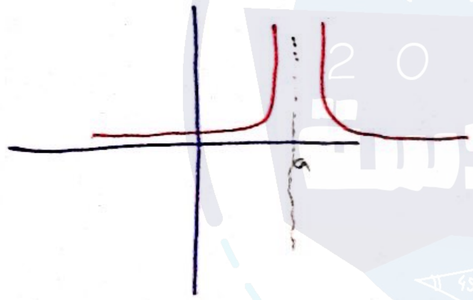


$\lim_{x\to a^+} f = 2$] Right limit

$\lim_{x\to a^-} f = 1$] Left limit

$\lim_{x\to a^+} f \neq \lim_{x\to a^-} f$ } ⇒ : limit does not exist

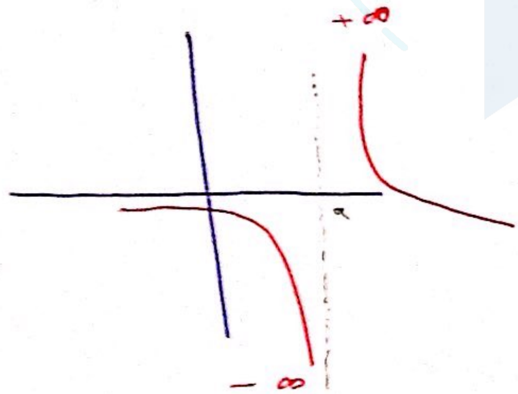
⑤



$\lim_{x\to a} f = \infty$

Limit does not exist since $\infty \notin \mathbb{R}$

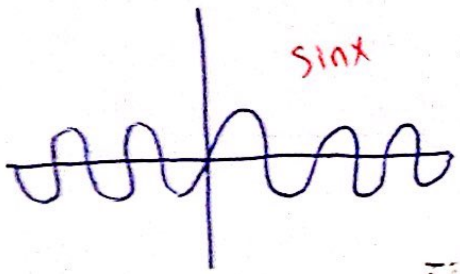
⑥



$\lim_{x\to a^+} f = +\infty$

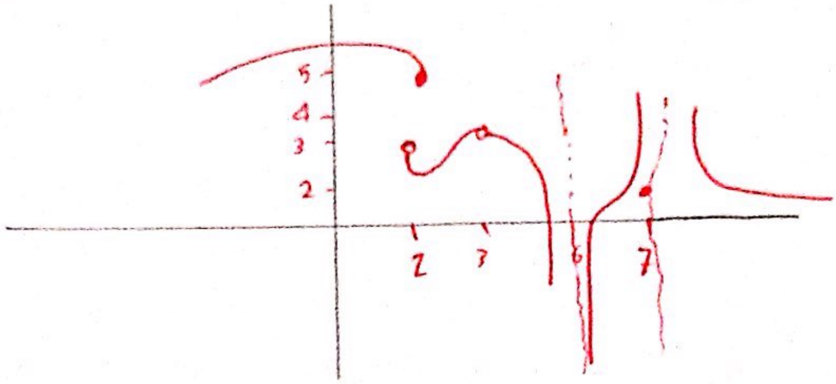
$\lim_{x\to a^-} f = -\infty$

→



$\underbrace{\lim_{x\to\infty}}_{\text{عندما x تأخذ قيم كبيرة}} \text{Sin}$ ⇒ many answers ⇒ limit doesn't exist

⇒ Limits



Find: 1) $f(2) = 5$ 2) $\lim\limits_{x \to 2^-} f = 5$ 3) $\lim\limits_{x \to 2^+} f = 3$

4) $\lim\limits_{x \to 2} f$ = doesn't exist 5) $f(3)$ = undefined 6) $\lim\limits_{x \to 3} f = 4$

7) $\lim\limits_{x \to 6^-} f = -\infty$ 8) $\lim\limits_{x \to 6^+} f = -\infty$ 9) $\lim\limits_{x \to 6} f = -\infty$

10) $f(6)$ = undefined 11) $f(7) = 2$

⇒ 1) $\lim\limits_{x \to 3} \frac{1}{x} = \frac{1}{3}$ 2) $\lim\limits_{x \to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$ 3) $\lim\limits_{x \to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$

$* \frac{1}{0^\pm} = \pm\infty$ in limits

4) $\lim\limits_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$ 5) $\lim\limits_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = -0 = 0$

$* \frac{1}{\pm\infty} = 0$ in Limits

⇒ **Ex:** 1) $\lim_{x\to 1} \frac{x^2-9}{x-3} = \frac{1^2-9}{1-3} = 4$

2) $\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{0}{0}$!! (indeterminate)

$= \lim_{x\to 3} \frac{(\cancel{x-3})(x+3)}{\cancel{x-3}} = 6$

3) $f(x) = \begin{cases} x^2 & , x \geq 0 \\ x-2 & , x < 0 \end{cases}$

1) $f(0) = 0$ 2) $f(2) = 4$

3) $f(-1) = -3$ 4) $\lim_{x\to 3} f = 3^2 = 9$

5) $\lim_{x\to -4} f = \lim_{x\to -4} x-2 = -6$

6) $\lim_{x\to 0} f$ → $\lim_{x\to 0^+} x^2 = 0$

→ $\lim_{x\to 0^-} x-2 = -2$

∴ $\lim_{x\to 0} f$ doesn't exist

4) $f(x) = \begin{cases} \frac{x^3+8}{x+2} & , x \neq -2 \\ 1 & , x = -2 \end{cases}$

$\lim_{x\to -2} f = \lim_{x\to -2} \frac{x^3+8}{x+2} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}} = 12$

⇒ L'Hopital ⇒ $\lim_{x\to -2} \frac{x^3+8}{x+2} = \frac{3x^2}{1} = 3 \times 4 = 12$

* $\frac{a}{\infty} = 0$, $\frac{\infty}{0} = \infty$, $\frac{a}{\infty} = 0$, $\frac{\infty}{a} = \infty$, $\frac{a}{0^{\pm}} = \pm\infty$

* $\infty + \infty = \infty$, $-\infty - \infty = -(\infty + \infty) = -\infty$

* Indeterminate forms! الكميات غير المحددة / غير المعينة

$$\underbrace{\frac{0}{0} \,,\, \frac{\infty}{\infty}}_{LH} \,,\; 0 \cdot \infty \,,\; 1^{\infty} \,,\; 0^{0} \,,\; \infty^{0}$$

$$\Rightarrow \lim_{x \to \infty} 1^{x} = 1^{\infty}$$

$$\hookrightarrow = \lim_{x \to \infty} 1 = 1$$

$$\Rightarrow \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = (1 + 0)^{\infty} = 1^{\infty} = e^{1}$$

$\Rightarrow$ **Ex:**

$$f(x) = \begin{cases} 1 + \sin x & , x < 0 \\ \cos x & , 0 \le x \le \pi \\ \sin x & , x > \pi \end{cases}$$

$\Rightarrow$ for what values of "a" $\lim_{x \to a} f(x)$ exists

$$\begin{bmatrix} a \le 0 \\ 0 \le a < \pi \\ a > \pi \end{bmatrix}$$

$\mathbb{R} - \{\pi\}$

$\Rightarrow$ **Ex:**

$$\lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \frac{0}{0} \,!!$$

$$\lim_{t \to -3} \frac{(t-3)\cancel{(t+3)}}{\cancel{(t+3)}(2t+1)} = \frac{6}{5}$$

$$LH \Rightarrow \frac{2t}{4t+7} = \frac{-6}{-5} = \frac{6}{5}$$

⇒ <u>Limits</u>

1) $\lim_{t\to 2} \dfrac{\sqrt{4t+1}-3}{t-2} = \frac{0}{0}$!!

$\lim_{t\to 2} \dfrac{\sqrt{4t+1}-3}{t-2} \times \dfrac{\sqrt{4t+1}+3}{\sqrt{4t+1}+3} = \lim_{t\to 2} \dfrac{4(\cancel{t-2})}{(\cancel{t-2})(\sqrt{4t+1}+3)}$

$= \frac{2}{3}$

2) $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \dfrac{\cancel{3-x}}{3x(\cancel{x-3})} = \dfrac{-1}{9}$

3) $\lim_{x\to -6} \dfrac{2x+12}{|x+6|}$

$x+6=0 \Rightarrow x=-6$

$\begin{array}{c|c} -(x+6) & x+6 \\ \hline \multicolumn{2}{c}{-6} \end{array}$

$\Rightarrow \lim_{x\to -6^+} \dfrac{2(\cancel{x+6})}{\cancel{x+6}} = 2$

$\lim_{x\to -6^-} \dfrac{2(\cancel{x+6})}{-\cancel{(x+6)}} = -2$

$\Rightarrow \therefore \lim_{x\to -6} \dfrac{2x+12}{|x+6|}$ = doesn't exist

4) $\lim_{x\to -6} \dfrac{2x-12}{|x+6|} = \dfrac{-24}{0} = -\infty$

* 1) $\lim_{x\to a} (f \mp g) = \lim_{x\to a} f \mp \lim_{x\to a} g = L \mp M$

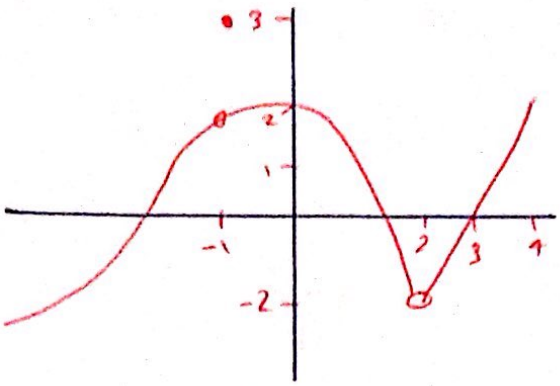
$\lim_{x\to a} f = L$

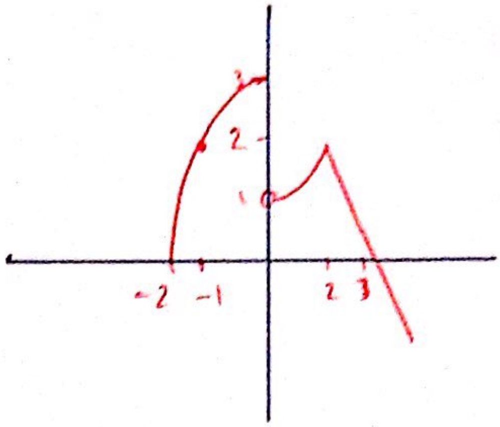
$\lim_{x\to a} g = M$

2) $\lim_{x\to a} \frac{f}{g} = \dfrac{L}{M}$, $M \neq 0$

3) $\lim_{x\to a} f.g = L.M$

⇒ **Ex:**





$f(-1) = 3$, $\lim_{x \to -1} f = 2$

⇒ 1) $\lim_{x \to 2} (f+g) = \lim_{x \to 2} f + \lim_{x \to 2} g = -2 + 2 = \boxed{0}$

2) $\lim_{x \to 3} \frac{f}{g} = \dfrac{\lim_{x \to 3} f}{\lim_{x \to 3} g} = \dfrac{0}{\frac{1}{2}} = \boxed{0}$

3) $\lim_{x \to 3} \frac{g}{f} = \dfrac{\frac{1}{2}}{0} = \boxed{\infty}$

4) $\lim_{x \to 2} (x^2 \cdot f(x)) = \lim_{x \to 2} x^2 \cdot \lim_{x \to 2} f(x) = 4 \cdot -2 = \boxed{-8}$

5) $f(-1) + \lim_{x \to -1} g = 3 + 2 = \boxed{5}$

6) $\lim_{x \to 0^+} f(x) + g(x) = 2 + 1 = \boxed{3}$

7) $\lim_{x \to 0^-} f(x) + g(x) = 2 + 3 = \boxed{5}$

8) $\lim_{x \to -1} f \cdot g = 2 \cdot 2 = 4$

Ex: $\lim_{x\to 0} \sin\frac{1}{x} = \sin\frac{1}{0} = \sin\infty$ (many answers) ∴ doesn't exist

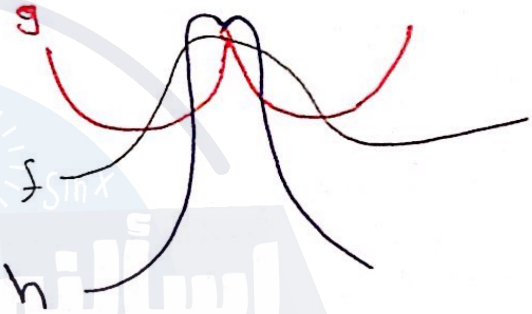
$(-1 \le \sin x \le 1)$

$\Rightarrow \lim_{x\to 0} x^2 \sin\frac{1}{x}$ = doesn't exist . 0

* Squeeze theorem

f, g & h are 3-functions

$g \le f \le h$, around $x=a$



If $\lim_{x\to a} g = \lim_{x\to a} h = L$

$\therefore \lim_{x\to a} f = L$

* إذا كانت نهاية الطرفين متساوية وتساوي رقم معين فإن نهاية مابينهما تساوي ذلك الرقم

$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x}$ $= \frac{\text{doesn't exist}}{\infty}$ (determinate)

* $a \cdot \infty = \infty$
* $a \cdot$ doesn't exist = doesn't exist

$-1 \le \sin x \le 1$

$\frac{-1}{x} \le \frac{\sin x}{x} \le 1$

$\lim_{x\to\infty} \frac{-1}{x} = \frac{-1}{\infty} = -0 = 0$

$\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$

$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$ by the squeeze thm.

$\Rightarrow \lim_{x \to 0} x^2 \sin\frac{1}{x}$

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\left.\begin{array}{l} \lim_{x \to 0} -x^2 = 0 \\ \lim_{x \to 0} x^2 = 0 \end{array}\right] \therefore \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$

by the squeeze thm.

⇒ Vertical & Horizontal asymptotes.

* V. asy.

$$\left.\begin{array}{l} 1)\ \lim\limits_{x\to a^+} f = \pm\infty \\ \\ 2)\ \lim\limits_{x\to a^-} f = \pm\infty \end{array}\right\} \Rightarrow x = a \text{ is a vertical asymptote}$$

⇒ Ex: $f(x) = \dfrac{x+1}{x^2-1}$ ⇒ find V. asy. if exists.

⇒ check $x=1$, $x=-1$

1) $\lim\limits_{x\to 1} \dfrac{x+1}{x^2-1} = \dfrac{2}{0} = \pm\infty$ ∴ $x=1$ is a vertical asymptote.

2) $\lim\limits_{x\to -1} \dfrac{x+1}{x^2-1} = \dfrac{0}{0}$!! ⇒ $\lim\limits_{x\to -1} \dfrac{1}{x-1} = \dfrac{-1}{2} \neq \infty$ ∴ $x=-1$ isn't V. asy.

Ex: $f(x) = \dfrac{x^2+1}{3x-2x^2}$ ⇒ Find V. asy.

$3x - 2x^2 = 0$
$x(3-2x) = 0$
$x = 0, \frac{3}{2}$

1) $\lim\limits_{x\to 0} \dfrac{x^2+1}{3x-2x^2} = \dfrac{1}{0} = \pm\infty$ ∴ $x=0$ is a V. asy.

2) $\lim\limits_{x\to \frac{3}{2}} \dfrac{x^2+1}{3x-2x^2} = \dfrac{\frac{13}{4}}{0} = \pm\infty$ ∴ $x=\frac{3}{2}$ is another V. asy.

Ex: $f(x) = \sin \frac{1}{x}$

$\lim_{x \to 0} \sin\frac{1}{x} = \sin \infty =$ doesn't exist $\neq \infty$ it isn't a V. asy.

Ex: $y = \ln x$

$\lim_{x \to 0^+} \ln x = -\infty \quad \therefore \quad x=0$ is a Vertical asymptote (y-axis)

* نأخذ صفر ما داخل Ln لمعرفة (Vertical asy.)

$\Rightarrow f(x) = \begin{cases} 2 & , x \geq 1 \\ \frac{1}{x-1} & , x < 1 \end{cases}$

$\lim_{x \to 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty \quad \therefore \quad x=1$ is a V. asy.

$\Rightarrow$ Ex: $f(x) = \begin{cases} 3x & , x \geq 1 \\ \frac{1}{x-2} & , x < 1 \end{cases}$

x=2 isn't a Vertical asymptome

$\Rightarrow$ Section 2.6: Horizontal asymptote.

1) $\lim_{x \to \infty} f = b < \infty \Rightarrow$ then $y=b$ is a H. asy.

finite عدد منتهي

$\Rightarrow \lim_{x \to -\infty} f(x) = c < \infty \quad \therefore y = c$ is another H. asy.

$\Rightarrow$ Ex: $f(x) = \frac{2}{x+1}$; is "f" have an H. asy.

1) $\lim_{x \to \infty} \frac{2}{x+1} = \frac{2}{\infty} = 0$

2) $\lim_{x \to -\infty} \frac{2}{x+1} = \frac{2}{-\infty+1} = \frac{2}{-\infty} = 0$

$\therefore y=0$ (x-axis) is a H. asy.

Ex: $y = \tan^{-1}(x)$ find: H. asy.

1) $\lim_{x \to \infty} \tan^{-1}x = \tan^{-1}\infty = \frac{\pi}{2} \Rightarrow \therefore y = \frac{\pi}{2}$ is H. asy.

2) $\lim_{x \to -\infty} \tan^{-1}x = \tan^{-1}-\infty = -\tan^{-1}\infty = -\frac{\pi}{2} \quad \therefore y = -\frac{\pi}{2}$ is another H. asy.

Ex: $\lim_{x \to 100} x^4 - 2x + 3$

$= \lim_{x \to \infty} x^4 = \infty$

* في الـPolynomials عندما $x \to \pm\infty$ نبقي على المتغير الذي يحمل القوة الأكبر ونحذف البقية

$\Rightarrow \lim_{x \to \infty} -3x^4 + 2x^2 + 1 = \lim_{x \to \infty} -3x^4 = -\infty$

$$\Rightarrow \lim_{x\to\infty} \frac{-10x^5+3x^2+1}{2x^5-x^2+10x+3}$$

$$= \lim_{x\to\infty} \frac{-10\cancel{x^5}}{2\cancel{x^5}} = -5 \qquad \therefore y=-5 \text{ is H. asy.}$$

$$\Rightarrow \lim_{x\to\infty} \frac{3x^4+10x+1}{-2x+5} = \lim_{x\to\infty} \frac{3x^4}{-2x} = \lim_{x\to\infty} \frac{-3x^3}{2} = -\infty$$

7) $f(5) = 4$

8) $\lim\limits_{x \to 5} f = \frac{1}{2}$

} f isn't continuous at $x=5$

9) $f(6) = 3$

10) $\lim\limits_{x \to 6^-} f = +\infty$

} not continuous

⇒ Def: "f" is continuous at $x=a$, if:

1) f is defined at $x=a$

2) $\lim\limits_{x \to a} f$ exists

3) $\lim\limits_{x \to a} f = f(a)$

⇒ "أنواع الانفصال"

* Removable discontinuity:

if $\lim\limits_{x \to a} f$ exists but either f isn't defined at $x=a$

or $\lim\limits_{x \to a} f \neq f(a)$

* Unremovable discontinuity

1) jump: if Right Limit $\neq$ Left Limit

2) infinite if $\lim\limits_{x \to a} f = \pm \infty$

Ex: 1) $f(x) = \begin{cases} \cos x & , x < 0 \\ 0 & , x = 0 \\ 1 - x^2 & , x > 0 \end{cases}$ $\Rightarrow$ is f continuous at $x=0$

$f(0) = 0$

$$\left.\begin{aligned} \lim_{x \to 0^+} f &= \lim_{x \to 0^+} 1 - x^2 = 1 - 0 = 1 \\ \lim_{x \to 0^-} f &= \lim_{x \to 0^-} \cos x = \cos 0 = 1 \end{aligned}\right] \lim_{x \to 0} f = 1$$

$\lim_{x \to 0} f$ exists but it isn't equal $f(0)$ $\therefore$ f isn't continuous at $x=0$

f has a removable discontinuity at $x=0$, Since $\lim_{x \to 0} f \neq f(0)$

2) $f(x) = \begin{cases} x^2 + 1 & , x \leq 0 \\ x + 1 & , x > 0 \end{cases}$ is f continuous at $x=0$

$$\left.\begin{aligned} f(0) = \lim_{x \to 0^-} x^2 + 1 &= 1 \\ \lim_{x \to 0^+} x + 1 &= 1 \end{aligned}\right\}$$ f is continuous at $x=0$

f is cont. everywhere

3) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x < 2 \\ ax^2 - bx + 3 & , 2 \le x < 3 \\ 2x - a + b & , x \ge 3 \end{cases}$

$\Rightarrow$ Find: a & b So that f is continuous everywhere.

$\lim_{x \to 2^-} f = 4$

$\therefore \lim_{x \to 2^+} f = 4 = \lim_{x \to 3^-} f$

$ax^2 - bx + 3 = 4$

$4a - 2b + 3 = 4$

$4a - 2b = 1 \quad \ldots\ldots (1) \quad \Rightarrow \quad 4a + 4 - 2a = 1$

$2a = -3$

$a = \frac{-3}{2}$

$\lim_{x \to 3^+} f = f(3) = 4$

$b = -2 - \frac{3}{2}$

$2x - a + b = 4$

$6 - a + b = 4$

$\boxed{b = \frac{-7}{2}}$

$b - a = -2 \quad \ldots\ldots (2) \quad \boxed{b = -2 + a}$

$\Rightarrow f(x) = \begin{cases} x - 1 & , x \le 1 \\ c(x-1)^2 & , x > 1 \end{cases}$ Find c So that the function is continuous everywhere.

$\lim_{x \to 1^-} x - 1 = 0 = \lim_{x \to 1^+} f$

$c(x-1)^2 = 0$

$c \cdot 0 = 0$

$0 = 0$ true independent of c

$c \in \mathbb{R}$

Ex: Find the discontinuity for "f" & classify them as removable, jump or vertical asymptote.

1) $f(x) = \dfrac{x-2}{x^2+x-6}$

$x^2+x-6=0$

$x = 2, -3$ are points of discontinuity.

$\Rightarrow x=2$ removable $\lim\limits_{x\to 2} f = \frac{1}{5}$ exists

$\Rightarrow x=-3$ vertical asymptote $\lim\limits_{x\to -3} f = \frac{1}{0} = \infty \Rightarrow$ unremovable

$x=-3$ is a vertical asymptote.

2) $f(x) = \dfrac{2+x}{2-|x|}$

$2-|x|=0$

$|x|=2$

$\boxed{x=\pm 2}$ points of discontinuity

$\Rightarrow x=2 \quad \lim\limits_{x\to 2} \dfrac{2+x}{2-x} = \frac{4}{0} = \infty$ $\quad x=2$ is vertical asymptote.

$\Rightarrow x=-2 \quad \lim\limits_{x\to -2} \frac{0}{0}!! \Rightarrow \dfrac{2+x}{2+x} = 1$ exists

$\therefore$ removable at $x=-2$

Ex: $f(x) = \begin{cases} x+1 & , x \geq 2 \\ 2x & , x < 2 \end{cases}$

$f(2) = 3$

$\lim_{x \to 2^+} f = 3$, $\lim_{x \to 2^-} f = 4$ $\Big]$ $\lim_{x \to 2} f$ doesn't exist

unremovable at $x=2$

jump discontinuity at $x=2$

Ex: $f(x) = 1 - \sqrt{1 - x^2}$, show "f" is continuous on $[-1, 1]$

1) f is continuous on $(-1, 1)$; domain

2) f is continuous from the right at $x=-1$

$\lim_{x \to -1^+} f = f(-1) = 1$

3) f is continuous from the left at $x=1$

$\lim_{x \to 1^-} f = f(1) = 1$

$\therefore f$ is continuous on $[-1, 1]$

* Thm. $\lim_{x \to a} g(x) = L$ & f is continuous at "L", then

$\lim_{x \to a} f(g(x))$ exists & $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$

Ex: 1) find the limit:

$$\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1} \lim_{x\to 1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

2) $g(2)=3$ f is continuous every where.

$\lim_{x\to 1} g(x) = 4$ & $f(4)=5$

$f(3)=7$

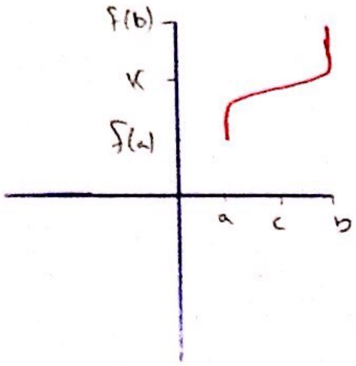
then $\lim_{x\to 1} f(g(x)) =$

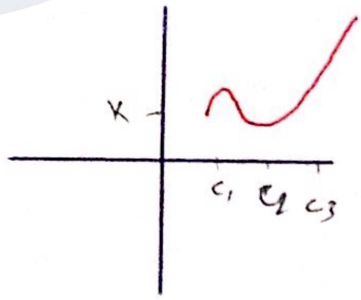
$= f\left(\lim_{x\to 1} g(x)\right) = f(4) = 5$

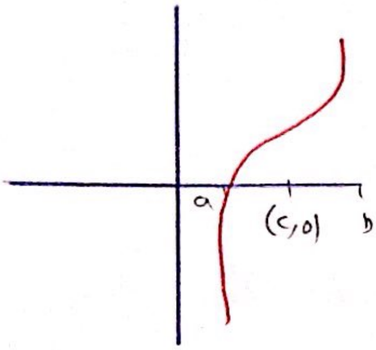
* Thm: The Intermediate Value thm. (I.V.T)

Let $f(x)$ be continuous on $[a,b]$ & let k between $f(a)$ & $f(b)$

then there is at least $c \in (a,b) \Rightarrow f(c)=k$







$f(a) < 0$

$f(b) > 0$

$f(a) \cdot f(b)$ is negative

$\Downarrow$

$f(x) = 0$ has at least one real solution.

Ex: Show that the equation : $x^4 + x = 3$ has at least one real solution.

$x^4 + x - 3 = 0$

$F(x) = x^4 + x - 3$

$f(0) = -3 < 0$

$f(2) = 15 > 0$

$f(1) = -1 < 0$

$f(1) \cdot f(2) < 0 \xRightarrow[\text{I.V.T}]{\text{By}}$ there is at least $c \in (1,2) \Rightarrow f(c) = 0$

eqn. has at least one real root (solution)

Ex: $\cos x = x^3$, Show that this eqn. has at least one real root in $(0, \pi)$

$\cos x - x^3 = 0$

$f(x) = \cos x - x^3$ on $[0, \pi]$ f is continuous.

$f(0) = 1 > 0$

$f(\pi) = -1 - \pi^3 < 0$

$f(0) \cdot f(\pi) < 0 \xRightarrow[\text{I.V.T}]{\text{By}}$ there is $c : f(c) = 0$

that is $\cos x = x^3$ has at least one real root

Ex: $f(a) \cdot f(b) < 0$

$\cancel{+}$ f is continuous on (a, b)

$\Rightarrow$ How many roots are there for f in (a, b)

1) at least 1 ✓

2) at most 1

3) exactly 1

Ex: Find: 1) $\lim\limits_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \tan^{-1}\left(\frac{1}{0^+}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

2) $\lim\limits_{x \to \infty} e^x = e^{\infty} = \infty$

* $\lim\limits_{x \to \infty} a^x = \begin{cases} \infty & a > 1 \\ 0 & 0 < a < 1 \end{cases}$

$\lim\limits_{x \to \infty} b^{-x} = \frac{1}{b^{\infty}} = \frac{1}{\infty} = 0$

3) $\lim\limits_{x \to \infty} e^{-x} = e^{-\infty} = \frac{1}{\infty} = 0$

4) $\lim\limits_{x \to \infty} \sin x = \sin \infty$ $\begin{matrix} 1 \\ -1 \end{matrix}$ $\rightarrow$ it has many answers

this limit doesn't exist

5) $\lim_{x\to\infty} 3x^5 + 2x^2 - 1 = -\infty$

$\lim_{x\to-\infty} -3x^5 + 2x^2 - 1 = -1 \cdot -\infty = \infty$

6) $\lim_{x\to-\infty} \sqrt{x^2+1} - x = \infty - (-\infty) = \infty$

7) $\lim_{x\to\infty} \sqrt{x^2+1} - x = (\infty - \infty)$!!

$$\lim_{x\to\infty} \sqrt{x^2+1} - x \times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{x^2+1 - x^2}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x}$$

$$= \frac{1}{\infty + \infty} = \frac{1}{\infty} = 0$$

Ex $f = \frac{\sqrt{4x^2+1}}{2x-5}$, Find H.asy. and V.asy.

1) $\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{2x-5} = \frac{\infty}{\infty}$!! $= \lim_{x\to\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}}{2x-5} = \frac{2}{2} = 1$

$\therefore y = 1$ is H.asy.

2) $\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{2x-5} = -1$ $\quad y = -1$ is another H.asy

V.asy أصفار المقام

$2x - 5 = 0$

$x = \frac{5}{2}$

$\Rightarrow \lim_{x\to\frac{5}{2}} \frac{\sqrt{4x^2+1}}{2x-5} = \frac{\sqrt{26}}{0} = \infty$

$\therefore x = \frac{5}{2}$ is a V.asy.

⇒ Derivatives

Def: f is differentiable at $x = a$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists} \quad = f'(a)$$

Ex: $f(x) = x^2$

Find: $f'(4)$ by ~~...~~ using the def.

$$\lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \frac{x^2 - 16}{x - 4} = x + 4 = 8$$

2) $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\Rightarrow f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$

3) $f'(x) = \frac{df}{dx}$

Ex: $\sqrt{2x-1}$ Find: $f'(x)$ by using the def.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \frac{\sqrt{2z-1} - \sqrt{2x-1}}{z - x} \times \frac{\sqrt{2z+1} + \sqrt{2x+1}}{\sqrt{2z+1} + \sqrt{2x-1}}$$

$$= \lim_{z \to x} \frac{2(\cancel{z-x})}{(\cancel{z-x})(\sqrt{2z-1} - \sqrt{2x+1})} = \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$$

⇒ Differentiation rules

1) $f \mp g = f' \mp g'$

2) $f \cdot g = f \cdot g' + g \cdot f'$

3) $\frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$

4) $\frac{1}{g} = \frac{-g'}{g^2}$

5) $c \cdot f = c \cdot f'$ (c ↓ const.)

$f(x)$	$f'(x)$
c	0
$x^r \Rightarrow r \in \mathbb{R}$	$r x^{r-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\csc x$	$-\csc x \cot x$
e^x	$e^x \ln e$
a^x	$a^x \ln a$
2^x	$2^x \ln 2$
$\ln x$	$\frac{1}{x}$
$\ln f$	$\frac{f'}{f}$
$\log_a x$	$\frac{1}{x \ln a}$
$\log_3 x$	$\frac{1}{x \ln 3}$

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$

$\Rightarrow$ Ex : 1) $f(x) = \ln|x^2-3x| + \log_4(x+1)$; find f'??

$$f' = \frac{2x-3}{x^2-3x} + \frac{1}{(x+1)\ln 4}$$

2) $x^2 e^x + 3^x$; find $\frac{df}{dx}$??

$$\frac{df}{dx} = x^2 \cdot e^x + e^x \cdot 2x + 3^x \ln 3$$

3) $F(x) = 2^{\cos x^3}$; find $f'(x)$??

$$f'(x) = 2^{\cos x^3} \ln 2 \cdot (-\sin x^3) \cdot 3x^2$$

* Thm: If f is diff. at $x=a$, then f is continuous at $x=a$

$\therefore \lim_{x\to a} f = f(a)$

$$f(x) - f(a) = f(x) - f(a)$$

$$f(x) - f(a) = \frac{f(x) - f(a)}{x-a} \cdot x-a$$

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a)$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} f(a)$$

diff $\rightarrow$ Cont

Not Cont $\rightarrow$ Not diff

$$\lim_{x\to a} f(x) = f(a)$$

* Study cont. then study diff.

$\Rightarrow$ Ex: $f(x) = \begin{cases} x^2+1 & , x \leq 0 \\ 3x & , x > 0 \end{cases}$

A) Is F diff at x=0 ??

1) cont?

$\lim\limits_{x \to 0^+} f = 0$
$\lim\limits_{x \to 0^-} f = 1$] f isn't cont. at $x=0$
$\therefore$ f isn't diff at $x=0$

B) Find F'

$F' = \begin{cases} 2x & , x < 0 \\ 3 & , x > 0 \end{cases}$

Ex: $f(x) = |x|$; Is F diff at x=0 ??

$f(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$\lim\limits_{x \to 0^+} f(x) = 0$
$\lim\limits_{x \to 0^-} f(x) = 0$
$f(0) = 0$] f is cont. at $x=0$

$F'(x) = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \end{cases}$

$f'_+(0) = 1$
$f'_-(0) = -1$] f isn't diff at $x=0$

$\Rightarrow$ **Ex:** $f(x) = \begin{cases} x^3 & , x \le 0 \\ x^2 & , x > 0 \end{cases}$

A] Is f diff. at $x=0$

$$\left.\begin{aligned} \lim_{x \to 0^+} f(x) &= 0 \\ \lim_{x \to 0^-} f(x) &= 0 \\ f(0) &= 0 \end{aligned}\right] \quad f \text{ is cont. at } x=0$$

$$f'(x) = \begin{cases} 3x^2 & , x \le 0 \\ 2x & , x > 0 \end{cases}$$

$$\left.\begin{aligned} f'_+(0) &= 0 \\ f'_-(0) &= 0 \end{aligned}\right] f \text{ is diff. at } x=0$$

Ex: $f(x) = \begin{cases} x \sin \frac{1}{x} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$ Is "f" diff at $x=0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$ by the squeeze thm

$-1 \le \sin \frac{1}{x} \le 1$

$-x \le x \sin \frac{1}{x} \le x$

$f(0) = 0$

$$\Rightarrow \quad f'(0) = \frac{f(x) - f(0)}{x - 0} = \frac{x \sin\frac{1}{x}}{x} = \sin\frac{1}{x} = \text{doesn't exist}$$

$\therefore$ f isn't diff at $x = 0$

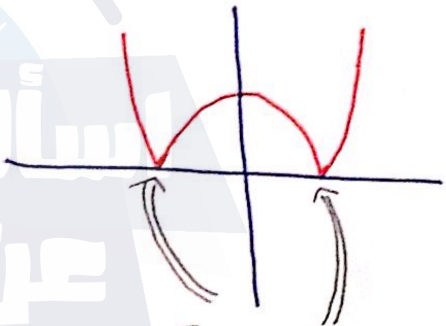
Ex: $f(x) = |x^2 - 1|$; Find $f'(x)$

$x^2 - 1 = 0$

$x = \pm 1$

x^2-1	$1-x^2$	x^2-1
	-1 1	

$$f(x) = \begin{cases} x^2 - 1 & , \; x \leq -1, \; x \geq 1 \\ 1 - x^2 & , \; -1 < x < 1 \end{cases}$$



Corner

$\therefore$ not differentiable

$$f'(x) = \begin{cases} 2x & , \; x < -1, \; x > 1 \\ -2x & , \; -1 < x < 1 \end{cases}$$

$\left.\begin{aligned} f'_+(1) &= 2 \\ f'_-(1) &= -2 \end{aligned}\right]$ $f'(1)$ doesn't exist

$\left.\begin{aligned} f'_+(-1) &= 2 \\ f'_-(-1) &= -2 \end{aligned}\right]$ $f'(-1)$ doesn't exist

* Higher derivatives :

$y = f(x)$

$y' = f'(x) = \dfrac{df}{dx}$

$y'' = \dfrac{d^2 y}{dx^2} = 2^{nd}$ derivative

$y''' = \dfrac{d^3 y}{dx^3} = 3^{rd}$ derivative

$\dfrac{d^4 y}{dx^4} = f^{(4)}$, 4^{th} derivative

$$\Rightarrow \frac{dy^n}{dx^n} = y^{(n)} = n^{th} \text{ derivative}$$

Ex: $f(x) = x^5 + 3x^2 + 1$

$$f'(x) = 5x^4 + 6x$$

$$f''(x) = 5.4\,x^3 + 6$$

$$f'''(x) = 5.4.3\,x^2$$

$$f^{(4)}(x) = 5.4.3.2\,x$$

$$f^{(5)}(x) = 5.4.3.2.1 = 5!$$

$$f^{(6)} = 0$$

$0! = 1$

* $f(x) = x^n$; n is a positive integer

$$f^{(n)}(x) = n!$$

$$f^{(n+1)}(x) = 0$$

Ex: $\lim\limits_{x \to 0} \frac{3^x - 1}{x} =$

LH: $3^x \ln 3 = \ln 3$

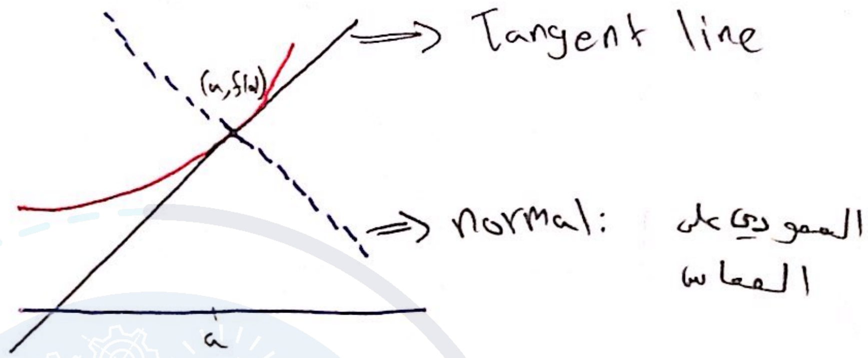
Let $f(x) = 3^x$

$$f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{3^x - 1}{x} = 3^x \ln 3 = \ln 3$$

* Geometrical meaning of $f'(a)$:

$f'(a)$ is the slope of the tangent line at point $(a, f(a))$

$$T_{\text{line}}: \; y - f(a) = f'(a)(x-a)$$



$$\text{Normal line}: \; y - f(a) = \frac{-1}{f'(a)}(x-a)$$

Ex: $f(x) = x^2$; Find the equation of the tangent and Normal lines of the function at $x=3$

$f(3) = 3^2 = 9 \quad \therefore$ pt $(3, 9)$

$f'(x) = 2x$

$f'(3) = 2 \cdot 3 = 6$

$\therefore$ T line: $\Rightarrow y - 9 = 6(x-3)$

$\Rightarrow y - 6x + 9 = 0$

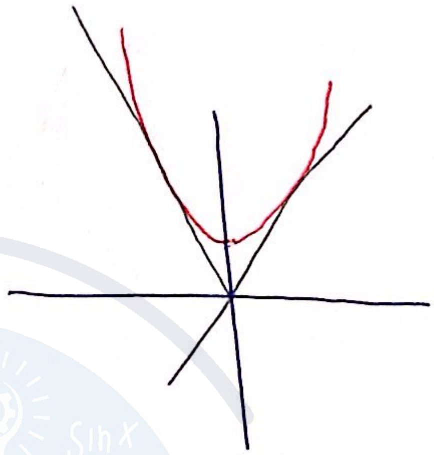
$\Rightarrow 9 - 6x + y = 0$

$\Rightarrow y + 9 = 6x$

N line: $y - 9 = \frac{-1}{6}(x-3)$

* إذا x, y بنفس طرف المعادلة ∴ الميل = $\frac{-\text{معامل س}}{\text{معامل ص}}$

Ex: Write equations of tangents of $f(x) = x^2+1$ that passes throw the point $(0,0)$



$$m = \frac{\Delta y}{\Delta x} = f'(x)$$

$$\frac{x^2+1}{x} = 2x$$

$$2x^2 = x^2+1$$

$$x^2 = 1$$

$$\boxed{x = \pm 1}$$

$\therefore$ P.t $(1, 2)$, $(-1, 2)$

$T_1: \; y - 0 = 2(x - 0)$

$$\boxed{y = 2x}$$

$T_2: \; y - 0 = -2(x - 0)$

$$\boxed{y = -2x}$$

Ex: $y = x^{\frac{3}{2}}$; write equations to the tangent and Normal lines, if the tangent of this curve is parallel to the line $y - 3x = 1$

Tangent // line

$\therefore \quad m_{tangent} = m_L$

$y' = 3$

$\frac{3}{2} x^{\frac{1}{2}} = 3$

$\boxed{x = 4} \quad \boxed{y = 8}$

T line: $y - 8 = 3(x - 4)$

N line: $y - 8 = \frac{-1}{3}(x - 4)$

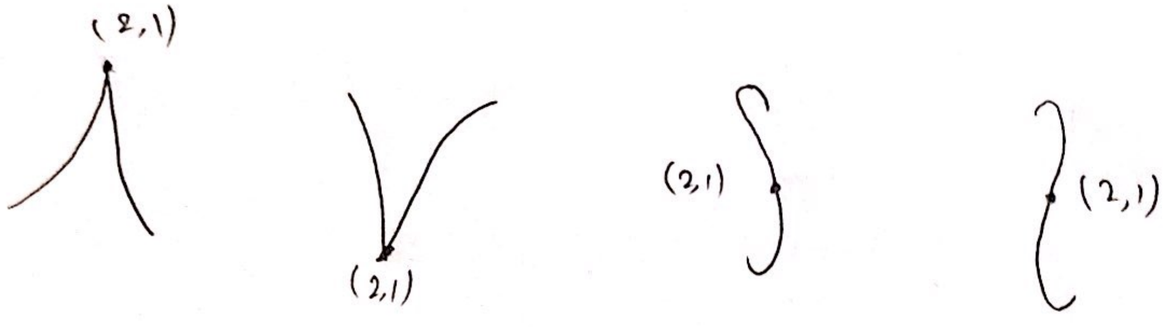
Ex: $f(x) = (x-2)^{\frac{2}{3}} + 1$; Find the equation of the tangent line at $\boxed{x = 2}$

$x = 2, \; y = 1 \Rightarrow$ p.t $(2, 1)$

$f'(x) = \dfrac{2}{3(x-2)^{\frac{1}{3}}}$

$f' \xrightarrow[x \to 2]{} \infty$

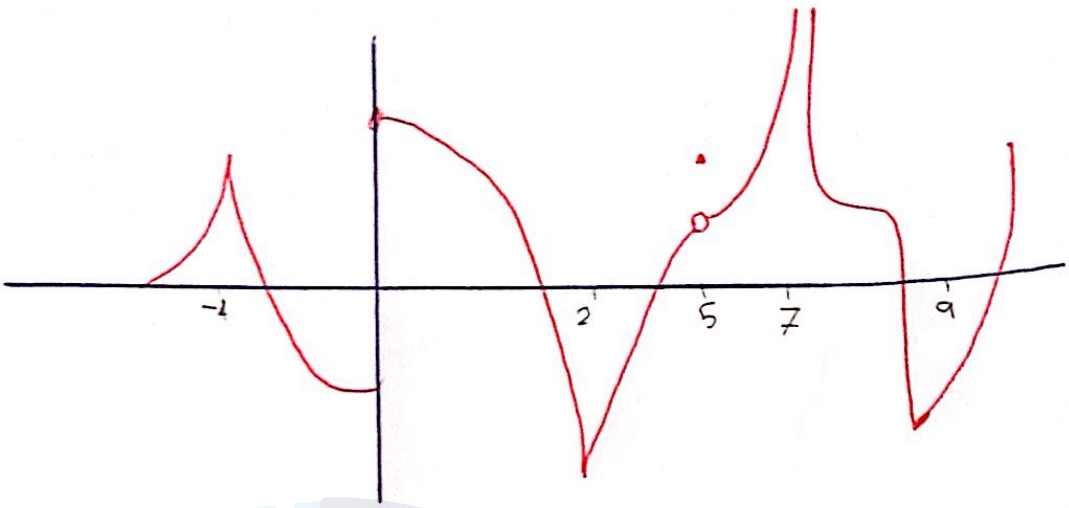
$\therefore$ vertical tangent $(x = 2)$



For

$f'(2)$ doesn't exist at this points $\} \Rightarrow f'(x) \xrightarrow[x \to 2]{} \infty$ V. tangent

Ex: State thenumbers of which "f" isn't differentiable.



$\Rightarrow$ $X = -4$: V. tangent

$x = 0$: Not Continuous

$x = 2$: V. t

$x = 5$: Not continuous

$x = 7$: Not continuous (undifined)

$x = 9$: Corner (not V. tangent) لانه ما في انحناء

$\Rightarrow$ 3. 4 & 3. 5 : The chain rule & implicit differentiation

* Chain rule:

1) $F(x) = g(h(x))$

$$\frac{df(x)}{dx} = f'(x) = g'(h(x)) \cdot h'(x)$$

$\Rightarrow$ Ex: $y = \tan^3(x^2+3x)$; Find y' ??

$$\frac{dy}{dx} = 3\tan^2(x^2+3x) \cdot \sec^2(x^2+3x) \cdot (2x+3)$$

2) $y = f(u)$ & u is a function of x $u = g(x)$

$$y \to u \to x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex: $y = \cos x^2$, $x = t^3$; Find $\frac{dy}{dt}$??

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= -2x \sin x^2 \cdot 3t^2$$

Ex: $y = 2^x$, Find: $\frac{dy}{dx^2}$

Let $u = x^2$

$$\frac{dy}{dx^2} = \frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} \rightarrow = \frac{1}{\frac{du}{dx}}$$

$$= 2^x \ln 2 \cdot \frac{1}{2x} = \frac{2^x \ln 2}{2x}$$

Ex: $h(x) = \sqrt{4 + 3f(x)}$, $f(1) = 7$, $f'(1) = 4$, Find $h'(1)$

$$h'(x) = \frac{3f'(x)}{2\sqrt{4+3f(x)}} = \frac{3 \cdot 4}{2 \cdot 5} = \frac{12}{10} = \frac{6}{5}$$

→ Implicit Differentiation

when the relation between the variables is implicit (equation) then the differentiation is implicit.

Ex: 1) $x^2 + 3xy^2 + y = 5$; find $\frac{dy}{dx} = y'$??

* diff. both sides to x ??

$2x + 3y^2 + 6xyy' + y' = 0$

$$\boxed{y' = \frac{-2x - 3y^2}{6xy + 1}}$$

2) $e^{\frac{x}{y}} = x - y$; y' ??

$e^{\frac{x}{y}} \cdot \frac{y - xy'}{y^2} = 1 - y'$

$y^2 - y^2y' = ye^{\frac{x}{y}} - xy'e^{\frac{x}{y}}$

$y'(xe^{\frac{x}{y}} - y^2) = ye^{\frac{x}{y}} - y^2$

$$\boxed{y' = \frac{ye^{\frac{x}{y}} - y^2}{xe^{\frac{x}{y}} - y^2}}$$

Ex: $f(x) + x^2(f(x))^3 = 10$ & $f(1) = 2$; find $f'(1)$??

$f'(x) + 2x(f(x))^3 + 3x^2(f(x))^2 \cdot f'(x) = 0$

$f'(1) + 16 + 12f'(x) = 0$

$f'(x) = \frac{-16}{13}$

$\Rightarrow$ **Ex:** $g(x) + x\sin(g(x)) = x^2$; Find $g'(0)$??

$$g'(x) + \sin(g(x)) + x\cos(g(x)).g'(x) = 2x$$

$$g'(0) + 0 + 0 = 0$$

$$\boxed{g'(0) = 0}$$

$$g(0) \Rightarrow \quad g(0) + 0 = 0$$

$$\boxed{g(0) = 0}$$

Ex: $\sin y + \cos x = 1$; Find y' ??

$$\cos y \,.\, y' - \sin x = 0$$

$$y' = \frac{\sin x}{\cos y}$$

$$y'' = \frac{\cos y.\cos x + \sin x\,\sin y.\,y'}{\cos^2 y}$$

$$y'' = \frac{\cos y\,.\,\cos x + \sin x.\sin y.\,\frac{\sin x}{\cos y}}{\cos^2 y}$$

$$y'' = \frac{\cos x}{\cos y} + \frac{\sin^2 x}{\cos^2 y}\,.\,\tan y$$

Ex: $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$; Find eqn of the tangent line at $(0, \frac{1}{2})$

$$2x + 2yy' = 2(2x^2 + 2y^2 - x).(4x + 4yy' - 1)$$

$$0 + y' = 2(0 + \tfrac{1}{2} - 0).(0 + 2y' - 1)$$

$$\boxed{y' = 1}$$

$$T_{line}: \quad y - \tfrac{1}{2} = x$$

* $(f^{-1})'_{(y)} = \dfrac{1}{f'(x)}$, $y = f(x)$

صورة

أصل الصورة

* $(f^{-1})'_{(x)} = \dfrac{1}{f'(\underbrace{f^{-1}(x)}))}$

صورة

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صورة

$\Rightarrow y = \sin x$

$(f^{-1})'_{(1)} = \dfrac{1}{f'(f^{-1}(1))}$

Ex: $f(4) = 5$, $f'(4) = \frac{2}{3}$, $f'(5) = 2$; find $(f^{-1})'(5)$??

$(f^{-1})'_{(5)} = \dfrac{1}{f'(4)} = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$

Ex: $f(x) = x^3 + x + 1$, find $\left.\dfrac{df^{-1}(x)}{dx}\right|_{x=3}$

صورة

$(f^{-1})'_{(3)} = \dfrac{1}{f'(1)} = \dfrac{1}{4}$

$\Rightarrow$ Ex: Find y' ??

1) $y = \sin^{-1}(x^3 + 2x)$

$$y' = \frac{3x^2 + 2}{\sqrt{1 - (x^3 + 2x)^2}}$$

2) $y = \sqrt{\tan^{-1}(x^3)}$

$$y' = \frac{3x^2}{2\sqrt{\tan^{-1}x^3} \cdot (1 + (x^3)^2)}$$

3) $y = \ln(e^{-x} + xe^{-x})$

$$= \frac{-xe^{-x}}{e^{-x} + xe^{-x}} = \frac{-x}{1+x}$$

4) $y = \ln(1 + \ln x)$

$$y' = \frac{\frac{1}{x}}{1 + \ln x}$$

5) $y = \ln\left(\frac{\sqrt{x^2+1} \cdot (1+x)^2}{x^3 (e^{4x})}\right)$

$$y = \ln\sqrt{x^2+1} + \ln(1+x)^2 - \ln x^3 - \ln e^{4x}$$

$$y' = \frac{2x}{2x^2+2} + \frac{2}{1+x} - \frac{3}{x} - 4$$

Ex: Write the equation of the tangent line at the point $(3,0)$ for $y = \ln(x^2 - 3x + 1)$

$$y' = \frac{2x - 3}{x^2 - 3x + 1} = \frac{6 - 3}{9 - 9 + 1} = 3 = m$$

T line: $y = 3(x - 3)$

N_{line}: $y = \frac{-1}{3}(x - 3)$

⇒ Logarithmic differentiation § 3.6

Ex: $y = \dfrac{\sqrt{x} \cdot e^{x^2-x} \cdot (x+1)^{\frac{2}{3}}}{(x^2+3)^2}$; Find y'??

$\ln y = \ln\sqrt{x} + \ln e^{x^2-x} + \ln (x+1)^{\frac{2}{3}} - \ln (x^2+3)^2$

$\dfrac{y'}{y} = \dfrac{1}{2x} + (2x-1) + \dfrac{2}{3(x+1)} - 2 \cdot \dfrac{2x}{(x^2+3)}$

$y' = \left(\dfrac{1}{2x} + (2x-1) + \dfrac{2}{3(x+1)} - \dfrac{4x}{x^2+3} \right) \cdot \left(\dfrac{\sqrt{x} \cdot e^{x^2-x} \cdot (x+1)^{\frac{2}{3}}}{(x^2+3)^2} \right)$

⇒ $y = x^2$, $y = 2^x$, $y = x^x$

$y' = 2x$, $y' = 2^x \ln 2$, $\ln y = x \ln x$

$y' = (\ln x + 1) \cdot x^x$

⇒ $y = x^{\sin x}$

$\ln y = \sin x \ln x$

$\dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x}$

$y' = \left(\cos x \ln x + \dfrac{\sin x}{x}\right) \cdot x^{\sin x}$

⇒ $x^y = y^x$

$y \ln x = x \ln y$

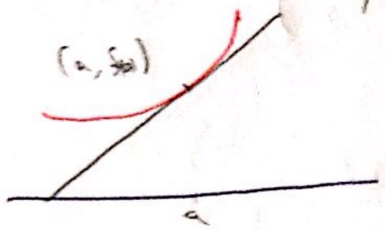
$\dfrac{y}{x} + y' \ln x = \ln y + \dfrac{x y'}{y}$

$y' = \dfrac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$

§ 3.10 Linear approximation "تقريب خطي"

$$y_f \cong y_T$$

$y = f(x)$



$$\underbrace{f(x)}_{\text{الجديدة}} \cong f(a) + f'(a)(x-a)$$

T: $y - f(a) = f'(x)(x-a)$

$y = f(a) + f'(x)(x-a)$

$L(x) = f(a) + f'(a)(x-a) \Rightarrow L(x) =$ Linearisation of "f"

Ex: Find the linearisation of $f(x) = \sqrt{x}$ at $x = 4$

$\boxed{y=2}$ $\quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

1) $y - 2 = \frac{1}{4}(x-4)$

$y = \frac{x}{4} + 1$

2) $L(x) = f(4) + f'(4)(x-4)$

$L(x) = 2 + \frac{1}{4}(x-4)$

$L(x) = \frac{x}{4} + 1$

3) approximate $f(x)$ linearly at $x = 4.1$

$\sqrt{x} \simeq 2 + \frac{1}{4}(x-4)$

$\sqrt{4.1} \simeq 2 + \frac{1}{40}$

⇒ Linear approximation

Ex: approximate Linearly

1) $\sqrt[3]{1001}$

let $f(x) = \sqrt[3]{x}$, $a \cong 1000$

$f(1000) = 10$

$f'(x) = \frac{1}{3x^{\frac{2}{3}}} \Rightarrow f'(1000) = \frac{1}{300}$

$f(x) \cong f(a) + f'(a)\underbrace{(x-a)}_{\Delta x} \Rightarrow \Delta x = 1001 - 1000 = 1$

$\sqrt[3]{1001} = 10 + \frac{1}{300}$

2) $\cos(28°)$

let $f(x) = \cos x$, $a = 30°$

$f(a) = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin x \Rightarrow f'(30) = \frac{-1}{2}$

$\cos 28° \cong \frac{\sqrt{3}}{2} - \frac{1}{2}\cdot(-2)\cdot\frac{\pi}{180}$

$\cong \frac{\sqrt{3}}{2} + \frac{\pi}{180}$

$\Delta x = 28° - 30°$
$= -2°$

في الاقترانات المثلثية

$\boxed{\Delta x \cdot \frac{\pi}{180}}$

Ex: Verify: $e^x \cos x \cong 1 + x$ at $x = 0$

let $f(x) = e^x \cos x$

$f(0) = 1$

$f'(x) = e^x \cos x - e^x \sin x \Rightarrow f'(0) = 1 - 0 = 1$

$$e^x \cos x \cong 1 + 1(x - 0)$$
$$\cong 1 + x$$

§ 3.11 Hyperbolic functions

1) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

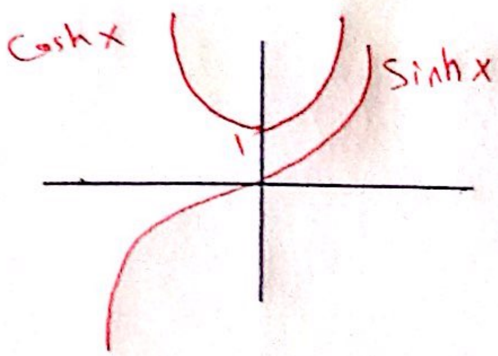
2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

3) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

4) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

5) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

6) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

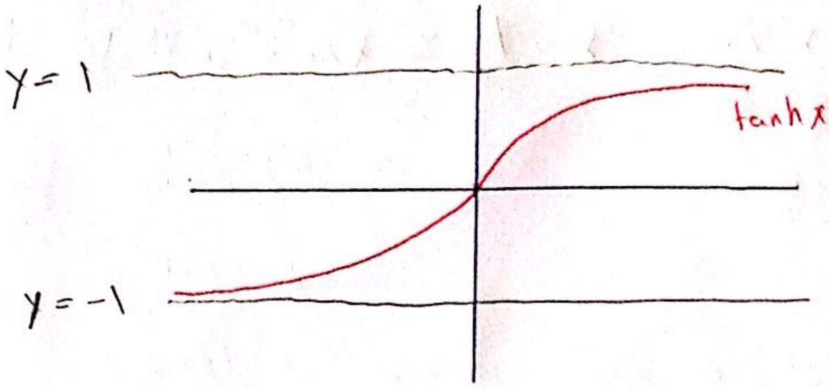


$\Rightarrow D_{\cosh x} = \mathbb{R}$, $R_{\cosh x} = [1, \infty)$

$\Rightarrow D_{\sinh x} = R_{\sinh x} = \mathbb{R}$

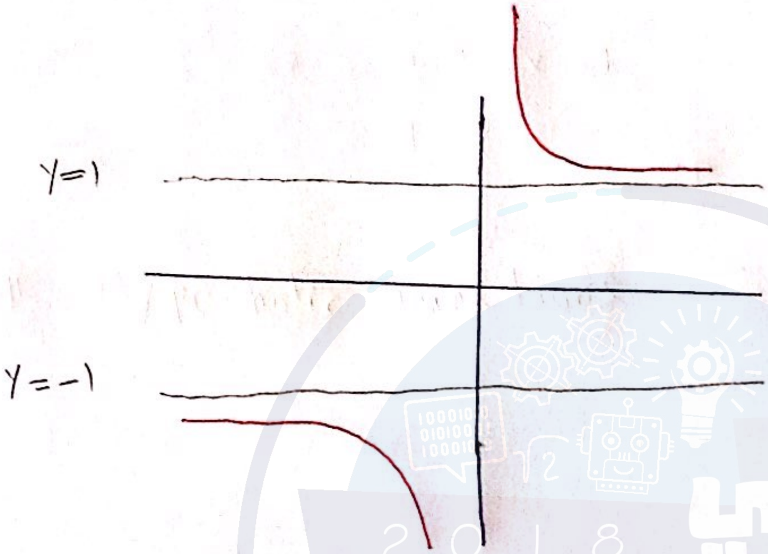
$\Rightarrow \sinh x < \cosh x$ دائماً

$\Rightarrow$ Not periodic



$D_{\tanh x} = \mathbb{R}$

$R_{\tanh x} = (-1, 1)$

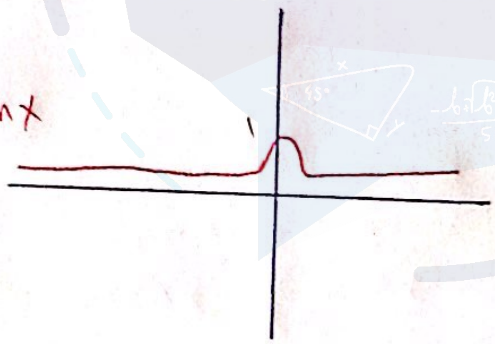


Coth x

$D_{\coth x} = \mathbb{R} - \{0\}$

$R_{\coth x} = \mathbb{R} - [-1, 1]$

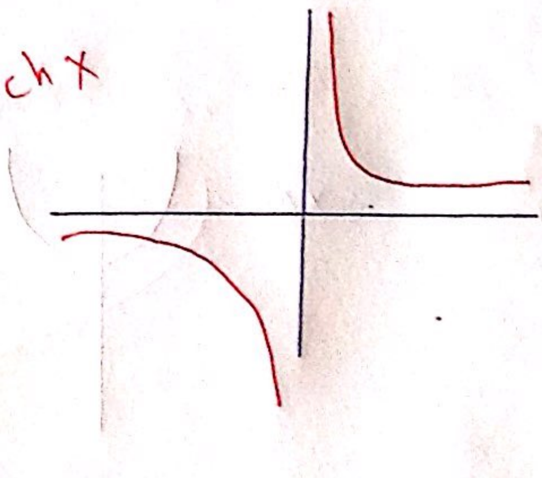
Sechx



$D_{\operatorname{sech} x} = \mathbb{R}$

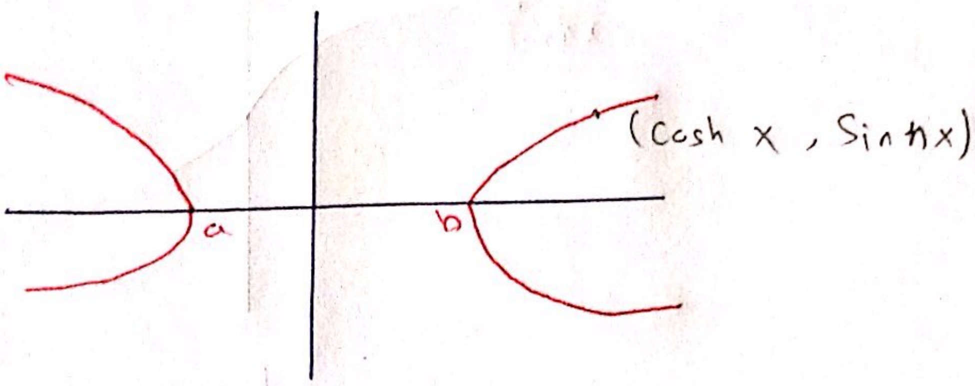
$R_{\operatorname{sech} x} = (0, 1)$

cschx



$D_{\operatorname{csch} x} = R_{\operatorname{csch} x} = \mathbb{R} - \{0\}$

* hyperbola



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow$ Unit hyperbola $(a=b=1) \Rightarrow x^2 - y^2 = 1$

$$\cosh^2 x - \sinh^2 x = 1$$

prove $\Rightarrow$ $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}}{4} = 1$

* **Identities**

1) $\cosh^2 x - \sinh^2 x = 1$

2) $1 - \tanh^2 x = \operatorname{sech}^2 x$

3) $\coth^2 x - 1 = \operatorname{csch}^2 x$

4) $\cosh(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y$

Ex: Find: $\cosh(\ln 5)$

$$\frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 + \frac{1}{5}}{2} = \frac{13}{5} = 2.6$$

* $\cosh x + \sinh x = e^{x}$
* $\cosh x - \sinh x = e^{-x}$

Ex: 1) $(\cosh x + \sinh x)^{10} = (e^{x})^{10} = e^{10x}$

2) $(\cosh(\ln 2) + \sinh(\ln 2))^{4} = (e^{\ln 2})^{4} = 2^{4} = 16$

Ex: if $\cosh x = \frac{5}{4}$, $x > 0$; Find :

1) $\sinh x$

$\cosh^2 x - \sinh^2 x = 1$

$\frac{25}{16} - \sinh^2 x = 1$

$\sinh x = +\frac{3}{4}$

Ex: $\tanh x = \frac{-3}{5}$; Find 1) $\sinh x$:

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\operatorname{sech} x = \frac{4}{5}$

$\boxed{\cosh x = \frac{5}{4}}$

$\frac{25}{16} - \sinh^2 x = 1$

$\sinh x = \frac{-3}{4}$

OR

$\coth^2 x - 1 = \operatorname{csch}^2 x$

$\frac{25}{9} - 1 = \operatorname{csch}^2 x$

$\frac{4}{3} = \operatorname{csch}^2 x$

$\operatorname{csch} x = \frac{-4}{3}$

$\sinh x = \frac{-3}{4}$

2) $\cosh x$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\frac{-3}{5} = \frac{-3}{4\cosh x}$

$\cosh x = \frac{5}{4}$

f	f'
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \cdot \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \cdot \coth x$

Ex: Find y' ??

1) $y = \ln^{\cosh x} + e^{\cosh 3x}$

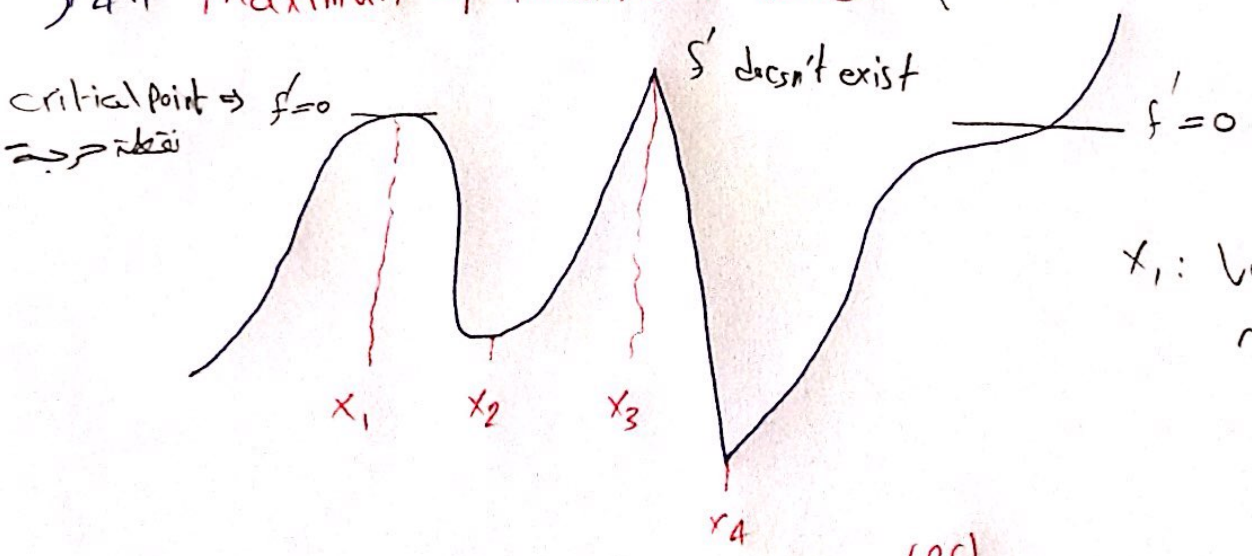
$y' = \tanh x + 3 \sinh 3x \cdot e^{\cosh 3x}$

2) $y = \sin(\cosh x^2)$

$y' = \cos(\cosh x^2) \cdot (\sinh x^2) \cdot 2x$

Chapter 4: Applications of differentiation

§4.1 Maximum & Minimum Values (extreme values قيم قصوى)



x_1: local or relative Maximum value

* **Def:** $x_0 \in D_f$ at x_0 ; f has a critical number (point) if $f'(x_0) = 0$ or $f'(x_0)$ doesn't exist.

* **Def:** 1) f is increasing on the interval I if $f' > 0$ on I

2) " " decreasing " " " " " $f' < 0$ " "

$I \equiv$ open interval

$\Rightarrow$ To classify the criticals for local Maximum and local minimum use the 1^{st} derivative test.

* 1^{st} derivative test

1) If f' changes from +ve to -ve around x_1 $\Rightarrow$ $f(x_1)$ is a local (relative) Maximum value
(inc → dec)

$+ \frown -$ at x_1

2) If f' changes from -ve to +ve $\Rightarrow$ $f(x)$ is a local minimum value.

$- \smile +$ at x

$f'' \ominus$

$f'' \oplus$

* 2^{nd} derivative test.

To classify the critical for local maximum and minimum.

$$f'(x_1)=0$$

1) if $f''(x_1)>0 \Longrightarrow f(x_1)$ is a local minimum value

2) if $f''(x_1)<0 \Longrightarrow f(x_1)$ is a local maximum value

3) $f''(x_1)=0 \Longrightarrow$ test fails

Ex: $f(x)=x^2$ Find inc. dec. intervals & criticals

$f'(x)=2x$ $\boxed{x=0}$ critical point

f' $2x$ $\quad - \quad \underset{0}{|} \quad +$

$\therefore$ f is inc. on $(0,\infty)$

f is dec. on $(-\infty,0)$ $\Big\} \Longrightarrow f(0)=0$ is a local minimum value