1 □ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

1. The functions $f(x) = x + \sqrt{2-x}$ and $g(u) = u + \sqrt{2-u}$ give exactly the same output values for every input value, so f and g are equal.

2. $f(x) = \dfrac{x^2 - x}{x-1} = \dfrac{x(x-1)}{x-1} = x$ for $x - 1 \neq 0$, so f and g [where $g(x) = x$] are not equal because $f(1)$ is undefined and $g(1) = 1$.

3. (a) The point $(1, 3)$ is on the graph of f, so $f(1) = 3$.

(b) When $x = -1$, y is about -0.2, so $f(-1) \approx -0.2$.

(c) $f(x) = 1$ is equivalent to $y = 1$. When $y = 1$, we have $x = 0$ and $x = 3$.

(d) A reasonable estimate for x when $y = 0$ is $x = -0.8$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-2 \leq x \leq 4$, or $[-2, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-1 \leq y \leq 3$, or $[-1, 3]$.

(f) As x increases from -2 to 1, y increases from -1 to 3. Thus, f is increasing on the interval $[-2, 1]$.

4. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

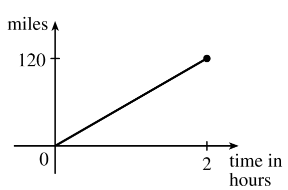
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \leq x \leq 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.

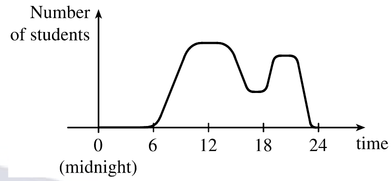
(f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

5. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. Written in interval notation, we get $[-85, 115]$.

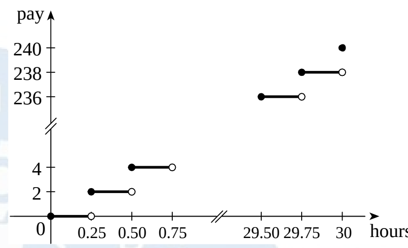
6. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



7. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

8. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

9. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

10. No, the curve is not the graph of a function since for $x = 0$, ± 1, and ± 2, there are infinitely many points on the curve.

11. (a) When $t = 1950$, $T \approx 13.8^\circ\text{C}$, so the global average temperature in 1950 was about 13.8°C.

(b) When $T = 14.2^\circ\text{C}$, $t \approx 1990$.

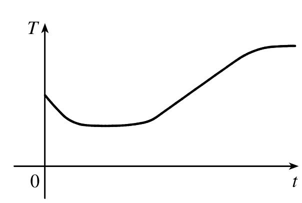
(c) The global average temperature was smallest in 1910 (the year corresponding to the lowest point on the graph) and largest in 2005 (the year corresponding to the highest point on the graph).

(d) When $t = 1910$, $T \approx 13.5^\circ\text{C}$, and when $t = 2005$, $T \approx 14.5^\circ\text{C}$. Thus, the range of T is about $[13.5, 14.5]$.

12. (a) The ring width varies from near 0 mm to about 1.6 mm, so the range of the ring width function is approximately $[0, 1.6]$.

(b) According to the graph, the earth gradually cooled from 1550 to 1700, warmed into the late 1700s, cooled again into the late 1800s, and has been steadily warming since then. In the mid-19th century, there was variation that could have been associated with volcanic eruptions.

13. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

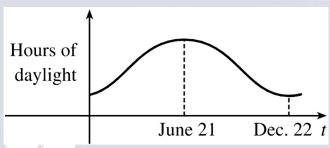


14. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

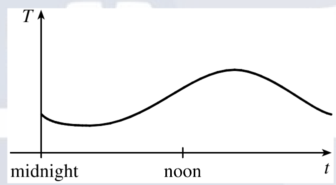
15. (a) The power consumption at 6 AM is $500\ \text{MW}$, which is obtained by reading the value of power P when $t = 6$ from the graph. At 6 PM we read the value of P when $t = 18$, obtaining approximately $730\ \text{MW}$.

(b) The minimum power consumption is determined by finding the time for the lowest point on the graph, $t = 4$, or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before $t = 12$, or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

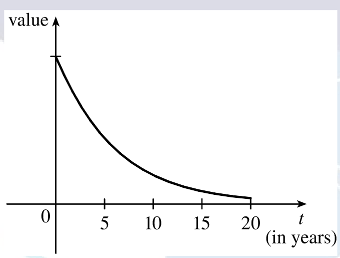
16. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



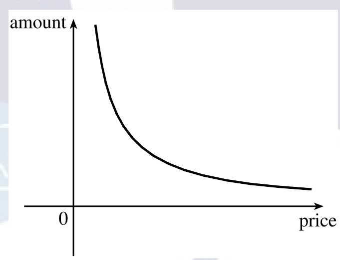
17. Of course, this graph depends strongly on the geographical location!



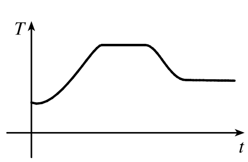
18. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



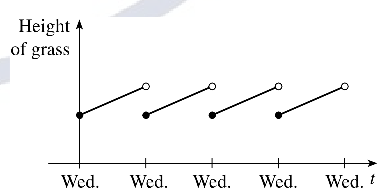
19. As the price increases, the amount sold decreases.



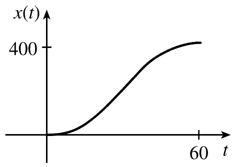
20. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



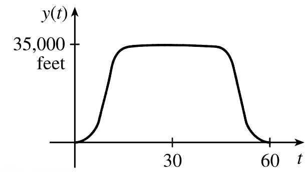
21.



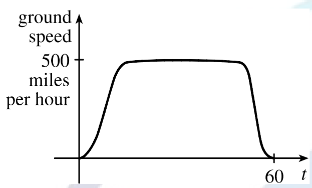
22. (a)



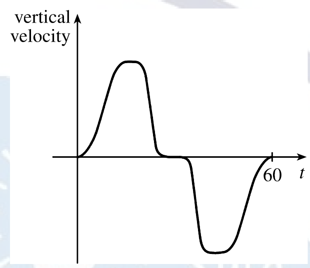
(b)



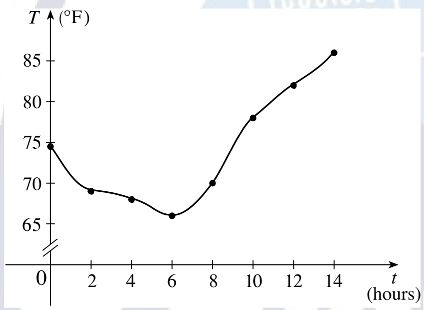
(c)



(d)

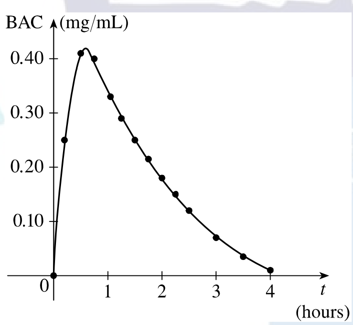


23. (a)



(b) 9:00 AM corresponds to $t = 9$. When $t = 9$, the temperature T is about $74°\mathrm{F}$.

24. (a)



(b) The blood alcohol concentration rises rapidly, then slowly decreases to near zero. Note that the BAC in this exercise is measured in $\mathrm{mg/mL}$, not percent.

25. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

26. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1)$.

27. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$,

and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h.$

28. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$,

and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2.$

29. $$\frac{f(x) - f(a)}{x - a} = \frac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \frac{\dfrac{a - x}{xa}}{x - a} = \frac{a - x}{xa(x - a)} = \frac{-1(x - a)}{xa(x - a)} = -\frac{1}{ax}$$

30. $$\frac{f(x) - f(1)}{x - 1} = \frac{\dfrac{x+3}{x+1} - 2}{x - 1} = \frac{\dfrac{x + 3 - 2(x+1)}{x+1}}{x - 1} = \frac{x + 3 - 2x - 2}{(x+1)(x-1)}$$
$$= \frac{-x + 1}{(x+1)(x-1)} = \frac{-(x-1)}{(x+1)(x-1)} = -\frac{1}{x+1}$$

31. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \quad \Leftrightarrow \quad 0 = (x+3)(x-3) \quad \Leftrightarrow \quad x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

32. $f(x) = (2x^3 - 5)/(x^2 + x - 6)$ is defined for all x except when $0 = x^2 + x - 6 \quad \Leftrightarrow \quad 0 = (x+3)(x-2) \quad \Leftrightarrow$ $x = -3$ or 2, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 2\} = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

33. $f(t) = \sqrt[3]{2t - 1}$ is defined for all real numbers. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty, \infty)$.

34. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$ is defined when $3 - t \geq 0 \quad \Leftrightarrow \quad t \leq 3$ and $2 + t \geq 0 \quad \Leftrightarrow \quad t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2, 3]$.

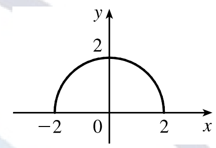
35. $h(x) = 1 \big/ \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x - 5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

36. $f(u) = \dfrac{u+1}{1+\dfrac{1}{u+1}}$ is defined when $u+1 \neq 0$ $[u \neq -1]$ and $1 + \dfrac{1}{u+1} \neq 0$. Since $1 + \dfrac{1}{u+1} = 0 \quad \Leftrightarrow$

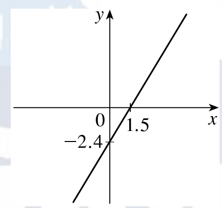
$\dfrac{1}{u+1} = -1 \quad \Leftrightarrow \quad 1 = -u-1 \quad \Leftrightarrow \quad u = -2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

37. $F(p) = \sqrt{2-\sqrt{p}}$ is defined when $p \geq 0$ and $2 - \sqrt{p} \geq 0$. Since $2 - \sqrt{p} \geq 0 \quad \Leftrightarrow \quad 2 \geq \sqrt{p} \quad \Leftrightarrow \quad \sqrt{p} \leq 2 \quad \Leftrightarrow$ $0 \leq p \leq 4$, the domain is $[0, 4]$.

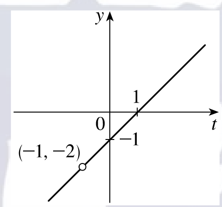
38. $h(x) = \sqrt{4-x^2}$. Now $y = \sqrt{4-x^2} \quad \Rightarrow \quad y^2 = 4 - x^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



39. The domain of $f(x) = 1.6x - 2.4$ is the set of all real numbers, denoted by $\mathbb{R}$ or $(-\infty, \infty)$. The graph of f is a line with slope 1.6 and y-intercept -2.4.

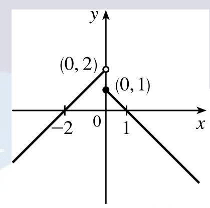


40. Note that $g(t) = \dfrac{t^2-1}{t+1} = \dfrac{(t+1)(t-1)}{t+1} = t - 1$ for $t + 1 \neq 0$, i.e., $t \neq -1$. The domain of g is the set of all real numbers except -1. In interval notation, we have $(-\infty, -1) \cup (-1, \infty)$. The graph of g is a line with slope 1, y-intercept -1, and a hole at $t = -1$.



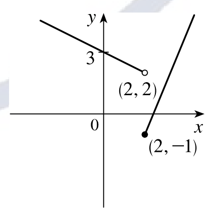
41. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

$f(-3) = -3 + 2 = -1$, $f(0) = 1 - 0 = 1$, and $f(2) = 1 - 2 = -1$.



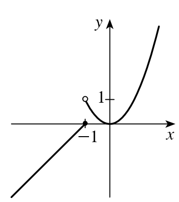
42. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x-5 & \text{if } x \geq 2 \end{cases}$

$f(-3) = 3 - \frac{1}{2}(-3) = \frac{9}{2}$, $f(0) = 3 - \frac{1}{2}(0) = 3$, and $f(2) = 2(2) - 5 = -1$.



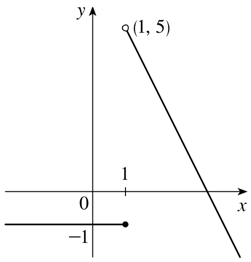
43. $f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

$f(-3) = -3 + 1 = -2$, $f(0) = 0^2 = 0$, and $f(2) = 2^2 = 4$.



44. $f(x) = \begin{cases} -1 & \text{if } x \le 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

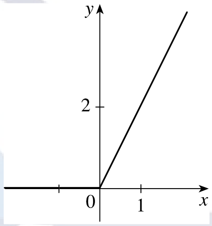
$f(-3) = -1$, $f(0) = -1$, and $f(2) = 7 - 2(2) = 3$.



45. $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$

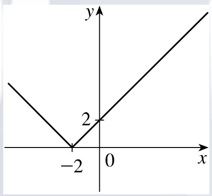
so $f(x) = x + |x| = \begin{cases} 2x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$

Graph the line $y = 2x$ for $x \ge 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



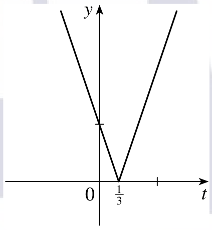
46. $f(x) = |x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \ge 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases}$

$$= \begin{cases} x + 2 & \text{if } x \ge -2 \\ -x - 2 & \text{if } x < -2 \end{cases}$$



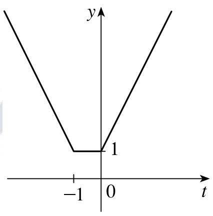
47. $g(t) = |1 - 3t| = \begin{cases} 1 - 3t & \text{if } 1 - 3t \ge 0 \\ -(1 - 3t) & \text{if } 1 - 3t < 0 \end{cases}$

$$= \begin{cases} 1 - 3t & \text{if } t \le \frac{1}{3} \\ 3t - 1 & \text{if } t > \frac{1}{3} \end{cases}$$



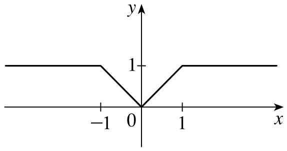
48. $|t| = \begin{cases} t & \text{if } t \ge 0 \\ -t & \text{if } t < 0 \end{cases}$ and

$|t + 1| = \begin{cases} t + 1 & \text{if } t + 1 \ge 0 \\ -(t + 1) & \text{if } t + 1 < 0 \end{cases} = \begin{cases} t + 1 & \text{if } t \ge -1 \\ -t - 1 & \text{if } t < -1 \end{cases}$



so $h(t) = |t| + |t + 1| = \begin{cases} t + (t + 1) & \text{if } t \ge 0 \\ -t + (t + 1) & \text{if } -1 \le t < 0 \\ -t + (-t - 1) & \text{if } t < -1 \end{cases} = \begin{cases} 2t + 1 & \text{if } t \ge 0 \\ 1 & \text{if } -1 \le t < 0 \\ -2t - 1 & \text{if } t < -1 \end{cases}$

49. To graph $f(x) = \begin{cases} |x| & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$, graph $y = |x|$ (Figure 16) for $-1 \le x \le 1$ and graph $y = 1$ for $x > 1$ and for $x < -1$.

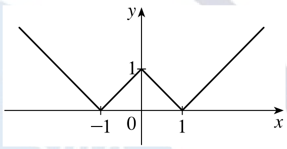


We could rewrite f as $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$.

50. $g(x) = \big||x| - 1\big| = \begin{cases} |x| - 1 & \text{if } |x| - 1 \ge 0 \\ -(|x| - 1) & \text{if } |x| - 1 < 0 \end{cases}$

$$= \begin{cases} |x| - 1 & \text{if } |x| \ge 1 \\ -|x| + 1 & \text{if } |x| < 1 \end{cases}$$

$$= \begin{cases} x - 1 & \text{if } |x| \ge 1 \text{ and } x \ge 0 \\ -x - 1 & \text{if } |x| \ge 1 \text{ and } x < 0 \\ -x + 1 & \text{if } |x| < 1 \text{ and } x \ge 0 \\ -(-x) + 1 & \text{if } |x| < 1 \text{ and } x < 0 \end{cases} = \begin{cases} x - 1 & \text{if } x \ge 1 \\ -x - 1 & \text{if } x \le -1 \\ -x + 1 & \text{if } 0 \le x < 1 \\ x + 1 & \text{if } -1 < x < 0 \end{cases}$$



51. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of the line segment joining the points $(1, -3)$ and $(5, 7)$ is $\dfrac{7 - (-3)}{5 - 1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x - 1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \le x \le 5$.

52. The slope of the line segment joining the points $(-5, 10)$ and $(7, -10)$ is $\dfrac{-10 - 10}{7 - (-5)} = -\dfrac{5}{3}$, so an equation is $y - 10 = -\frac{5}{3}[x - (-5)]$. The function is $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \le x \le 7$.

53. We need to solve the given equation for y. $x + (y - 1)^2 = 0 \Leftrightarrow (y - 1)^2 = -x \Leftrightarrow y - 1 = \pm\sqrt{-x} \Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

54. $x^2 + (y - 2)^2 = 4 \Leftrightarrow (y - 2)^2 = 4 - x^2 \Leftrightarrow y - 2 = \pm\sqrt{4 - x^2} \Leftrightarrow y = 2 \pm \sqrt{4 - x^2}$. The top half is given by the function $f(x) = 2 + \sqrt{4 - x^2}$, $-2 \le x \le 2$.

55. For $0 \le x \le 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \le 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x - 3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x + 3 & \text{if } 0 \le x \le 3 \\ 2x - 6 & \text{if } 3 < x \le 5 \end{cases}$$

56. For $-4 \le x \le -2$, the graph is the line with slope $-\frac{3}{2}$ passing through $(-2, 0)$; that is, $y - 0 = -\frac{3}{2}[x - (-2)]$, or $y = -\frac{3}{2}x - 3$. For $-2 < x < 2$, the graph is the top half of the circle with center $(0, 0)$ and radius 2. An equation of the circle is $x^2 + y^2 = 4$, so an equation of the top half is $y = \sqrt{4 - x^2}$. For $2 \le x \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through $(2, 0)$; that is, $y - 0 = \frac{3}{2}(x - 2)$, or $y = \frac{3}{2}x - 3$. So the function is

$$f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4 - x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } \quad 2 \le x \le 4 \end{cases}$$

57. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

58. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

59. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

60. Let the length, width, and height of the closed rectangular box be denoted by L, W, and H, respectively. The length is twice the width, so $L = 2W$. The volume V of the box is given by $V = LWH$. Since $V = 8$, we have $8 = (2W)WH \quad \Rightarrow$ $8 = 2W^2H \quad \Rightarrow \quad H = \dfrac{8}{2W^2} = \dfrac{4}{W^2}$, and so $H = f(W) = \dfrac{4}{W^2}$.

61. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

62. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad \Leftrightarrow \quad 2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

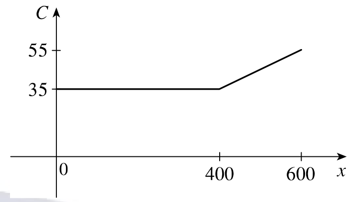
$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2 = 15x - \tfrac{4}{8}x^2 - \tfrac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right).$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad \Leftrightarrow$ $30 - x - \frac{1}{2}\pi x > 0 \quad \Leftrightarrow \quad 60 > 2x + \pi x \quad \Leftrightarrow \quad x < \dfrac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2 + \pi}$.

63. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$; $W > 0 \quad \Leftrightarrow \quad 12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

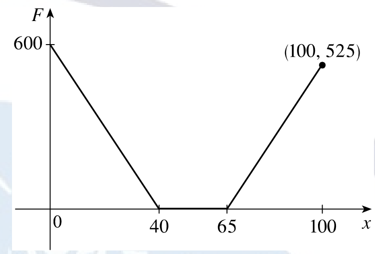
64. We can summarize the monthly cost with a piecewise defined function.

$$C(x) = \begin{cases} 35 & \text{if } 0 \le x \le 400 \\ 35 + 0.10(x - 400) & \text{if } x > 400 \end{cases}$$



65. We can summarize the amount of the fine with a piecewise defined function.

$$F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \le x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$$



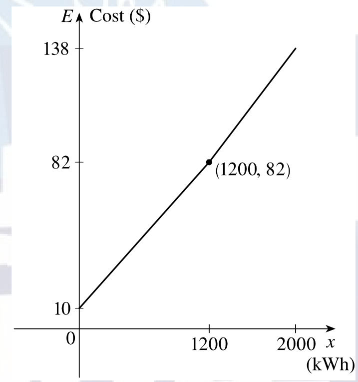
66. For the first 1200 kWh, $E(x) = 10 + 0.06x$.

For usage over 1200 kWh, the cost is

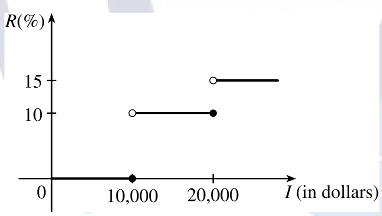
$E(x) = 10 + 0.06(1200) + 0.07(x - 1200) = 82 + 0.07(x - 1200)$.

Thus,

$$E(x) = \begin{cases} 10 + 0.06x & \text{if } 0 \le x \le 1200 \\ 82 + 0.07(x - 1200) & \text{if } x > 1200 \end{cases}$$



67. (a)

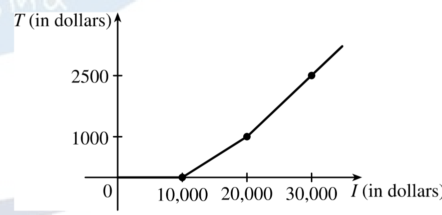


(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.

On \$26,000, tax is assessed on \$16,000, and

$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



68. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

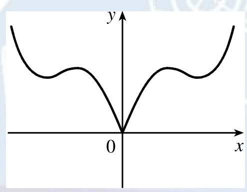
69. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

70. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

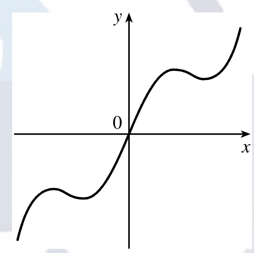
71. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

72. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



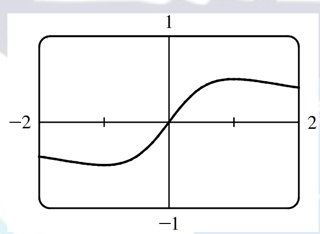
(b) If f is odd, we get the rest of the graph by rotating $180°$ about the origin.



73. $f(x) = \dfrac{x}{x^2+1}$.

$f(-x) = \dfrac{-x}{(-x)^2+1} = \dfrac{-x}{x^2+1} = -\dfrac{x}{x^2+1} = -f(x)$.

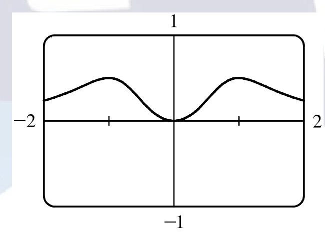
Since $f(-x) = -f(x)$, f is an odd function.



74. $f(x) = \dfrac{x^2}{x^4+1}$.

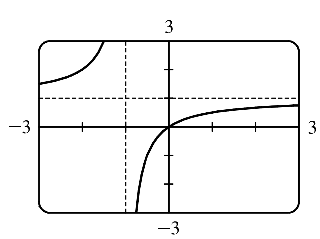
$f(-x) = \dfrac{(-x)^2}{(-x)^4+1} = \dfrac{x^2}{x^4+1} = f(x)$.

Since $f(-x) = f(x)$, f is an even function.



75. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

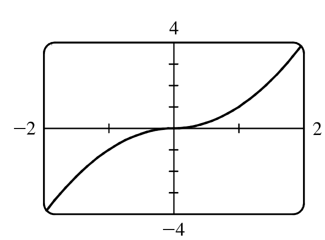
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



76. $f(x) = x\,|x|$.

$$f(-x) = (-x)\,|-x| = (-x)\,|x| = -(x\,|x|) = -f(x)$$

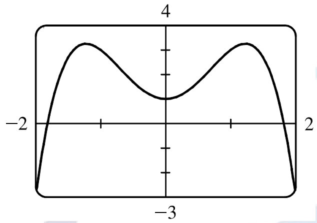
Since $f(-x) = -f(x)$, f is an odd function.



77. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

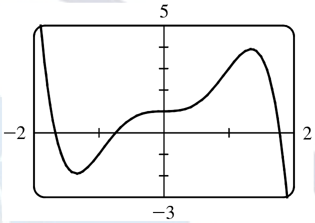
Since $f(-x) = f(x)$, f is an even function.



78. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5)$$
$$= 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



79. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f+g$ will be *odd*. If g is the zero function, then $f+g$ will be *even*.)

80. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \log_2 x$ is a logarithmic function.

(b) $g(x) = \sqrt[4]{x}$ is a root function with $n = 4$.

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ is a rational function because it is a ratio of polynomials.

(d) $u(t) = 1 - 1.1t + 2.54t^2$ is a polynomial of degree 2 (also called a *quadratic function*).

(e) $v(t) = 5^t$ is an exponential function.

(f) $w(\theta) = \sin\theta\,\cos^2\theta$ is a trigonometric function.

2. (a) $y = \pi^x$ is an exponential function (notice that x is the *exponent*).

(b) $y = x^\pi$ is a power function (notice that x is the *base*).

(c) $y = x^2(2 - x^3) = 2x^2 - x^5$ is a polynomial of degree 5.

(d) $y = \tan t - \cos t$ is a trigonometric function.

(e) $y = s/(1 + s)$ is a rational function because it is a ratio of polynomials.

(f) $y = \sqrt{x^3 - 1}/(1 + \sqrt[3]{x})$ is an algebraic function because it involves polynomials and roots of polynomials.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $[y = x^5]$ must be f. Since g is flatter than h near the origin, we must have (c) $[y = x^8]$ matched with g and (a) $[y = x^2]$ matched with h.

4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

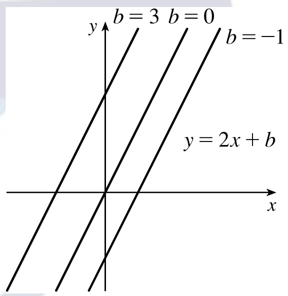
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

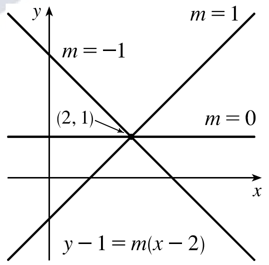
5. The denominator cannot equal 0, so $1 - \sin x \neq 0 \quad \Leftrightarrow \quad \sin x \neq 1 \quad \Leftrightarrow \quad x \neq \frac{\pi}{2} + 2n\pi$. Thus, the domain of $f(x) = \dfrac{\cos x}{1 - \sin x}$ is $\{x \mid x \neq \frac{\pi}{2} + 2n\pi,\, n \text{ an integer}\}$.

6. The denominator cannot equal 0, so $1 - \tan x \neq 0 \quad \Leftrightarrow \quad \tan x \neq 1 \quad \Leftrightarrow \quad x \neq \frac{\pi}{4} + n\pi$. The tangent function is not defined if $x \neq \frac{\pi}{2} + n\pi$. Thus, the domain of $g(x) = \dfrac{1}{1 - \tan x}$ is $\{x \mid x \neq \frac{\pi}{4} + n\pi,\, x \neq \frac{\pi}{2} + n\pi,\, n \text{ an integer}\}$.

7. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

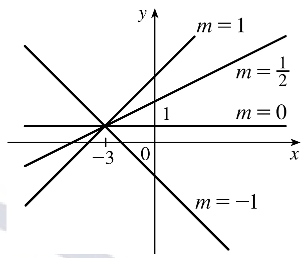


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $\quad y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

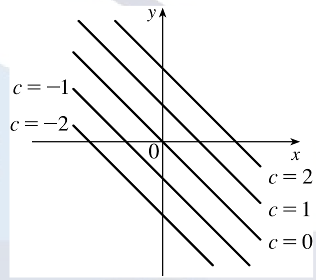


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

8. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



9. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



10. The vertex of the parabola on the left is $(3, 0)$, so an equation is $y = a(x - 3)^2 + 0$. Since the point $(4, 2)$ is on the parabola, we'll substitute 4 for x and 2 for y to find a. $2 = a(4 - 3)^2 \quad \Rightarrow \quad a = 2$, so an equation is $f(x) = 2(x - 3)^2$.

The y-intercept of the parabola on the right is $(0, 1)$, so an equation is $y = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ are on the parabola, we'll substitute -2 for x and 2 for y as well as 1 for x and -2.5 for y to obtain two equations with the unknowns a and b.

$(-2, 2)$: $\quad 2 = 4a - 2b + 1 \quad \Rightarrow \quad 4a - 2b = 1 \quad$ **(1)**

$(1, -2.5)$: $\quad -2.5 = a + b + 1 \quad \Rightarrow \quad a + b = -3.5 \quad$ **(2)**

$2 \cdot$ **(2)** $+$ **(1)** gives us $6a = -6 \quad \Rightarrow \quad a = -1$. From **(2)**, $-1 + b = -3.5 \quad \Rightarrow \quad b = -2.5$, so an equation is $g(x) = -x^2 - 2.5x + 1$.

11. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x - (-1)](x - 0)(x - 2)$, or $f(x) = ax(x + 1)(x - 2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.
$6 = a(1)(2)(-1) \quad \Rightarrow \quad -2a = 6 \quad \Rightarrow \quad a = -3$, so an equation for f is $f(x) = -3x(x + 1)(x - 2)$.

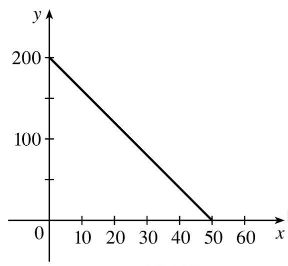
12. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\,^\circ\mathrm{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\mathrm{C}$ in the year 1900.

(b) $t = 2100 - 1900 = 200 \quad \Rightarrow \quad T = 0.02(200) + 8.50 = 12.50\,^\circ\mathrm{C}$

13. (a) $D = 200$, so $c = 0.0417D(a + 1) = 0.0417(200)(a + 1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

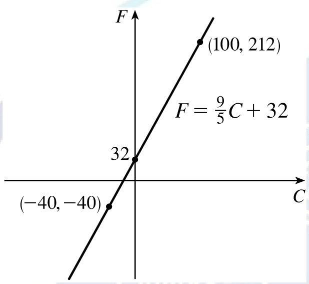
(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

14. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

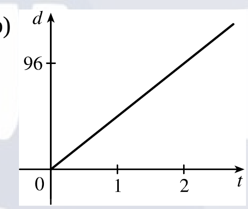
15. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1°C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

16. (a) Let $d =$ distance traveled (in miles) and $t =$ time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40 - 0}{\frac{5}{6} - 0} = 48$ and so $d = 48t$.

(b)



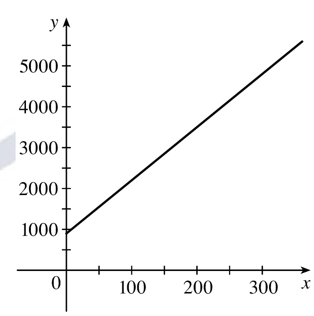
(c) The slope is 48 and represents the car's speed in mi/h.

17. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \quad \Leftrightarrow \quad T - 80 = \frac{1}{6}N - \frac{173}{6} \quad \Leftrightarrow \quad T = \frac{1}{6}N + \frac{307}{6} \quad \left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\text{F} \approx 76\,^\circ\text{F}$.

18. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \quad \Leftrightarrow$ $y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

19. (a) We are given $\dfrac{\text{change in pressure}}{10 \text{ feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \quad \Leftrightarrow \quad 0.434d = 85 \quad \Leftrightarrow \quad d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

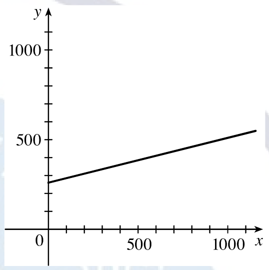
20. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$.

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \quad \Leftrightarrow \quad C - 460 = \frac{1}{4}d - 200 \quad \Leftrightarrow \quad C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

21. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

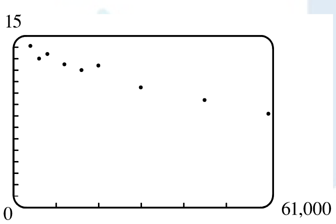
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

22. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Exercises 23–28: Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

23. (a)

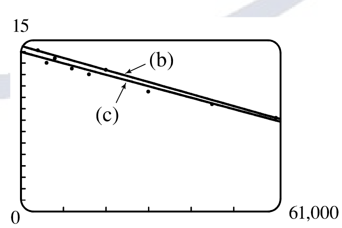


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

$y - 14.1 = \dfrac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently,

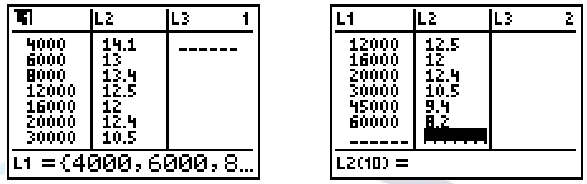
$y \approx -0.000105357x + 14.521429$.



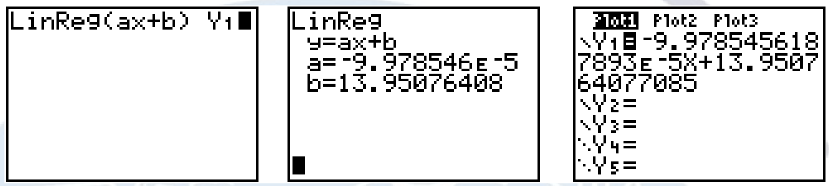
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

The following commands and screens illustrate how to find the least squares regression line on a TI-84 Plus.

Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

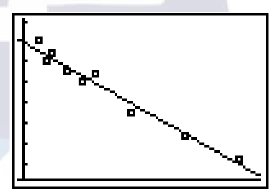


Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▶ 4 VARS ▶ 1 1 ENTER .



Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .

Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

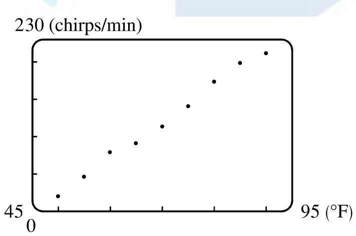


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

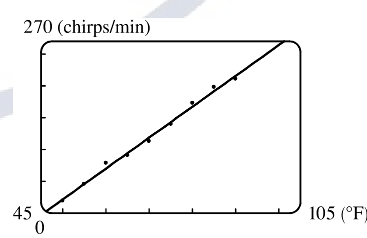
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

24. (a)



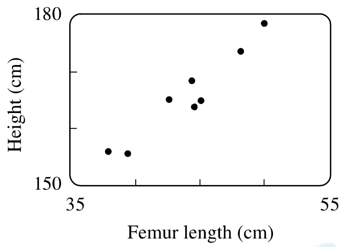
(b)



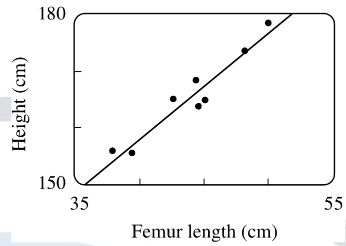
Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

(c) When $x = 100^\circ$F, $y = 264.7 \approx 265$ chirps/min.

25. (a)



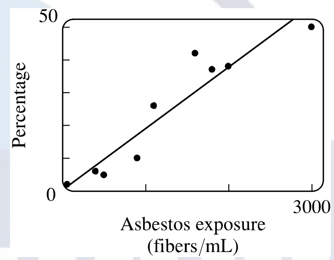
(b) Using a computing device, we obtain the regression line $y = 1.88074x + 82.64974$.



(c) When $x = 53$ cm, $y \approx 182.3$ cm.

26. (a) Using a computing device, we obtain the regression line $y = 0.01879x + 0.30480$.

(b) The regression line appears to be a suitable model for the data.

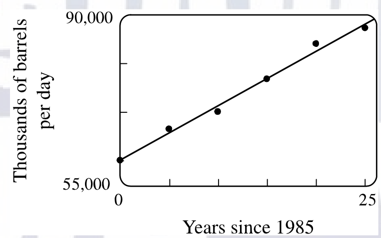


(c) The y-intercept represents the percentage of laboratory rats that develop lung tumors when *not* exposed to asbestos fibers.

27. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line $y = 1116.64x + 60{,}188.33$.

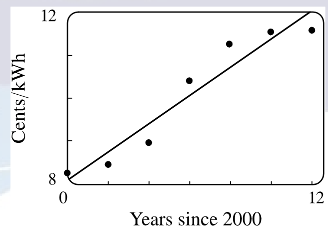
(c) For 2002, $x = 17$ and $y \approx 79{,}171$ thousands of barrels per day. For 2012, $x = 27$ and $y \approx 90{,}338$ thousands of barrels per day.



28. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line $y = 0.33089x + 8.07321$.

(c) For 2005, $x = 5$ and $y \approx 9.73$ cents/kWh. For 2013, $x = 13$ and $y \approx 12.37$ cents/kWh.



29. If x is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives us an illumination of $f\left(\frac{1}{2}x\right) = k\left(\frac{1}{2}x\right)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is 4 times as bright.

30. (a) If $A = 60$, then $S = 0.7A^{0.3} \approx 2.39$, so you would expect to find 2 species of bats in that cave.

(b) $S = 4 \;\Rightarrow\; 4 = 0.7A^{0.3} \;\Rightarrow\; \frac{40}{7} = A^{3/10} \;\Rightarrow\; A = \left(\frac{40}{7}\right)^{10/3} \approx 333.6$, so we estimate the surface area of the cave to be 334 m^2.

31. (a) Using a computing device, we obtain a power function $N = cA^b$, where $c \approx 3.1046$ and $b \approx 0.308$.

(b) If $A = 291$, then $N = cA^b \approx 17.8$, so you would expect to find 18 species of reptiles and amphibians on Dominica.

32. (a) $T = 1.000\,431\,227d^{\,1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

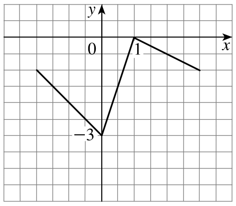
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

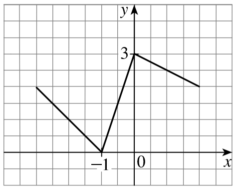
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

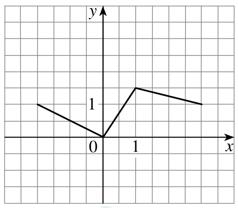
4. (a) $y = f(x) - 3$: Shift the graph of f 3 units down.



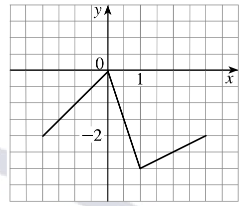
(b) $y = f(x + 1)$: Shift the graph of f 1 unit to the left.



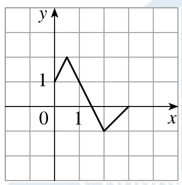
(c) $y = \frac{1}{2}f(x)$: Shrink the graph of f vertically by a factor of 2.



(d) $y = -f(x)$: Reflect the graph of f about the x-axis.

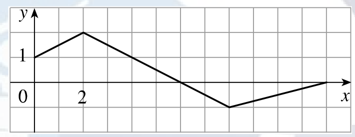


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



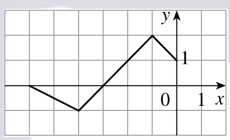
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



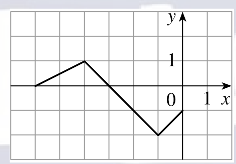
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

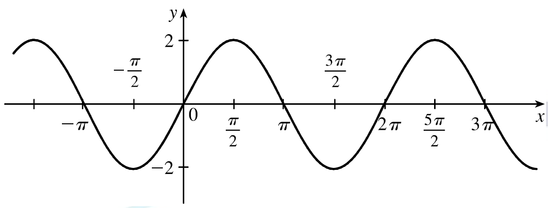
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f\ \underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

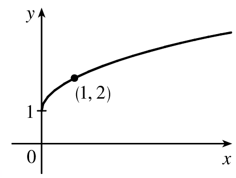
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 \\ &= -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

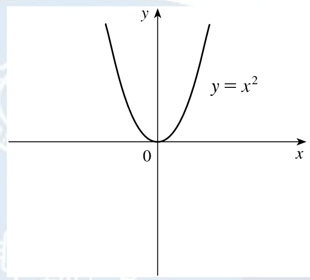
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.

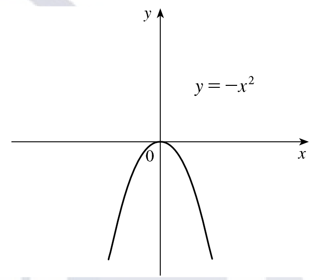


(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.

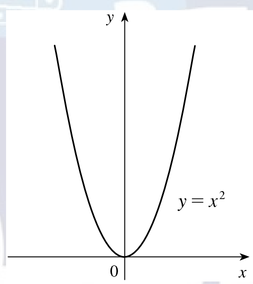


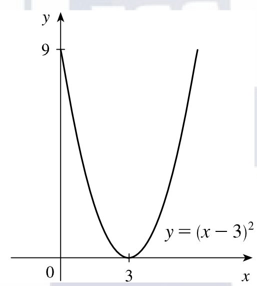
9. $y = -x^2$: Start with the graph of $y = x^2$ and reflect about the x-axis.



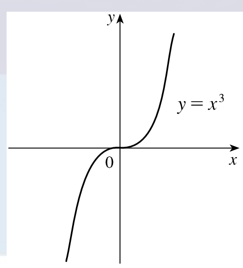


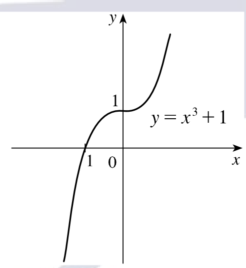
10. $y = (x-3)^2$: Start with the graph of $y = x^2$ and shift 3 units to the right.



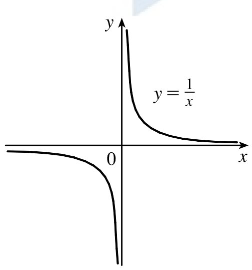


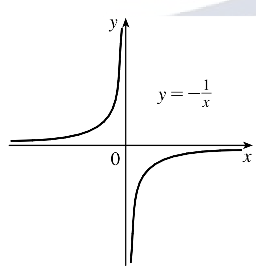
11. $y = x^3 + 1$: Start with the graph of $y = x^3$ and shift upward 1 unit.

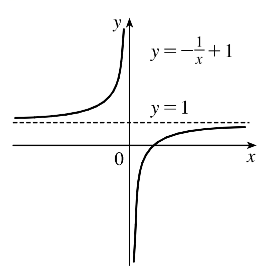




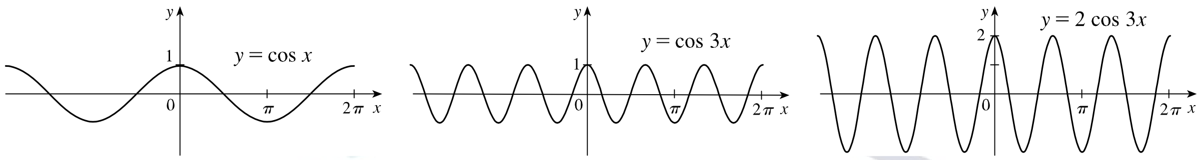
12. $y = 1 - \dfrac{1}{x} = -\dfrac{1}{x} + 1$: Start with the graph of $y = \dfrac{1}{x}$, reflect about the x-axis, and shift upward 1 unit.



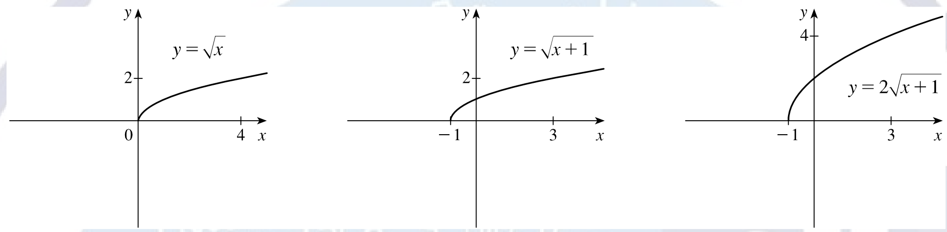




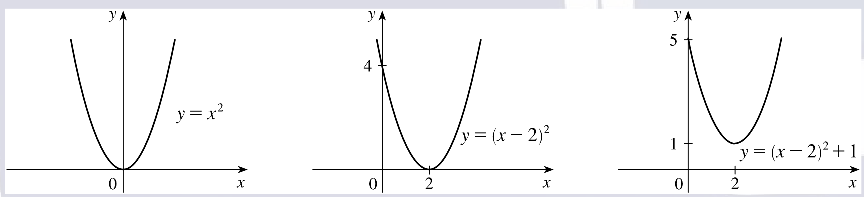
13. $y = 2\cos 3x$: Start with the graph of $y = \cos x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 2.



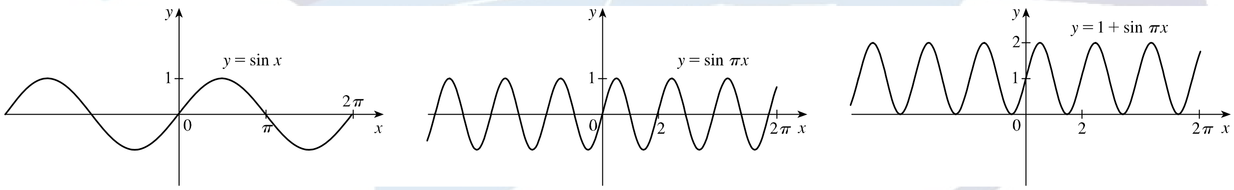
14. $y = 2\sqrt{x+1}$: Start with the graph of $y = \sqrt{x}$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



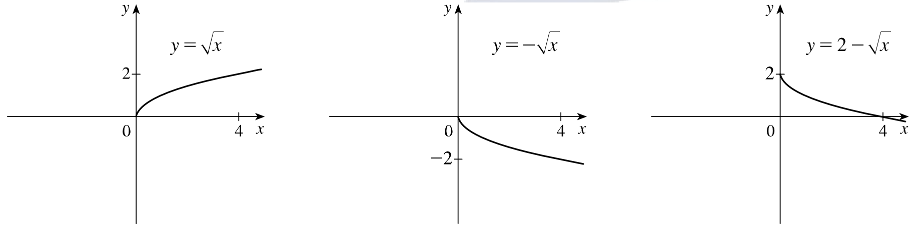
15. $y = x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 = (x-2)^2 + 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift upward 1 unit.



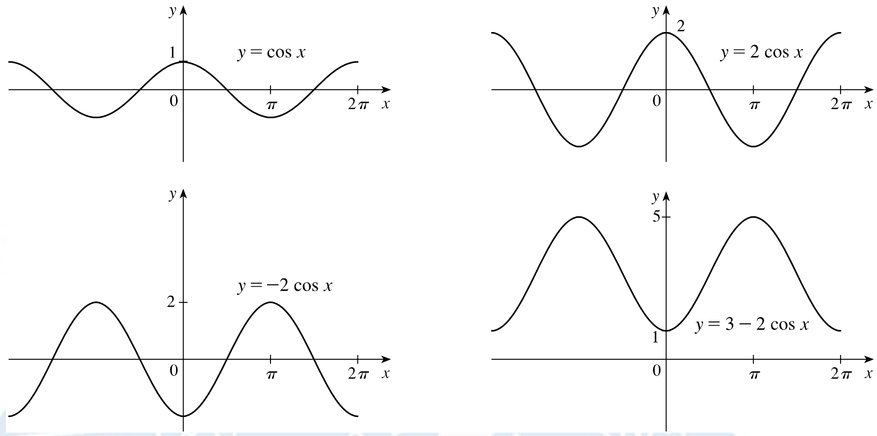
16. $y = 1 + \sin \pi x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of π, and then shift upward 1 unit.



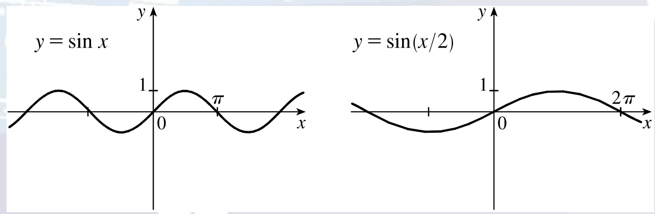
17. $y = 2 - \sqrt{x}$: Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and then shift 2 units upward.



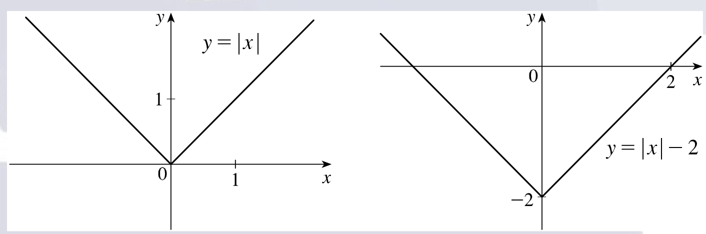
18. $y = 3 - 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, reflect about the x-axis, and then shift 3 units upward.



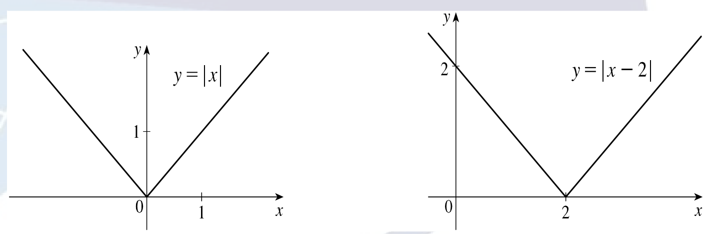
19. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



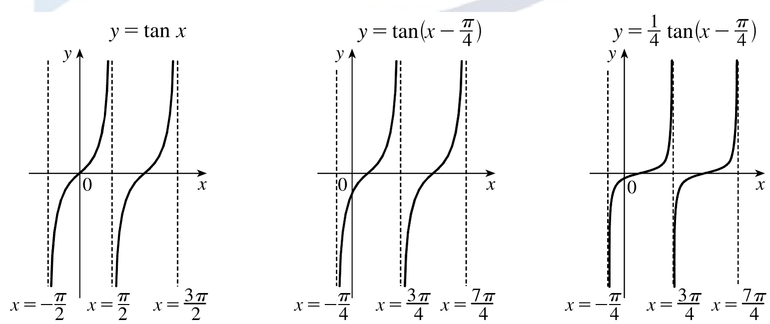
20. $y = |x| - 2$: Start with the graph of $y = |x|$ and shift 2 units downward.



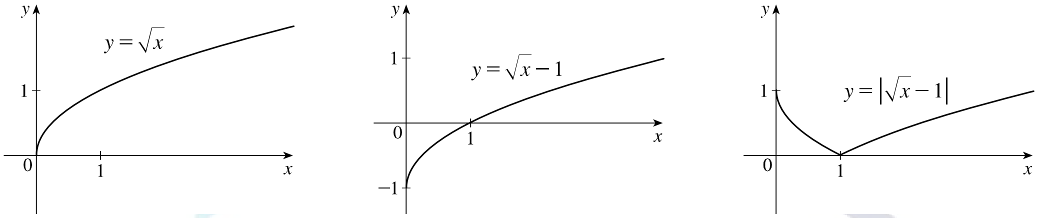
21. $y = |x - 2|$: Start with the graph of $y = |x|$ and shift 2 units to the right.



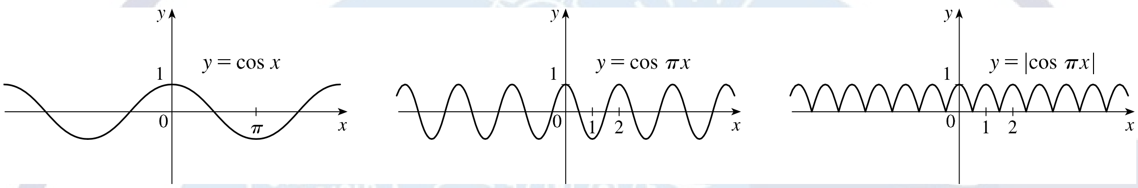
22. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sqrt{x} - 1|$: Start with the graph of $y = \sqrt{x}$, shift it 1 unit downward, and then reflect the portion of the graph below the x-axis about the x-axis.



24. $y = |\cos \pi x|$: Start with the graph of $y = \cos x$, shrink it horizontally by a factor of π, and reflect all the parts of the graph below the x-axis about the x-axis.



25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

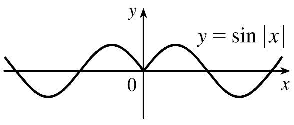
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. The water depth $D(t)$ can be modeled by a cosine function with amplitude $\dfrac{12 - 2}{2} = 5$ m, average magnitude $\dfrac{12 + 2}{2} = 7$ m, and period 12 hours. High tide occurred at time 6:45 AM ($t = 6.75$ h), so the curve begins a cycle at time $t = 6.75$ h (shift 6.75 units to the right). Thus, $D(t) = 5\cos\left[\frac{2\pi}{12}(t - 6.75)\right] + 7 = 5\cos\left[\frac{\pi}{6}(t - 6.75)\right] + 7$, where D is in meters and t is the number of hours after midnight.

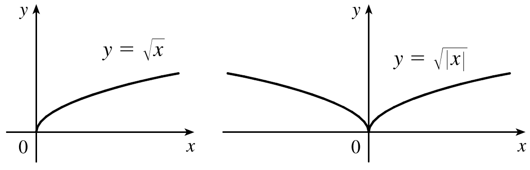
28. The total volume of air $V(t)$ in the lungs can be modeled by a sine function with amplitude $\dfrac{2500 - 2000}{2} = 250$ mL, average volume $\dfrac{2500 + 2000}{2} = 2250$ mL, and period 4 seconds. Thus, $V(t) = 250\sin\frac{2\pi}{4}t + 2250 = 250\sin\frac{\pi}{2}t + 2250$, where V is in mL and t is in seconds.

29. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

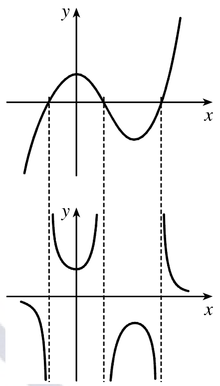
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



30. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

(a) $(f + g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1,\ D = (-\infty, \infty)$, or $\mathbb{R}$.

(b) $(f - g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1,\ D = \mathbb{R}$.

(c) $(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2,\ D = \mathbb{R}$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1},\ D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

32. $f(x) = \sqrt{3 - x},\ D = (-\infty, 3]$; $g(x) = \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, \infty)$.

(a) $(f + g)(x) = \sqrt{3 - x} + \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$, which is the intersection of the domains of f and g.

(b) $(f - g)(x) = \sqrt{3 - x} - \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$.

(c) $(fg)(x) = \sqrt{3 - x} \cdot \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{3 - x}}{\sqrt{x^2 - 1}},\ D = (-\infty, -1) \cup (1, 3]$. We must exclude $x = \pm 1$ since these values would make $\dfrac{f}{g}$ undefined.

33. $f(x) = 3x + 5$; $g(x) = x^2 + x$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 + x) = 3(x^2 + x) + 5 = 3x^2 + 3x + 5,\ D = \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = g(3x + 5) = (3x + 5)^2 + (3x + 5)$

$= 9x^2 + 30x + 25 + 3x + 5 = 9x^2 + 33x + 30,\ D = \mathbb{R}$.

(c) $(f \circ f) = f(f(x)) = f(3x + 5) = 3(3x + 5) + 5 = 9x + 15 + 5 = 9x + 20,\ D = \mathbb{R}$.

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 + x) = (x^2 + x)^2 + (x^2 + x)$

$= x^4 + 2x^3 + x^2 + x^2 + x = x^4 + 2x^3 + 2x^2 + x,\ D = \mathbb{R}$.

34. $f(x) = x^3 - 2$; $g(x) = 1 - 4x$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(1 - 4x) = (1 - 4x)^3 - 2$

$= (1)^3 - 3(1)^2(4x) + 3(1)(4x)^2 - (4x)^3 - 2 = 1 - 12x + 48x^2 - 64x^3 - 2$

$= -1 - 12x + 48x^2 - 64x^3,\ D = \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^3 - 2) = 1 - 4(x^3 - 2) = 1 - 4x^3 + 8 = 9 - 4x^3,\ D = \mathbb{R}$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^3 - 2) = (x^3 - 2)^3 - 2$

$$= (x^3)^3 - 3(x^3)^2(2) + 3(x^3)(2)^2 - (2)^3 - 2 = x^9 - 6x^6 + 12x^3 - 10, \;\; D = \mathbb{R}.$$

(d) $(g \circ g)(x) = g(g(x)) = g(1 - 4x) = 1 - 4(1 - 4x) = 1 - 4 + 16x = -3 + 16x, \;\; D = \mathbb{R}.$

35. $f(x) = \sqrt{x+1}$, $D = \{x \mid x \geq -1\}$; $g(x) = 4x - 3, \;\; D = \mathbb{R}.$

(a) $(f \circ g)(x) = f(g(x)) = f(4x - 3) = \sqrt{(4x-3)+1} = \sqrt{4x-2}$

The domain of $f \circ g$ is $\{x \mid 4x - 3 \geq -1\} = \{x \mid 4x \geq 2\} = \left\{x \mid x \geq \frac{1}{2}\right\} = \left[\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}\,) = 4\sqrt{x+1} - 3$

The domain of $g \circ f$ is $\{x \mid x$ is in the domain of f and $f(x)$ is in the domain of $g\}$. This is the domain of f, that is, $\{x \mid x + 1 \geq 0\} = \{x \mid x \geq -1\} = [-1, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x+1}\,) = \sqrt{\sqrt{x+1}+1}$

For the domain, we need $x + 1 \geq 0$, which is equivalent to $x \geq -1$, and $\sqrt{x+1} \geq -1$, which is true for all real values of x. Thus, the domain of $f \circ f$ is $[-1, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(4x - 3) = 4(4x - 3) - 3 = 16x - 12 - 3 = 16x - 15, \;\; D = \mathbb{R}.$

36. $f(x) = \sin x$; $g(x) = x^2 + 1$. $\quad D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1), \;\; D = \mathbb{R}.$

(b) $(g \circ f) = g(f(x)) = g(\sin x) = (\sin x)^2 + 1 = \sin^2 x + 1, \;\; D = \mathbb{R}.$

(c) $(f \circ f)(x) = f(f(x)) = f(\sin x) = \sin(\sin x), \;\; D = \mathbb{R}.$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2, \;\; D = \mathbb{R}.$

37. $f(x) = x + \dfrac{1}{x}, \;\; D = \{x \mid x \neq 0\}; \quad g(x) = \dfrac{x+1}{x+2}, \;\; D = \{x \mid x \neq -2\}$

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x+1}{x+2}\right) = \dfrac{x+1}{x+2} + \dfrac{1}{\dfrac{x+1}{x+2}} = \dfrac{x+1}{x+2} + \dfrac{x+2}{x+1}$

$$= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{\left(x^2 + 2x + 1\right) + \left(x^2 + 4x + 4\right)}{(x+2)(x+1)} = \frac{2x^2 + 6x + 5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$,
the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(x + \dfrac{1}{x}\right) = \dfrac{\left(x + \dfrac{1}{x}\right) + 1}{\left(x + \dfrac{1}{x}\right) + 2} = \dfrac{\dfrac{x^2 + 1 + x}{x}}{\dfrac{x^2 + 1 + 2x}{x}} = \dfrac{x^2 + x + 1}{x^2 + 2x + 1} = \dfrac{x^2 + x + 1}{(x+1)^2}$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$,
the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{1}{\frac{x^2+1}{x}} = x + \frac{1}{x} + \frac{x}{x^2+1}$

$$= \frac{x(x)(x^2+1) + 1\,(x^2+1) + x(x)}{x(x^2+1)} = \frac{x^4 + x^2 + x^2 + 1 + x^2}{x(x^2+1)}$$

$$= \frac{x^4 + 3x^2 + 1}{x(x^2+1)}, \quad D = \{x \mid x \neq 0\}$$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \frac{x+1+x+2}{x+1+2x+4} = \frac{2x+3}{3x+5}$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$,

the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

38. $f(x) = \frac{x}{1+x}$, $D = \{x \mid x \neq -1\}$; $g(x) = \sin 2x$, $D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin 2x) = \frac{\sin 2x}{1 + \sin 2x}$

Domain: $1 + \sin 2x \neq 0 \;\Rightarrow\; \sin 2x \neq -1 \;\Rightarrow\; 2x \neq \frac{3\pi}{2} + 2\pi n \;\Rightarrow\; x \neq \frac{3\pi}{4} + \pi n$ [n an integer].

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{1+x}\right) = \sin\left(\frac{2x}{1+x}\right)$.

Domain: $\{x \mid x \neq -1\}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{\frac{x}{1+x}}{1 + \frac{x}{1+x}} = \frac{\left(\frac{x}{1+x}\right)\cdot(1+x)}{\left(1 + \frac{x}{1+x}\right)\cdot(1+x)} = \frac{x}{1+x+x} = \frac{x}{2x+1}$

Since $f(x)$ is not defined for $x = -1$, and $f(f(x))$ is not defined for $x = -\frac{1}{2}$,

the domain of $(f \circ f)(x)$ is $D = \{x \mid x \neq -1, -\frac{1}{2}\}$.

(d) $(g \circ g)(g) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x)$.

Domain: $\mathbb{R}$

39. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$

40. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(2^{\sqrt{x}}) = \left|2^{\sqrt{x}} - 4\right|$

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3 + 2)) = f[(x^3+2)^2]$

$$= f(x^6 + 4x^3 + 4) = \sqrt{(x^6 + 4x^3 + 4) - 3} = \sqrt{x^6 + 4x^3 + 1}$$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \tan\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right)$

43. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)$.

44. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

45. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}} = F(x)$.

46. Let $g(x) = \dfrac{x}{1+x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1+x}\right) = \sqrt[3]{\dfrac{x}{1+x}} = G(x)$.

47. Let $g(t) = t^2$ and $f(t) = \sec t \tan t$. Then $(f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2)\tan(t^2) = v(t)$.

48. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = f(\tan t) = \dfrac{\tan t}{1+\tan t} = u(t)$.

49. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 1) = \sqrt{\sqrt{x} - 1} = R(x)$.

50. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[8]{2 + |x|} = H(x)$.

51. Let $h(t) = \cos t$, $g(t) = \sin t$, and $f(t) = t^2$. Then

$(f \circ g \circ h)(t) = f(g(h(t))) = f(g(\cos t)) = f(\sin(\cos t)) = [\sin(\cos t)]^2 = \sin^2(\cos t) = S(t)$.

52. (a) $f(g(1)) = f(6) = 5$

(b) $g(f(1)) = g(3) = 2$

(c) $f(f(1)) = f(3) = 4$

(d) $g(g(1)) = g(6) = 3$

(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$

(f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

53. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

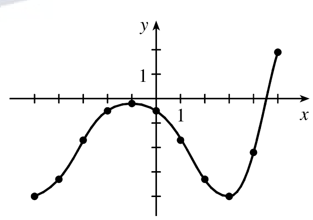
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

54. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



55. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

(b) $A = \pi r^2 \quad \Rightarrow \quad (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

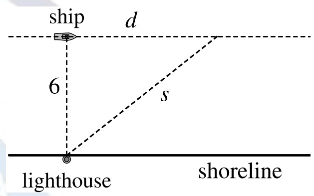
56. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2 \text{ cm/s})(t \text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.

The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).

57. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s.

By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \quad \Rightarrow \quad s = f(d) = \sqrt{d^2 + 36}$.



(b) Using $d = rt$, we get $d = (30 \text{ km/h})(t \text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

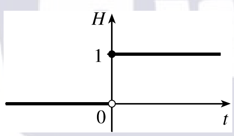
(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

58. (a) $d = rt \quad \Rightarrow \quad d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

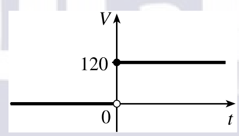
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

59. (a)



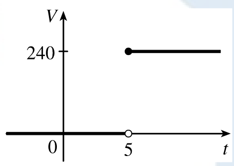
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

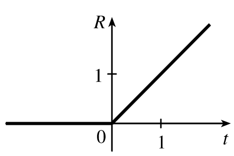
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

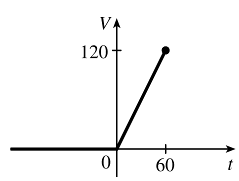
60. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



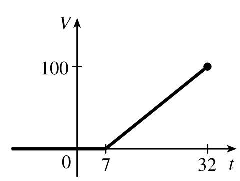
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t)$, $t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4\,(t - 7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t - 7)H(t - 7)$, $t \leq 32$.



61. If $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1.$

So $f \circ g$ is a linear function with slope m_1m_2.

62. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2x,$

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2x) = 1.04(1.04)^2x = (1.04)^3x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3x) = 1.04(1.04)^3x, = (1.04)^4x.$

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A's}(x) = (1.04)^nx$.

63. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

64. We need a function g so that $g(f(x)) = g(x + 4) = h(x) = 4x - 1 = 4(x + 4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

65. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

66. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

1.4 Exponential Functions

1. (a) $\dfrac{4^{-3}}{2^{-8}} = \dfrac{2^8}{4^3} = \dfrac{2^8}{(2^2)^3} = \dfrac{2^8}{2^6} = 2^{8-6} = 2^2 = 4$

(b) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

2. (a) $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$

(b) $x(3x^2)^3 = x \cdot 3^3(x^2)^3 = 27x \cdot x^6 = 27x^7$

3. (a) $b^8(2b)^4 = b^8 \cdot 2^4b^4 = 16b^{12}$

(b) $\dfrac{(6y^3)^4}{2y^5} = \dfrac{6^4(y^3)^4}{2y^5} = \dfrac{1296y^{12}}{2y^5} = 648y^7$

4. (a) $\dfrac{x^{2n}\cdot x^{3n-1}}{x^{n+2}} = \dfrac{x^{2n+3n-1}}{x^{n+2}} = \dfrac{x^{5n-1}}{x^{n+2}} = x^{4n-3}$

(b) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}} = \dfrac{\sqrt{a}\sqrt{\sqrt{b}}}{\sqrt[3]{a}\sqrt[3]{b}} = \dfrac{a^{1/2}b^{1/4}}{a^{1/3}b^{1/3}} = a^{(1/2-1/3)}b^{(1/4-1/3)} = a^{1/6}b^{-1/12}$

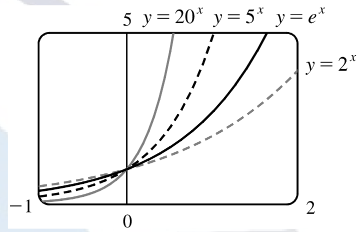
5. (a) $f(x) = b^x,\ \ b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.

6. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

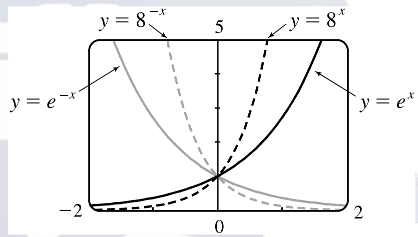
(b) $e \approx 2.71828$ (c) $f(x) = e^x$

7. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.

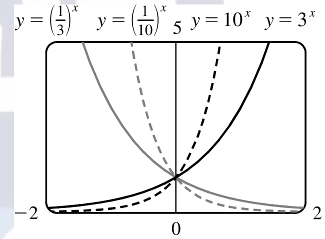
Note: The notation "$x \to \infty$" can be thought of as "x becomes large" at this point. More details on this notation are given in Chapter 2.



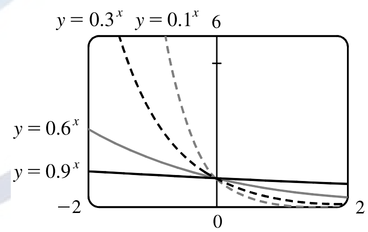
8. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



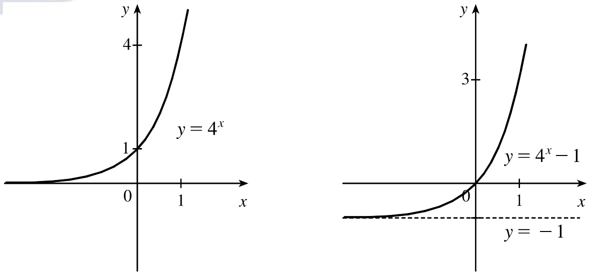
9. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x\right.$ and $\left.\left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



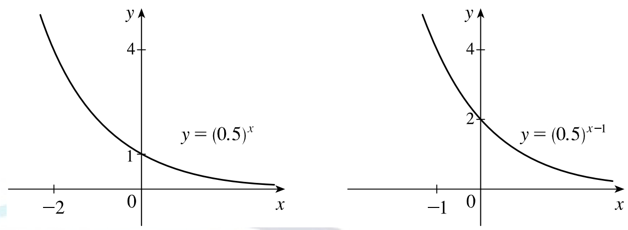
10. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



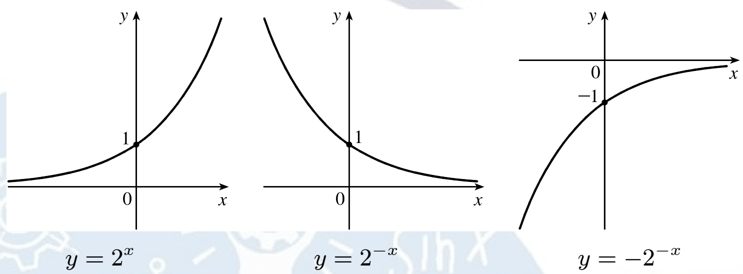
11. We start with the graph of $y = 4^x$ (Figure 3) and shift it 1 unit down to obtain the graph of $y = 4^x - 1$.



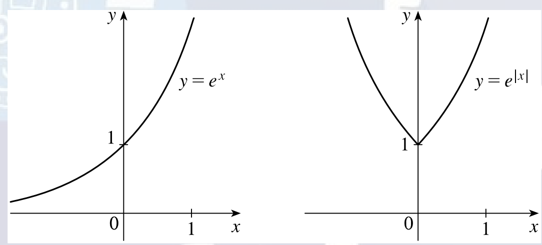
12. We start with the graph of $y = (0.5)^x$ (Figure 3) and shift it 1 unit to the right to obtain the graph of $y = (0.5)^{x-1}$.



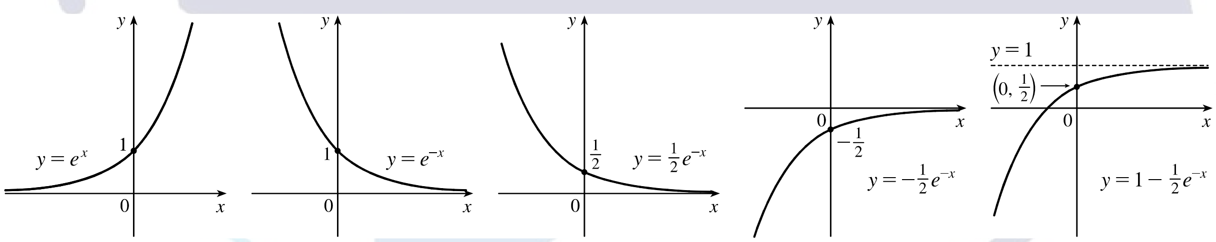
13. We start with the graph of $y = 2^x$ (Figure 16), reflect it about the y-axis, and then about the x-axis (or just rotate $180°$ to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



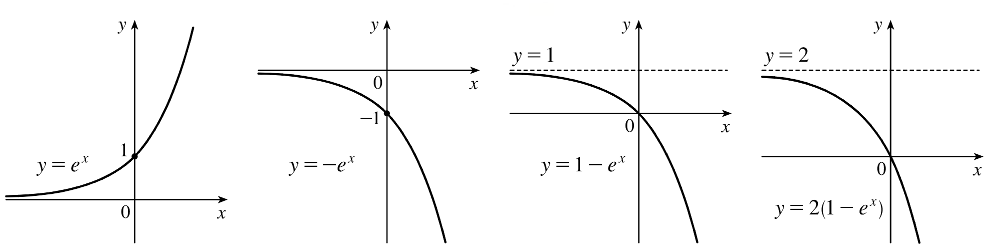
14. We start with the graph of $y = e^x$ (Figure 16) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



15. We start with the graph of $y = e^x$ (Figure 16) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph upward one unit to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



16. We start with the graph of $y = e^x$ (Figure 13) and reflect about the x-axis to get the graph of $y = -e^x$. Then shift the graph upward one unit to get the graph of $y = 1 - e^x$. Finally, we stretch the graph vertically by a factor of 2 to obtain the graph of $y = 2(1 - e^x)$.



17. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

18. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

19. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad \Leftrightarrow \quad e^{1-x^2} = 1 \quad \Leftrightarrow \quad 1 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1 + x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

20. (a) The function $g(t) = \sqrt{10^t - 100}$ has domain $\{t \mid 10^t - 100 \geq 0\} = \{t \mid 10^t \geq 10^2\} = \{t \mid t \geq 2\} = [2, \infty)$.

(b) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^t - 1)$ also has domain $\mathbb{R}$.

21. Use $y = Cb^x$ with the points $(1, 6)$ and $(3, 24)$. $\;6 = Cb^1 \quad [C = \frac{6}{b}]$ and $24 = Cb^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{b}\right) b^3 \quad \Rightarrow$ $4 = b^2 \quad \Rightarrow \quad b = 2 \quad$ [since $b > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

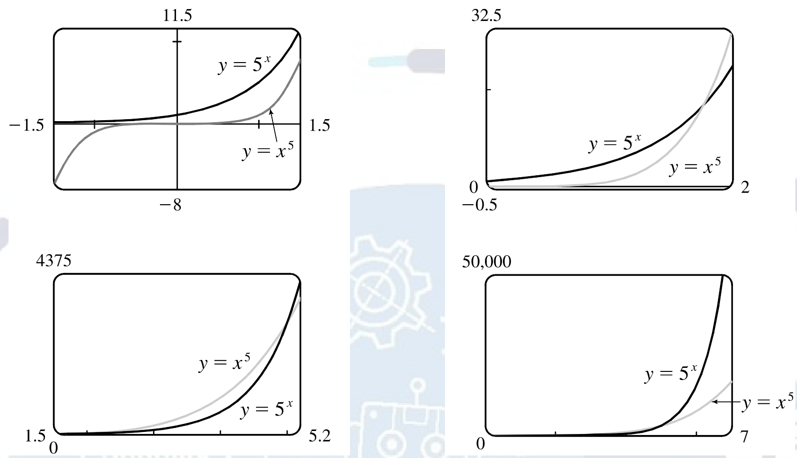
22. Use $y = Cb^x$ with the points $(-1, 3)$ and $\left(1, \frac{4}{3}\right)$. From the point $(-1, 3)$, we have $3 = Cb^{-1}$, hence $C = 3b$. Using this and the point $\left(1, \frac{4}{3}\right)$, we get $\frac{4}{3} = Cb^1 \quad \Rightarrow \quad \frac{4}{3} = (3b)b \quad \Rightarrow \quad \frac{4}{9} = b^2 \quad \Rightarrow \quad b = \frac{2}{3} \quad$ [since $b > 0$] and $C = 3(\frac{2}{3}) = 2$. The function is $f(x) = 2(\frac{2}{3})^x$.

23. If $f(x) = 5^x$, then $\dfrac{f(x + h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x (5^h - 1)}{h} = 5^x \left(\dfrac{5^h - 1}{h}\right)$.

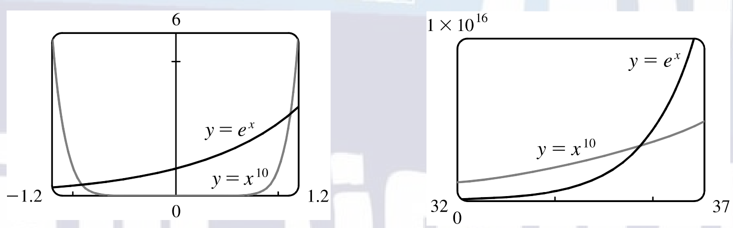
24. Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134{,}217{,}728$ cents, or \$1,342,177.28. Clearly, the second method of payment results in a larger amount for any month.

25. 2 ft $= 24$ in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $\quad g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

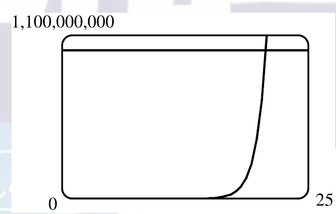
26. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



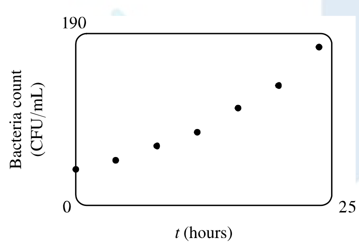
27. The graph of g finally surpasses that of f at $x \approx 35.8$.



28. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.

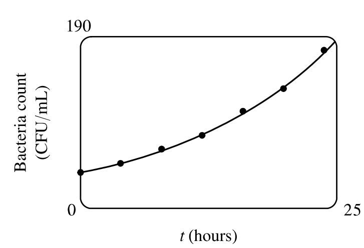


29. (a)



(b) Using a graphing calculator, we obtain the exponential curve $f(t) = 36.89301(1.06614)^t$.

(c) Using the TRACE and zooming in, we find that the bacteria count doubles from 37 to 74 in about 10.87 hours.

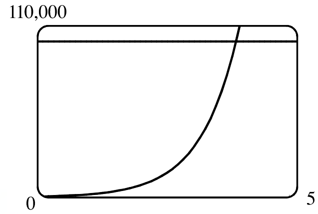


30. (a) Three hours represents 6 doubling periods (one doubling period is 30 minutes). $500 \cdot 2^6 = 32{,}000$

(b) In t hours, there will be $2t$ doubling periods. The initial population is 500, so the population y at time t is $y = 500 \cdot 2^{2t}$.

(c) $t = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad y = 500 \cdot 2^{2(2/3)} \approx 1260$

(d) We graph $y_1 = 500 \cdot 2^{2t}$ and $y_2 = 100{,}000$. The two curves intersect at $t \approx 3.82$, so the population reaches 100,000 in about 3.82 hours.

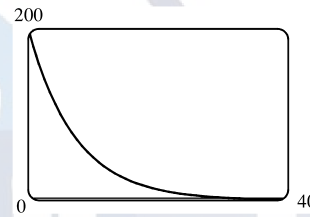


31. (a) Fifteen days represents 3 half-life periods (one half-life period is 5 days). $200\left(\frac{1}{2}\right)^3 = 25$ mg

(b) In t hours, there will be $t/5$ half-life periods. The initial amount is 200 mg, so the amount remaining after t days is $y = 200\left(\frac{1}{2}\right)^{t/5}$ mg, or equivalently, $y = 200 \cdot 2^{-t/5}$ mg.

(c) $t = 3$ weeks $= 21$ days $\quad \Rightarrow \quad y = 200 \cdot 2^{-21/5} \approx 10.9$ mg

(d) We graph $y_1 = 200 \cdot 2^{-t/5}$ and $y_2 = 1$. The two curves intersect at $t \approx 38.2$, so the mass will be reduced to 1 mg in about 38.2 days.

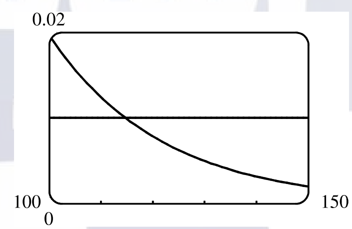


32. (a) Sixty hours represents 4 half-life periods. $2 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ g

(b) In t hours, there will be $t/15$ half-life periods. The initial mass is 2 g, so the mass y at time t is $y = 2 \cdot \left(\frac{1}{2}\right)^{t/15}$.

(c) 4 days $= 4 \cdot 24 = 96$ hours. $t = 96 \quad \Rightarrow \quad y = 2 \cdot \left(\frac{1}{2}\right)^{96/15} \approx 0.024$ g

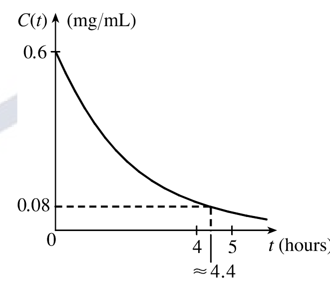
(d) $y = 0.01 \quad \Rightarrow \quad t \approx 114.7$ hours



33. From the table, we see that $V(1) = 76$. In Figure 11, we estimate that $V = 38$ (half of 76) when $t \approx 4.5$. This gives us a half-life of $4.5 - 1 = 3.5$ days.

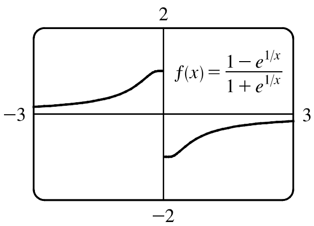
34. (a) The exponential decay model has the form $C(t) = a\left(\frac{1}{2}\right)^{t/1.5}$, where t is the number of hours after midnight and $C(t)$ is the BAC. When $t = 0$, $C(t) = 0.6$, so $0.6 = a\left(\frac{1}{2}\right)^0 \quad \Leftrightarrow \quad a = 0.6$. Thus, the model is $C(t) = 0.6\left(\frac{1}{2}\right)^{t/1.5}$.

(b) From the graph, we estimate that the BAC is 0.08 mg/mL when $t \approx 4.4$ hours. (Note that the legal limit is often 0.08%, which is not 0.08 mg/mL.)



35. Let $t = 0$ correspond to 1950 to get the model $P = ab^t$, where $a \approx 2614.086$ and $b \approx 1.01693$. To estimate the population in 1993, let $t = 43$ to obtain $P \approx 5381$ million. To predict the population in 2020, let $t = 70$ to obtain $P \approx 8466$ million.

36. Let $t = 0$ correspond to 1900 to get the model $P = ab^t$, where $a \approx 80.8498$ and $b \approx 1.01269$. To estimate the population in 1925, let $t = 25$ to obtain $P \approx 111$ million. To predict the population in 2020, let $t = 120$ to obtain $P \approx 367$ million.

37. 

From the graph, it appears that f is an odd function (f is undefined for $x = 0$).

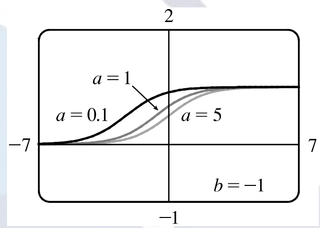
To prove this, we must show that $f(-x) = -f(x)$.

$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

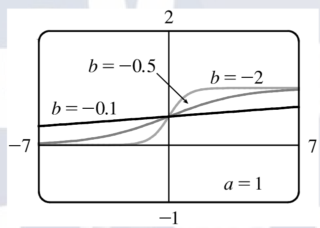
$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

so f is an odd function.

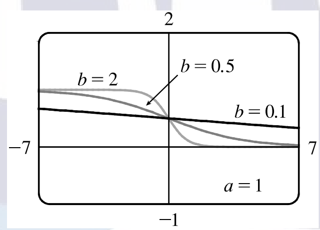
38. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1 + ae^{bx}}$ for $a = 0.1$, 1, and 5.



From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.



As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)



If b is positive, the graph of f is reflected through the y-axis.

Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1 + a)$.

1.5 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one because it never takes on the same value twice.

5. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

6. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

7. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

8. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

9. The graph of $f(x) = 2x - 3$ is a line with slope 2. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $2x_1 \neq 2x_2 \quad\Rightarrow\quad 2x_1 - 3 \neq 2x_2 - 3 \quad\Rightarrow\quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = x^4 - 16$ is symmetric with respect to the y-axis. Pick any x-values equidistant from 0 to find two equal function values. For example, $f(-1) = -15$ and $f(1) = -15$, so f is not one-to-one.

11. $g(x) = 1 - \sin x$. $\quad g(0) = 1$ and $g(\pi) = 1$, so g is not one-to-one.

12. The graph of $g(x) = \sqrt[3]{x}$ passes the Horizontal Line Test, so g is one-to-one.

13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

14. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. (a) Since f is 1-1, $f(6) = 17 \quad\Leftrightarrow\quad f^{-1}(17) = 6$.

(b) Since f is 1-1, $f^{-1}(3) = 2 \quad\Leftrightarrow\quad f(2) = 3$.

16. First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 1$, then $f(1) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 1$. If f is a 1-1 function, then $f(f^{-1}(a)) = a$, so $f(f^{-1}(2)) = 2$.

17. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

18. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) \approx -1.7$.

19. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad\Rightarrow\quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad\Rightarrow\quad \frac{9}{5}C + 32 \geq -459.67 \quad\Rightarrow\quad \frac{9}{5}C \geq -491.67 \quad\Rightarrow\quad C \geq -273.15$, the domain of the inverse function.

20. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad\Rightarrow\quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad\Rightarrow\quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad\Rightarrow\quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad\Rightarrow\quad v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$.

This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

21. $y = f(x) = 1 + \sqrt{2 + 3x} \quad (y \geq 1) \quad\Rightarrow\quad y - 1 = \sqrt{2 + 3x} \quad\Rightarrow\quad (y - 1)^2 = 2 + 3x \quad\Rightarrow\quad (y - 1)^2 - 2 = 3x \quad\Rightarrow\quad x = \frac{1}{3}(y - 1)^2 - \frac{2}{3}$. Interchange x and y: $\quad y = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. So $f^{-1}(x) = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. Note that the domain of f^{-1} is $x \geq 1$.

22. $y = f(x) = \dfrac{4x-1}{2x+3} \quad\Rightarrow\quad y(2x+3) = 4x-1 \quad\Rightarrow\quad 2xy+3y = 4x-1 \quad\Rightarrow\quad 3y+1 = 4x-2xy \quad\Rightarrow$

$3y+1 = (4-2y)x \quad\Rightarrow\quad x = \dfrac{3y+1}{4-2y}$. Interchange x and y: $y = \dfrac{3x+1}{4-2x}$. So $f^{-1}(x) = \dfrac{3x+1}{4-2x}$.

23. $y = f(x) = e^{2x-1} \quad\Rightarrow\quad \ln y = 2x-1 \quad\Rightarrow\quad 1+\ln y = 2x \quad\Rightarrow\quad x = \frac{1}{2}(1+\ln y)$.

Interchange x and y: $y = \frac{1}{2}(1+\ln x)$. So $f^{-1}(x) = \frac{1}{2}(1+\ln x)$.

24. $y = f(x) = x^2 - x \quad (x \geq \frac{1}{2}) \quad\Rightarrow\quad y = x^2 - x + \frac{1}{4} - \frac{1}{4} \quad\Rightarrow\quad y = (x-\frac{1}{2})^2 - \frac{1}{4} \quad\Rightarrow$

$y + \frac{1}{4} = (x-\frac{1}{2})^2 \quad\Rightarrow\quad x - \frac{1}{2} = \sqrt{y+\frac{1}{4}} \quad\Rightarrow\quad x = \frac{1}{2} + \sqrt{y+\frac{1}{4}}$. Interchange x and y: $y = \frac{1}{2} + \sqrt{x+\frac{1}{4}}$. So

$f^{-1}(x) = \frac{1}{2} + \sqrt{x+\frac{1}{4}}$.

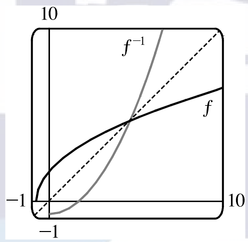
25. $y = f(x) = \ln(x+3) \quad\Rightarrow\quad x+3 = e^y \quad\Rightarrow\quad x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

26. $y = f(x) = \dfrac{1-e^{-x}}{1+e^{-x}} \quad\Rightarrow\quad y(1+e^{-x}) = 1-e^{-x} \quad\Rightarrow\quad y + ye^{-x} = 1 - e^{-x} \quad\Rightarrow\quad ye^x + y = e^x - 1$ [multiply each term by e^x] $\quad\Rightarrow\quad ye^x - e^x = -y-1 \quad\Rightarrow\quad e^x(y-1) = -y-1 \quad\Rightarrow\quad e^x = \dfrac{1+y}{1-y} \quad\Rightarrow\quad x = \ln\left(\dfrac{1+y}{1-y}\right)$.

Interchange x and y: $y = \ln\left(\dfrac{1+x}{1-x}\right)$. So $f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)$.

27. $y = f(x) = \sqrt{4x+3} \;\; (y \geq 0) \quad\Rightarrow\quad y^2 = 4x+3 \quad\Rightarrow\quad x = \dfrac{y^2-3}{4}$.

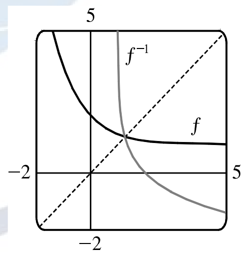
Interchange x and y: $y = \dfrac{x^2-3}{4}$. So $f^{-1}(x) = \dfrac{x^2-3}{4}$ $(x \geq 0)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



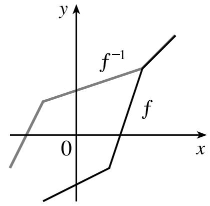
28. $y = f(x) = 1 + e^{-x} \quad\Rightarrow\quad e^{-x} = y-1 \quad\Rightarrow\quad -x = \ln(y-1) \quad\Rightarrow$

$x = -\ln(y-1)$. Interchange x and y: $y = -\ln(x-1)$.

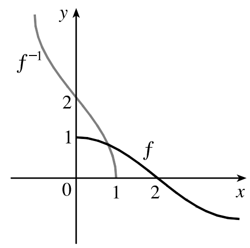
So $f^{-1}(x) = -\ln(x-1)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



29. Reflect the graph of f about the line $y = x$. The points $(-1,-2)$, $(1,-1)$, $(2,2)$, and $(3,3)$ on f are reflected to $(-2,-1)$, $(-1,1)$, $(2,2)$, and $(3,3)$ on f^{-1}.



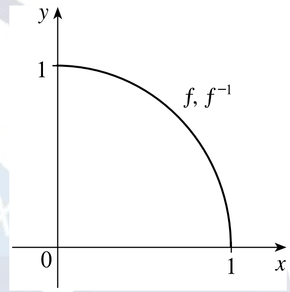
30. Reflect the graph of f about the line $y = x$.



31. (a) $y = f(x) = \sqrt{1 - x^2}$ $\quad (0 \le x \le 1$ and note that $y \ge 0) \quad \Rightarrow$

$y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y^2 \quad \Rightarrow \quad x = \sqrt{1 - y^2}$. So

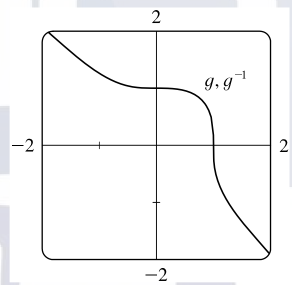
$f^{-1}(x) = \sqrt{1 - x^2}$, $0 \le x \le 1$. We see that f^{-1} and f are the same function.



(b) The graph of f is the portion of the circle $x^2 + y^2 = 1$ with $0 \le x \le 1$ and $0 \le y \le 1$ (quarter-circle in the first quadrant). The graph of f is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $f^{-1} = f$.

32. (a) $y = g(x) = \sqrt[3]{1 - x^3} \quad \Rightarrow \quad y^3 = 1 - x^3 \quad \Rightarrow \quad x^3 = 1 - y^3 \quad \Rightarrow$

$x = \sqrt[3]{1 - y^3}$. So $g^{-1}(x) = \sqrt[3]{1 - x^3}$. We see that g and g^{-1} are the same function.



(b) The graph of g is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $g^{-1} = g$.

33. (a) It is defined as the inverse of the exponential function with base b, that is, $\log_b x = y \quad \Leftrightarrow \quad b^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

34. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 13.

35. (a) $\log_2 32 = \log_2 2^5 = 5$ by (7).

(b) $\log_8 2 = \log_8 8^{1/3} = \frac{1}{3}$ by (7).

Another method: Set the logarithm equal to x and change to an exponential equation.

$\log_8 2 = x \quad \Leftrightarrow \quad 8^x = 2 \quad \Leftrightarrow \quad (2^3)^x = 2 \quad \Leftrightarrow \quad 2^{3x} = 2^1 \quad \Leftrightarrow \quad 3x = 1 \quad \Leftrightarrow \quad x = \frac{1}{3}$.

36. (a) $\log_5 \dfrac{1}{125} = \log_5 \dfrac{1}{5^3} = \log_5 5^{-3} = -3$ by (7).

(b) $\ln(1/e^2) = \ln e^{-2} = -2$ by (9).

37. (a)
$$\begin{aligned} \log_{10} 40 + \log_{10} 2.5 &= \log_{10} [(40)(2.5)] && \text{[by Law 1]} \\ &= \log_{10} 100 \\ &= \log_{10} 10^2 = 2 && \text{[by (7)]} \end{aligned}$$

(b) $\log_8 60 - \log_8 3 - \log_8 5 = \log_8 \frac{60}{3} - \log_8 5$ [by Law 2]

$= \log_8 20 - \log_8 5$

$= \log_8 \frac{20}{5}$ [by Law 2]

$= \log_8 4 = \log_8 8^{2/3} = \frac{2}{3}$ [by (7)]

38. (a) $e^{-\ln 2} = \dfrac{1}{e^{\ln 2}} = \dfrac{1}{2}$ by (9). *Or:* $e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \dfrac{1}{2}$

(b) $e^{\ln(\ln e^3)} = e^{\ln 3}$ [by (9)] $= 3$ by (9).

39. $\ln 10 + 2\ln 5 = \ln 10 + \ln 5^2$ [by Law 3]

$= \ln[(10)(25)]$ [by Law 1]

$= \ln 250$

40. $\ln b + 2\ln c - 3\ln d = \ln b + \ln c^2 - \ln d^3$ [by Law 3]

$= \ln bc^2 - \ln d^3$ [by Law 1]

$= \ln \dfrac{bc^2}{d^3}$ [by Law 2]

41. $\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right] = \ln[(x+2)^3]^{1/3} + \frac{1}{2}\ln \dfrac{x}{(x^2+3x+2)^2}$ [by Laws 3, 2]

$= \ln(x+2) + \ln \dfrac{\sqrt{x}}{x^2+3x+2}$ [by Law 3]

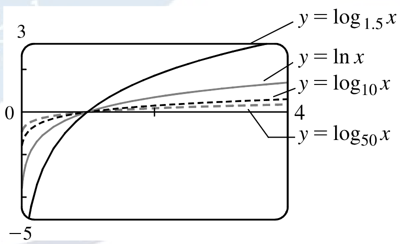
$= \ln \dfrac{(x+2)\sqrt{x}}{(x+1)(x+2)}$ [by Law 1]

$= \ln \dfrac{\sqrt{x}}{x+1}$

Note that since $\ln x$ is defined for $x > 0$, we have $x+1$, $x+2$, and x^2+3x+2 all positive, and hence their logarithms are defined.

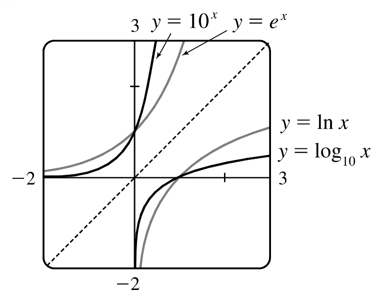
42. (a) $\log_5 10 = \dfrac{\ln 10}{\ln 5}$ [by (10)] ≈ 1.430677 (b) $\log_3 57 = \dfrac{\ln 57}{\ln 3}$ [by (10)] ≈ 3.680144

43. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$.



These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.

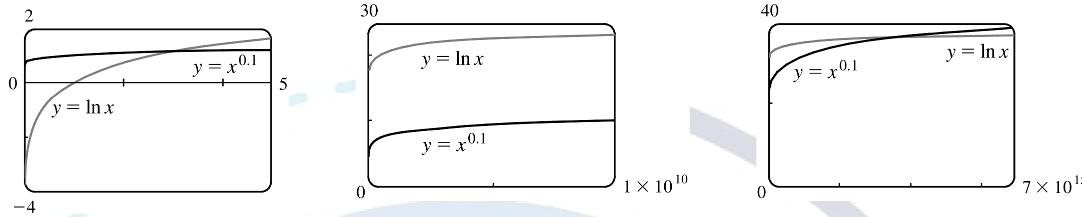
44. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

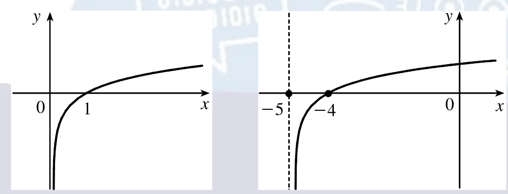
$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

46.



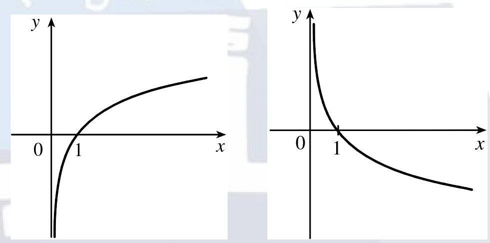
From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

47. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.



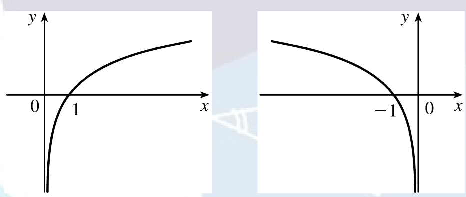
$y = \log_{10} x$ $\qquad$ $y = \log_{10}(x + 5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.



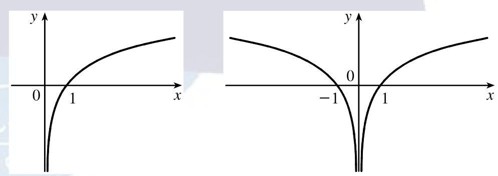
$y = \ln x$ $\qquad$ $y = -\ln x$

48. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.



$y = \ln x$ $\qquad$ $y = \ln(-x)$

(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

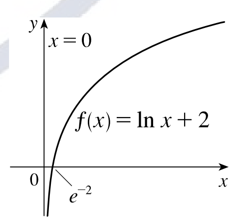


$y = \ln x$ $\qquad$ $y = \ln|x|$

49. (a) The domain of $f(x) = \ln x + 2$ is $x > 0$ and the range is $\mathbb{R}$.

(b) $y = 0 \quad \Rightarrow \quad \ln x + 2 = 0 \quad \Rightarrow \quad \ln x = -2 \quad \Rightarrow \quad x = e^{-2}$

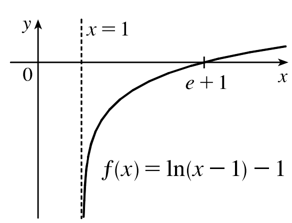
(c) We shift the graph of $y = \ln x$ two units upward.



50. (a) The domain of $f(x) = \ln(x - 1) - 1$ is $x > 1$ and the range is $\mathbb{R}$.

(b) $y = 0 \quad \Rightarrow \quad \ln(x - 1) - 1 = 0 \quad \Rightarrow \quad \ln(x - 1) = 1 \quad \Rightarrow$
$x - 1 = e^1 \quad \Rightarrow \quad x = e + 1$

(c) We shift the graph of $y = \ln x$ one unit to the right and one unit downward.



51. (a) $e^{7-4x} = 6 \quad \Leftrightarrow \quad 7 - 4x = \ln 6 \quad \Leftrightarrow \quad 7 - \ln 6 = 4x \quad \Leftrightarrow \quad x = \frac{1}{4}(7 - \ln 6)$

(b) $\ln(3x - 10) = 2 \quad \Leftrightarrow \quad 3x - 10 = e^2 \quad \Leftrightarrow \quad 3x = e^2 + 10 \quad \Leftrightarrow \quad x = \frac{1}{3}(e^2 + 10)$

52. (a) $\ln(x^2 - 1) = 3 \quad \Leftrightarrow \quad x^2 - 1 = e^3 \quad \Leftrightarrow \quad x^2 = 1 + e^3 \quad \Leftrightarrow \quad x = \pm\sqrt{1 + e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \;\Leftrightarrow\; (e^x - 1)(e^x - 2) = 0 \;\Leftrightarrow\; e^x = 1$ or $e^x = 2 \;\Leftrightarrow\; x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

53. (a) $2^{x-5} = 3 \quad \Leftrightarrow \quad \log_2 3 = x - 5 \quad \Leftrightarrow \quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad \Leftrightarrow \quad \ln\left(2^{x-5}\right) = \ln 3 \quad \Leftrightarrow \quad (x - 5)\ln 2 = \ln 3 \quad \Leftrightarrow \quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x - 1) = \ln(x(x - 1)) = 1 \quad \Leftrightarrow \quad x(x - 1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$.

54. (a) $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

(b) $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$

$ax - bx = \ln C \quad \Leftrightarrow \quad (a - b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a - b}$

55. (a) $\ln x < 0 \quad \Rightarrow \quad x < e^0 \quad \Rightarrow \quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad \Rightarrow \quad \ln e^x > \ln 5 \quad \Rightarrow \quad x > \ln 5$

56. (a) $1 < e^{3x-1} < 2 \quad \Rightarrow \quad \ln 1 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 0 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 1 < 3x < 1 + \ln 2 \quad \Rightarrow$

$\frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

(b) $1 - 2\ln x < 3 \quad \Rightarrow \quad -2\ln x < 2 \quad \Rightarrow \quad \ln x > -1 \quad \Rightarrow \quad x > e^{-1}$

57. (a) We must have $e^x - 3 > 0 \quad \Leftrightarrow \quad e^x > 3 \quad \Leftrightarrow \quad x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$.

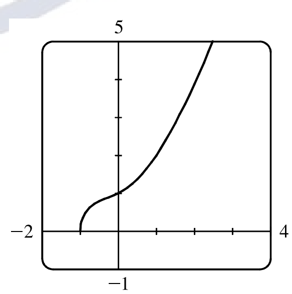
(b) $y = \ln(e^x - 3) \quad \Rightarrow \quad e^y = e^x - 3 \quad \Rightarrow \quad e^x = e^y + 3 \quad \Rightarrow \quad x = \ln(e^y + 3)$, so $f^{-1}(x) = \ln(e^x + 3)$.

Now $e^x + 3 > 0 \quad \Rightarrow \quad e^x > -3$, which is true for any real x, so the domain of f^{-1} is $\mathbb{R}$.

58. (a) By (9), $e^{\ln 300} = 300$ and $\ln(e^{300}) = 300$.

(b) A calculator gives $e^{\ln 300} = 300$ and an error message for $\ln(e^{300})$ since e^{300} is larger than most calculators can evaluate.

59. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1.



Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y.

Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:

$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$.

[continued]

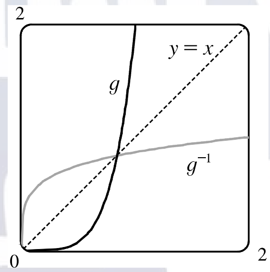
Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3}-8-2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

60. (a) If we use Derive, then solving $x = y^6 + y^4$ for y gives us six solutions of the form $y = \pm\frac{\sqrt{3}}{3}\sqrt{B-1}$, where $B \in \left\{-2\sin\dfrac{A}{3}, 2\sin\left(\dfrac{A}{3}+\dfrac{\pi}{3}\right), -2\cos\left(\dfrac{A}{3}+\dfrac{\pi}{6}\right)\right\}$ and $A = \sin^{-1}\left(\dfrac{27x-2}{2}\right)$. The inverse for $y = x^6 + x^4$ $(x \geq 0)$ is $y = \frac{\sqrt{3}}{3}\sqrt{B-1}$ with $B = 2\sin\left(\dfrac{A}{3}+\dfrac{\pi}{3}\right)$, but because the domain of A is $\left[0, \frac{4}{27}\right]$, this expression is only valid for $x \in \left[0, \frac{4}{27}\right]$.

Happily, Maple gives us the rest of the solution! We solve $x = y^6 + y^4$ for y to get the two real solutions $\pm\dfrac{\sqrt{6}}{6}\dfrac{\sqrt{C^{1/3}\left(C^{2/3}-2C^{1/3}+4\right)}}{C^{1/3}}$, where $C = 108x + 12\sqrt{3}\sqrt{x\,(27x-4)}$, and the inverse for $y = x^6 + x^4$ $(x \geq 0)$ is the positive solution, whose domain is $\left[\frac{4}{27}, \infty\right)$.

Mathematica also gives two real solutions, equivalent to those of Maple. The positive one is $\dfrac{\sqrt{6}}{6}\left(\sqrt[3]{4}D^{1/3} + 2\sqrt[3]{2}D^{-1/3} - 2\right)$, where $D = -2 + 27x + 3\sqrt{3}\sqrt{x}\sqrt{27x-4}$. Although this expression also has domain $\left[\frac{4}{27}, \infty\right)$, Mathematica is mysteriously able to plot the solution for all $x \geq 0$.

(b)



61. (a) $n = f(t) = 100 \cdot 2^{t/3} \;\Rightarrow\; \dfrac{n}{100} = 2^{t/3} \;\Rightarrow\; \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \;\Rightarrow\; t = 3\log_2\left(\dfrac{n}{100}\right)$. Using formula (10), we can write this as $t = f^{-1}(n) = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \;\Rightarrow\; t = f^{-1}(50{,}000) = 3 \cdot \dfrac{\ln\left(\frac{50{,}000}{100}\right)}{\ln 2} = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

62. (a) $Q = Q_0(1 - e^{-t/a}) \;\Rightarrow\; \dfrac{Q}{Q_0} = 1 - e^{-t/a} \;\Rightarrow\; e^{-t/a} = 1 - \dfrac{Q}{Q_0} \;\Rightarrow\; -\dfrac{t}{a} = \ln\left(1 - \dfrac{Q}{Q_0}\right) \;\Rightarrow$ $t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \;\Rightarrow\; t = -2\ln(1 - 0.9Q_0/Q_0) = -2\ln 0.1 \approx 4.6$ seconds.

63. (a) $\cos^{-1}(-1) = \pi$ because $\cos\pi = -1$ and π is in the interval $[0, \pi]$ (the range of $\cos^{-1}$).

(b) $\sin^{-1}(0.5) = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = 0.5$ and $\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the range of $\sin^{-1}$).

64. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (the range of $\tan^{-1}$).

(b) $\arctan(-1) = -\frac{\pi}{4}$ because $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (the range of $\arctan$).

65. (a) $\csc^{-1}\sqrt{2} = \frac{\pi}{4}$ because $\csc\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ (the range of $\csc^{-1}$).

(b) $\arcsin 1 = \frac{\pi}{2}$ because $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the range of arcsin).

66. (a) $\sin^{-1}(-1/\sqrt{2}) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}\left(\sqrt{3}/2\right) = \frac{\pi}{6}$ because $\cos\frac{\pi}{6} = \sqrt{3}/2$ and $\frac{\pi}{6}$ is in $[0, \pi]$.

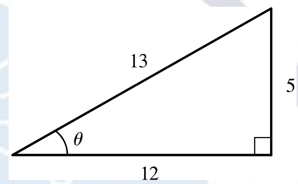
67. (a) $\cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$ because $\cot\frac{5\pi}{6} = -\sqrt{3}$ and $\frac{5\pi}{6}$ is in $(0, \pi)$ (the range of $\cot^{-1}$).

(b) $\sec^{-1} 2 = \frac{\pi}{3}$ because $\sec\frac{\pi}{3} = 2$ and $\frac{\pi}{3}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ (the range of $\sec^{-1}$).

68. (a) $\arcsin(\sin(5\pi/4)) = \arcsin\left(-1/\sqrt{2}\right) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) Let $\theta = \sin^{-1}\left(\frac{5}{13}\right)$ [see the figure].

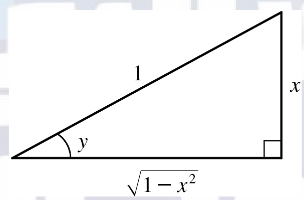
$$\cos\left(2\sin^{-1}\left(\tfrac{5}{13}\right)\right) = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\tfrac{12}{13}\right)^2 - \left(\tfrac{5}{13}\right)^2 = \tfrac{144}{169} - \tfrac{25}{169} = \tfrac{119}{169}$$



69. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

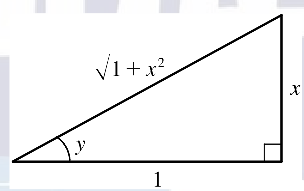
70. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1 - x^2}}.$$



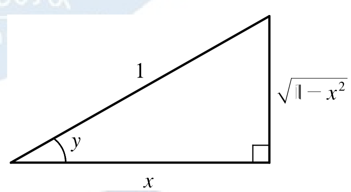
71. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1 + x^2}}.$$

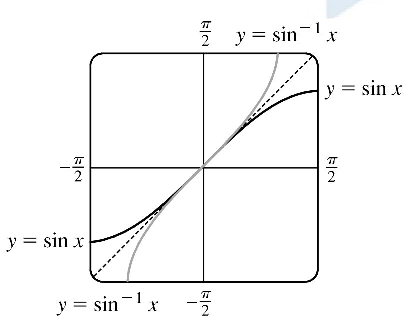


72. Let $y = \arccos x$. Then $\cos y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(2\arccos x) = \sin 2y = 2\sin y\,\cos y$$
$$= 2\left(\sqrt{1 - x^2}\right)(x) = 2x\sqrt{1 - x^2}$$

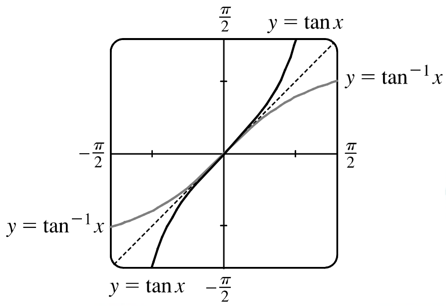


73.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

74.



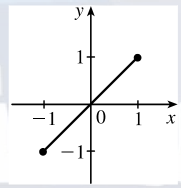
The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

75. $g(x) = \sin^{-1}(3x + 1)$.

Domain $(g) = \{x \mid -1 \le 3x + 1 \le 1\} = \{x \mid -2 \le 3x \le 0\} = \left\{x \mid -\frac{2}{3} \le x \le 0\right\} = \left[-\frac{2}{3}, 0\right]$.

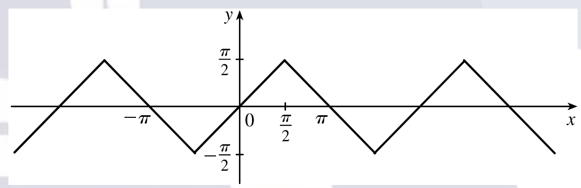
Range $(g) = \left\{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

76. (a) $f(x) = \sin\left(\sin^{-1} x\right)$



Since one function undoes what the other one does, we get the identity function, $y = x$, on the restricted domain $-1 \le x \le 1$.

(b) $g(x) = \sin^{-1}(\sin x)$



This is similar to part (a), but with domain $\mathbb{R}$.

Equations for g on intervals of the form $\left(-\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n\right)$, for any integer n, can be found using $g(x) = (-1)^n x + (-1)^{n+1} n\pi$.

The sine function is monotonic on each of these intervals, and hence, so is g (but in a linear fashion).

77. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

1 Review

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s+t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

8. True. We can divide by e^x since $e^x \neq 0$ for every x.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

10. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$. What *is* true, however, is that $\ln(x^6) = 6 \ln x$ for $x > 0$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

12. False. It is true that $\tan \frac{3\pi}{4} = -1$, but since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we must have $\tan^{-1}(-1) = -\frac{\pi}{4}$.

13. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

14. False. For example, if $x = -3$, then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.

(d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.

(f) f is not one-to-one since it fails the Horizontal Line Test.

(g) f is odd since its graph is symmetric about the origin.

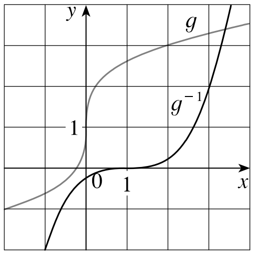
2. (a) When $x = 2$, $y = 3$. Thus, $g(2) = 3$.

(b) g is one-to-one because it passes the Horizontal Line Test.

(c) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(d) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

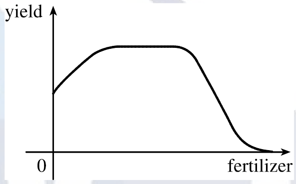
(e) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



3. $f(x) = x^2 - 2x + 3$, so $f(a+h) = (a+h)^2 - 2(a+h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and

$$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

4. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



5. $f(x) = 2/(3x - 1)$. Domain: $3x - 1 \neq 0 \;\Rightarrow\; 3x \neq 1 \;\Rightarrow\; x \neq \frac{1}{3}$. $D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.)

$R = (-\infty, 0) \cup (0, \infty)$

6. $g(x) = \sqrt{16 - x^4}$. Domain: $16 - x^4 \geq 0 \;\Rightarrow\; x^4 \leq 16 \;\Rightarrow\; |x| \leq \sqrt[4]{16} \;\Rightarrow\; |x| \leq 2$. $D = [-2, 2]$

Range: $y \geq 0$ and $y \leq \sqrt{16} \;\Rightarrow\; 0 \leq y \leq 4$.

$R = [0, 4]$

7. $h(x) = \ln(x + 6)$. Domain: $x + 6 > 0 \;\Rightarrow\; x > -6$. $D = (-6, \infty)$

Range: $x + 6 > 0$, so $\ln(x + 6)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.

$R = (-\infty, \infty)$

8. $y = F(t) = 3 + \cos 2t$. Domain: $\mathbb{R}$. $D = (-\infty, \infty)$

Range: $-1 \leq \cos 2t \leq 1 \;\Rightarrow\; 2 \leq 3 + \cos 2t \leq 4 \;\Rightarrow\; 2 \leq y \leq 4$.

$R = [2, 4]$

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

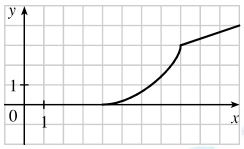
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

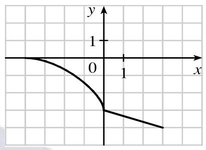
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

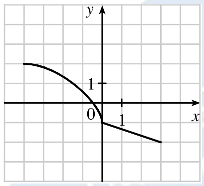
10. (a) To obtain the graph of $y = f(x-8)$, we shift the graph of $y = f(x)$ right 8 units.



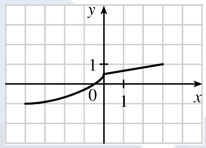
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



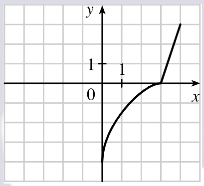
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



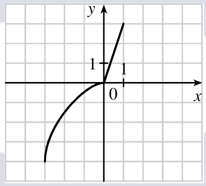
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



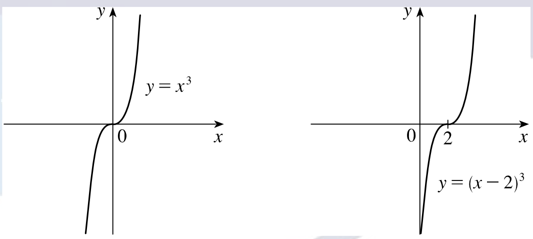
(e) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$.



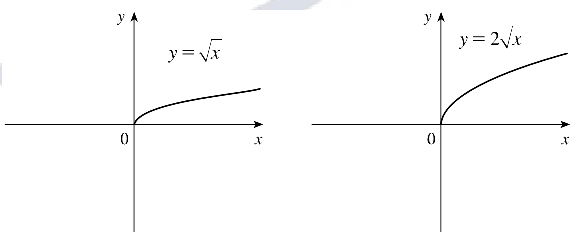
(f) To obtain the graph of $y = f^{-1}(x+3)$, we reflect the graph of $y = f(x)$ about the line $y = x$ [see part (e)], and then shift the resulting graph left 3 units.



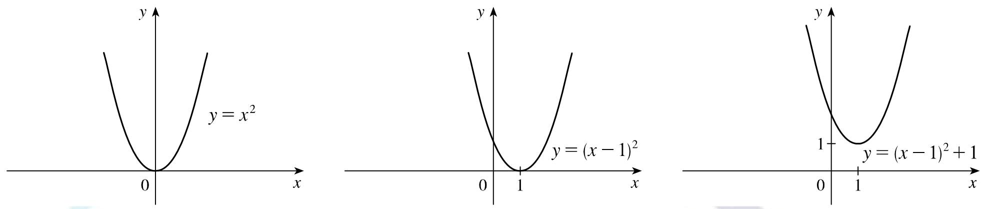
11. $y = (x-2)^3$: Start with the graph of $y = x^3$ and shift 2 units to the right.



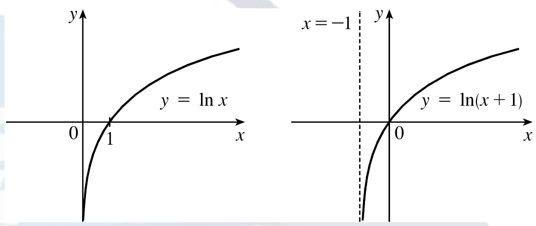
12. $y = 2\sqrt{x}$: Start with the graph of $y = \sqrt{x}$ and stretch vertically by a factor of 2.



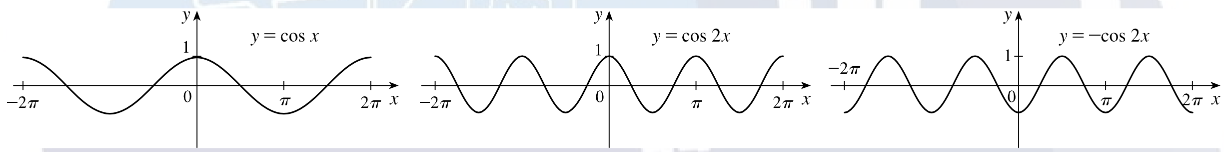
13. $y = x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x - 1)^2 + 1$: Start with the graph of $y = x^2$, shift 1 unit to the right, and shift 1 unit upward.



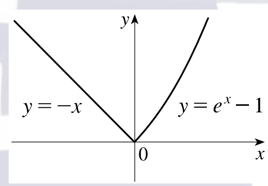
14. $y = \ln(x + 1)$: Start with the graph of $y = \ln x$ and shift left 1 unit.



15. $f(x) = -\cos 2x$: Start with the graph of $y = \cos x$, shrink horizontally by a factor of 2, and reflect about the x-axis.



16. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$



On $(-\infty, 0)$, graph $y = -x$ (the line with slope -1 and y-intercept 0) with open endpoint $(0, 0)$.

On $[0, \infty)$, graph $y = e^x - 1$ (the graph of $y = e^x$ shifted 1 unit downward) with closed endpoint $(0, 0)$.

17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

18. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\dfrac{0 - 2}{-1 + 2} = -2$, and an equation is $y - 0 = -2(x + 1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1 - x^2}$ (we have solved for positive y). Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{if } -1 < x \leq 1 \end{cases}$

19. $f(x) = \ln x$, $\quad D = (0, \infty)$; $\quad g(x) = x^2 - 9$, $\quad D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

Domain: $x^2 - 9 > 0 \;\Rightarrow\; x^2 > 9 \;\Rightarrow\; |x| > 3 \;\Rightarrow\; x \in (-\infty, -3) \cup (3, \infty)$

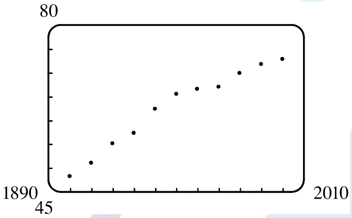
(b) $(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

(c) $(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \Rightarrow x > e^0 = 1$, or $(1, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

21.

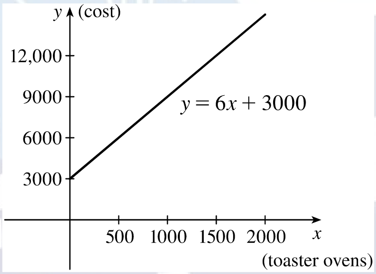


Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$, gives us an estimate of 77.6 years for the year 2010.

22. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line:

$$y - 9000 = \frac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \quad \Rightarrow$$

$$y = 6(x - 1000) + 9000 \quad \Rightarrow \quad y = 6x + 3000.$$



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

23. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

24. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \Rightarrow 2xy + x = y + 1 \Rightarrow 2xy - y = 1 - x \Rightarrow$

$y(2x - 1) = 1 - x \Rightarrow y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

25. (a) $e^{2\ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

(c) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(d) Let $\theta = \cos^{-1} \frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

26. (a) $e^x = 5 \Rightarrow x = \ln 5$

(b) $\ln x = 2 \Rightarrow x = e^2$

(c) $e^{e^x} = 2 \Rightarrow e^x = \ln 2 \Rightarrow x = \ln(\ln 2)$

(d) $\tan^{-1} x = 1 \Rightarrow \tan\tan^{-1} x = \tan 1 \Rightarrow x = \tan 1$ (≈ 1.5574)

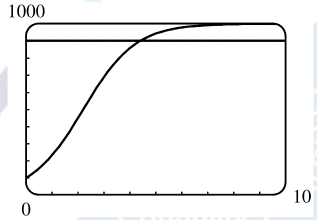
27. (a) After 4 days, $\frac{1}{2}$ gram remains; after 8 days, $\frac{1}{4}$ g; after 12 days, $\frac{1}{8}$ g; after 16 days, $\frac{1}{16}$ g.

(b) $m(4) = \dfrac{1}{2}$, $m(8) = \dfrac{1}{2^2}$, $m(12) = \dfrac{1}{2^3}$, $m(16) = \dfrac{1}{2^4}$. From the pattern, we see that $m(t) = \dfrac{1}{2^{t/4}}$, or $2^{-t/4}$.

(c) $m = 2^{-t/4} \quad \Rightarrow \quad \log_2 m = -t/4 \quad \Rightarrow \quad t = -4\log_2 m$; this is the time elapsed when there are m grams of ^{100}Pd.

(d) $m = 0.01 \quad \Rightarrow \quad t = -4\log_2 0.01 = -4\left(\dfrac{\ln 0.01}{\ln 2}\right) \approx 26.6$ days

28. (a)



The population would reach 900 in about 4.4 years.

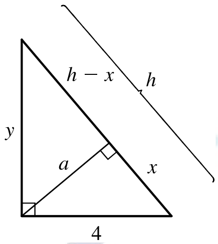
(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \Rightarrow \quad 100P + 900Pe^{-t} = 100{,}000 \Rightarrow \quad 900Pe^{-t} = 100{,}000 - 100P \quad \Rightarrow$

$e^{-t} = \dfrac{100{,}000 - 100P}{900P} \quad \Rightarrow \quad -t = \ln\left(\dfrac{1000 - P}{9P}\right) \quad \Rightarrow \quad t = -\ln\left(\dfrac{1000 - P}{9P}\right)$, or $\ln\left(\dfrac{9P}{1000 - P}\right)$;

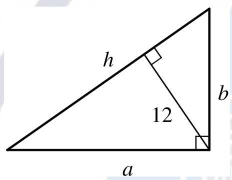
this is the time required for the population to reach a given number P.

(c) $P = 900 \quad \Rightarrow \quad t = \ln\left(\dfrac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

PRINCIPLES OF PROBLEM SOLVING

1. 

By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2 - 16}}{h}$.

2. 

Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle. In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4\,(6h)}{2P} \quad\Rightarrow\quad 2Ph = P^2 - 24h \quad\Rightarrow$

$2Ph + 24h = P^2 \quad\Rightarrow\quad h\,(2P + 24) = P^2 \quad\Rightarrow\quad h = \dfrac{P^2}{2P + 24}$.

3. $|2x - 1| = \begin{cases} 2x - 1 & \text{if } x \ge \frac{1}{2} \\ 1 - 2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x + 5| = \begin{cases} x + 5 & \text{if } x \ge -5 \\ -x - 5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \le x < \frac{1}{2}$, and $x \ge \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \quad\Leftrightarrow\quad x = 3$, which is false, since we are considering $x < -5$.

If $-5 \le x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \quad\Leftrightarrow\quad x = -\frac{7}{3}$.

If $x \ge \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \quad\Leftrightarrow\quad x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

4. $|x - 1| = \begin{cases} x - 1 & \text{if } x \ge 1 \\ 1 - x & \text{if } x < 1 \end{cases}$ and $|x - 3| = \begin{cases} x - 3 & \text{if } x \ge 3 \\ 3 - x & \text{if } x < 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \le x < 3$, and $x \ge 3$.

If $x < 1$, we must have $1 - x - (3 - x) \ge 5 \quad\Leftrightarrow\quad 0 \ge 7$, which is false.

If $1 \le x < 3$, we must have $x - 1 - (3 - x) \ge 5 \quad\Leftrightarrow\quad x \ge \frac{9}{2}$, which is false because $x < 3$.

If $x \ge 3$, we must have $x - 1 - (x - 3) \ge 5 \quad\Leftrightarrow\quad 2 \ge 5$, which is false.

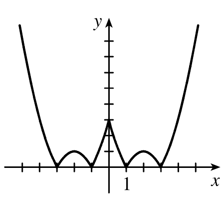
All three cases lead to falsehoods, so the inequality has no solution.

5. $f(x) = \left|x^2 - 4\,|x| + 3\right|$. If $x \ge 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x - 1)(x - 3)|$.

Case (i): If $0 < x \le 1$, then $f(x) = x^2 - 4x + 3$.

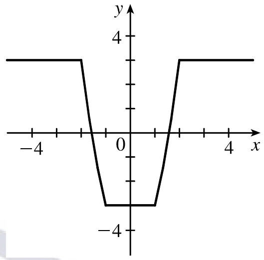
Case (ii): If $1 < x \le 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.



This enables us to sketch the graph for $x \ge 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \le x < -1$, and $-1 \le x < 0$.

6. $g(x) = |x^2 - 1| - |x^2 - 4|$.



$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \text{ and } |x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

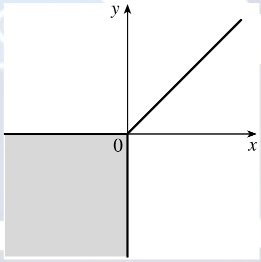
So for $0 \leq |x| < 1$, $g(x) = 1 - x^2 - (4 - x^2) = -3$, for $1 \leq |x| < 2$, $g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for $|x| \geq 2$, $g(x) = x^2 - 1 - (x^2 - 4) = 3$.

7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	



Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

Case 4 gives us the entire third quadrant.

8. $|x - y| + |x| - |y| \leq 2$ [call this inequality $(\star)$]

Case (i): $x \geq y \geq 0$. Then $(\star)$ $\Leftrightarrow$ $x - y + x - y \leq 2$ $\Leftrightarrow$ $x - y \leq 1$ $\Leftrightarrow$ $y \geq x - 1$.

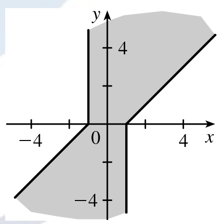
Case (ii): $y \geq x \geq 0$. Then $(\star)$ $\Leftrightarrow$ $y - x + x - y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

Case (iii): $x \geq 0$ and $y \leq 0$. Then $(\star)$ $\Leftrightarrow$ $x - y + x + y \leq 2$ $\Leftrightarrow$ $2x \leq 2$ $\Leftrightarrow$ $x \leq 1$.

Case (iv): $x \leq 0$ and $y \geq 0$. Then $(\star)$ $\Leftrightarrow$ $y - x - x - y \leq 2$ $\Leftrightarrow$ $-2x \leq 2$ $\Leftrightarrow$ $x \geq -1$.

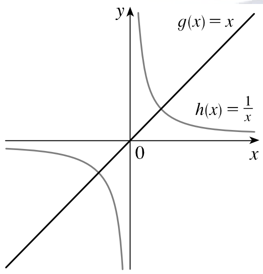
Case (v): $y \leq x \leq 0$. Then $(\star)$ $\Leftrightarrow$ $x - y - x + y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

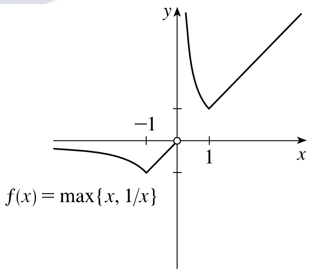
Case (vi): $x \leq y \leq 0$. Then $(\star)$ $\Leftrightarrow$ $y - x - x + y \leq 2$ $\Leftrightarrow$ $y - x \leq 1$ $\Leftrightarrow$ $y \leq x + 1$.



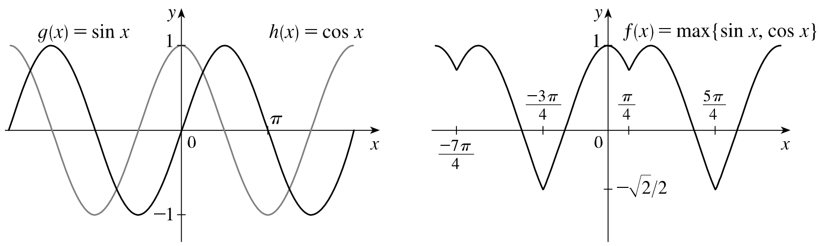
Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through 180° about the origin.

9. (a) To sketch the graph of $f(x) = \max\{x, 1/x\}$, we first graph $g(x) = x$ and $h(x) = 1/x$ on the same coordinate axes. Then create the graph of f by plotting the largest y-value of g and h for every value of x.

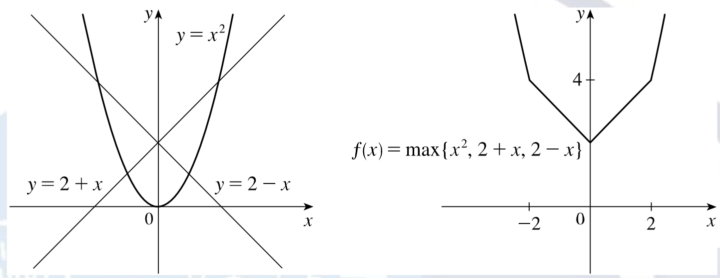




(b)

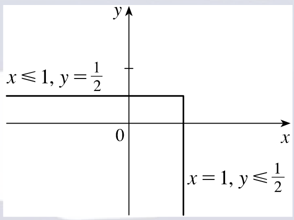


(c)

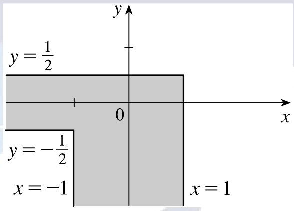


On the TI-84 Plus, max is found under `LIST`, then under `MATH`. To graph $f(x) = \max\{x^2, 2+x, 2-x\}$, use $\texttt{Y} = \max(x^2, \max(2+x, 2-x))$.

10. (a) If $\max\{x, 2y\} = 1$, then either $x = 1$ *and* $2y \le 1$ ***or*** $x \le 1$ *and* $2y = 1$. Thus, we obtain the set of points such that $x = 1$ *and* $y \le \frac{1}{2}$ [a vertical line with highest point $(1, \frac{1}{2})$] ***or*** $x \le 1$ *and* $y = \frac{1}{2}$ [a horizontal line with rightmost point $(1, \frac{1}{2})$].



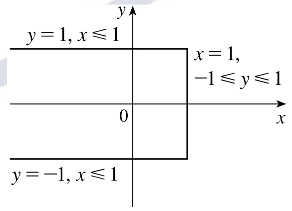
(b) The graph of $\max\{x, 2y\} = 1$ is shown in part (a), and the graph of $\max\{x, 2y\} = -1$ can be found in a similar manner. The inequalities in $-1 \le \max\{x, 2y\} \le 1$ give us all the points on or inside the boundaries.



(c) $\max\{x, y^2\} = 1 \quad \Leftrightarrow$

$x = 1$ *and* $y^2 \le 1$ $[-1 \le y \le 1]$

or $x \le 1$ *and* $y^2 = 1$ $[y = \pm 1]$.



11. $(\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{31} 32) = \left(\dfrac{\ln 3}{\ln 2}\right)\left(\dfrac{\ln 4}{\ln 3}\right)\left(\dfrac{\ln 5}{\ln 4}\right) \cdots \left(\dfrac{\ln 32}{\ln 31}\right) = \dfrac{\ln 32}{\ln 2} = \dfrac{\ln 2^5}{\ln 2} = \dfrac{5 \ln 2}{\ln 2} = 5$

12. (a) $f(-x) = \ln\left(-x + \sqrt{(-x)^2 + 1}\right) = \ln\left(-x + \sqrt{x^2+1} \cdot \dfrac{-x - \sqrt{x^2+1}}{-x - \sqrt{x^2+1}}\right)$

$$= \ln\left(\frac{x^2 - (x^2+1)}{-x - \sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-x - \sqrt{x^2+1}}\right) = \ln\left(\frac{1}{x + \sqrt{x^2+1}}\right)$$

$$= \ln 1 - \ln\left(x + \sqrt{x^2+1}\right) = -\ln\left(x + \sqrt{x^2-1}\right) = -f(x)$$

(b) $y = \ln\left(x + \sqrt{x^2+1}\right)$. Interchanging x and y, we get $x = \ln\left(y + \sqrt{y^2+1}\right) \Rightarrow e^x = y + \sqrt{y^2+1} \Rightarrow$

$e^x - y = \sqrt{y^2+1} \Rightarrow e^{2x} - 2ye^x + y^2 = y^2 + 1 \Rightarrow e^{2x} - 1 = 2ye^x \Rightarrow y = \dfrac{e^{2x}-1}{2e^x} = f^{-1}(x)$

13. $\ln\left(x^2 - 2x - 2\right) \le 0 \Rightarrow x^2 - 2x - 2 \le e^0 = 1 \Rightarrow x^2 - 2x - 3 \le 0 \Rightarrow (x-3)(x+1) \le 0 \Rightarrow x \in [-1, 3]$.

Since the argument must be positive, $x^2 - 2x - 2 > 0 \Rightarrow \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0 \Rightarrow$

$x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

14. Assume that $\log_2 5$ is rational. Then $\log_2 5 = m/n$ for natural numbers m and n. Changing to exponential form gives us $2^{m/n} = 5$ and then raising both sides to the nth power gives $2^m = 5^n$. But 2^m is even and 5^n is odd. We have arrived at a contradiction, so we conclude that our hypothesis, that $\log_2 5$ is rational, is false. Thus, $\log_2 5$ is irrational.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

16. Let $f(x) = \sin x$, $g(x) = x$, and $h(x) = x$. Then the left-hand side of the equation is $[f \circ (g + h)](x) = \sin(x + x) = \sin 2x = 2 \sin x \cos x$; and the right-hand side is $(f \circ g)(x) + (f \circ h)(x) = \sin x + \sin x = 2 \sin x$. The two sides are not equal, so the given statement is false.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

18. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then
  $1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$
  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

20. (a) $f_0(x) = 1/(2-x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2 - \dfrac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2 - \dfrac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2 - \dfrac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n+1-nx}{n+2-(n+1)x}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$.

Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

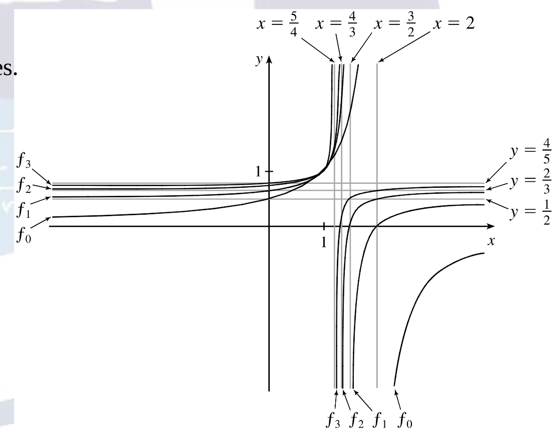
$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2 - \dfrac{k+1-kx}{k+2-(k+1)x}}$$

$$= \frac{k+2-(k+1)x}{2\left[k+2-(k+1)x\right]-(k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k+1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b) From the graph, we can make several observations:

- The values at each fixed $x = a$ keep increasing as n increases.
- The vertical asymptote gets closer to $x = 1$ as n increases.
- The horizontal asymptote gets closer to $y = 1$ as n increases.
- The x-intercept for f_{n+1} is the value of the vertical asymptote for f_n.
- The y-intercept for f_n is the value of the horizontal asymptote for f_{n+1}.



2 □ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

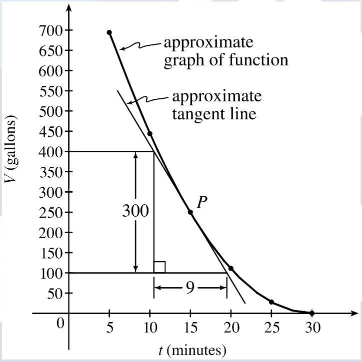
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $y = \dfrac{1}{1-x}$, $P(2, -1)$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	$(1.5, -2)$	2
(ii)	1.9	$(1.9, -1.111\,111)$	1.111 111
(iii)	1.99	$(1.99, -1.010\,101)$	1.010 101
(iv)	1.999	$(1.999, -1.001\,001)$	1.001 001
(v)	2.5	$(2.5, -0.666\,667)$	0.666 667
(vi)	2.1	$(2.1, -0.909\,091)$	0.909 091
(vii)	2.01	$(2.01, -0.990\,099)$	0.990 099
(viii)	2.001	$(2.001, -0.999\,001)$	0.999 001

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

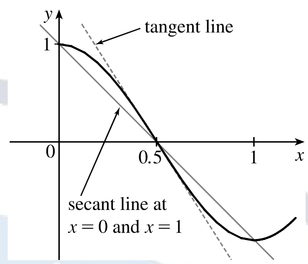
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{ave}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{ave}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{ave}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

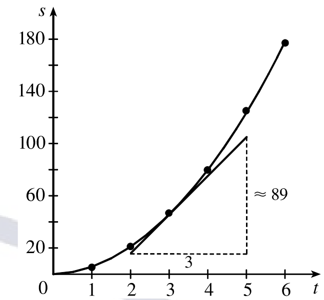
7. (a) (i) On the interval $[2, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{79.2 - 20.6}{2} = 29.3$ ft/s.

(ii) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{79.2 - 46.5}{1} = 32.7$ ft/s.

(iii) On the interval $[4, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(4)}{5 - 4} = \dfrac{124.8 - 79.2}{1} = 45.6$ ft/s.

(iv) On the interval $[4, 6]$, $v_{\text{ave}} = \dfrac{s(6) - s(4)}{6 - 4} = \dfrac{176.7 - 79.2}{2} = 48.75$ ft/s.

(b) Using the points $(2, 16)$ and $(5, 105)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\dfrac{105 - 16}{5 - 2} = \dfrac{89}{3} \approx 29.7$ ft/s.



8. (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval $[1, 2]$, $v_{\text{ave}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6$ cm/s.

(ii) On the interval $[1, 1.1]$, $v_{\text{ave}} = \dfrac{s(1.1) - s(1)}{1.1 - 1} \approx \dfrac{-3.471 - (-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval $[1, 1.01]$, $v_{\text{ave}} = \dfrac{s(1.01) - s(1)}{1.01 - 1} \approx \dfrac{-3.0613 - (-3)}{0.01} = -6.13$ cm/s.

(iv) On the interval $[1, 1.001]$, $v_{\text{ave}} = \dfrac{s(1.001) - s(1)}{1.001 - 1} \approx \dfrac{-3.00627 - (-3)}{0.001} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.3 cm/s.

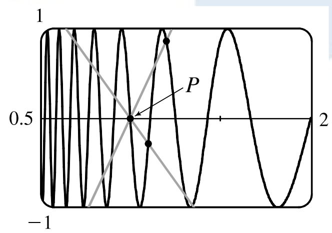
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) As x approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim\limits_{x \to 2^-} f(x) = 3$.

(b) As x approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 2^+} f(x) = 1$.

(c) $\lim\limits_{x \to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $x = 2$, $y = 3$, so $f(2) = 3$.

(e) As x approaches 4, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 4} f(x) = 4$.

(f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim\limits_{x \to 1} f(x) = 2$.

(b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 3^-} f(x) = 1$.

(c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 3^+} f(x) = 4$.

(d) $\lim\limits_{x \to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim\limits_{x \to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim\limits_{x \to -3^+} h(x) = 4$.

(c) $\lim\limits_{x \to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim\limits_{x \to 0^-} h(x) = 1$.

(f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim\limits_{x \to 0^+} h(x) = -1$.

(g) $\lim\limits_{x \to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

(i) Since $\lim\limits_{x \to 2^-} h(x) = 2$ and $\lim\limits_{x \to 2^+} h(x) = 2$, we have $\lim\limits_{x \to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

(k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x \to 5^+} h(x) = 3$.

(l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x \to 5^-} h(x)$ does not exist.

7. (a) $\lim_{t \to 0^-} g(t) = -1$

(b) $\lim_{t \to 0^+} g(t) = -2$

(c) $\lim_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \to 2^-} g(t) = 2$

(e) $\lim_{t \to 2^+} g(t) = 0$

(f) $\lim_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$

(h) $\lim_{t \to 4} g(t) = 3$

8. (a) $\lim_{x \to -3} A(x) = \infty$

(b) $\lim_{x \to 2} A(x)$ does not exist.

(c) $\lim_{x \to 2^-} A(x) = -\infty$

(d) $\lim_{x \to 2^+} A(x) = \infty$

(e) $\lim_{x \to -1} A(x) = -\infty$

(f) The equations of the vertical asymptotes are $x = -3$, $x = -1$ and $x = 2$.

9. (a) $\lim_{x \to -7} f(x) = -\infty$

(b) $\lim_{x \to -3} f(x) = \infty$

(c) $\lim_{x \to 0} f(x) = \infty$

(d) $\lim_{x \to 6^-} f(x) = -\infty$

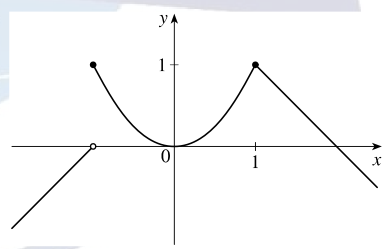
(e) $\lim_{x \to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t \to 12^-} f(t) = 150$ mg and $\lim_{t \to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. From the graph of

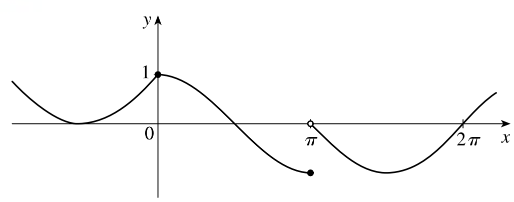
$$f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1, \\ 2 - x & \text{if } x \ge 1 \end{cases}$$



we see that $\lim_{x \to a} f(x)$ exists for all a except $a = -1$. Notice that the right and left limits are different at $a = -1$.

12. From the graph of

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi, \\ \sin x & \text{if } x > \pi \end{cases}$$

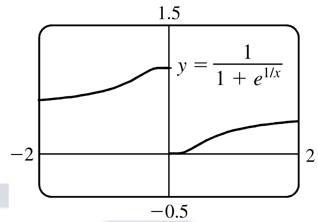


we see that $\lim_{x \to a} f(x)$ exists for all a except $a = \pi$. Notice that the right and left limits are different at $a = \pi$.

13. (a) $\lim_{x \to 0^-} f(x) = 1$

(b) $\lim_{x \to 0^+} f(x) = 0$

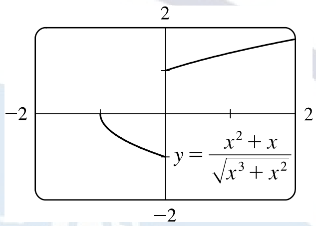
(c) $\lim_{x \to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



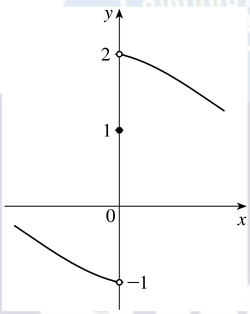
14. (a) $\lim_{x \to 0^-} f(x) = -1$

(b) $\lim_{x \to 0^+} f(x) = 1$

(c) $\lim_{x \to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

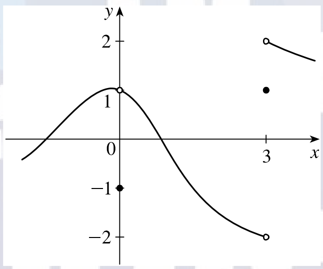


15. $\lim_{x \to 0^-} f(x) = -1, \quad \lim_{x \to 0^+} f(x) = 2, \quad f(0) = 1$



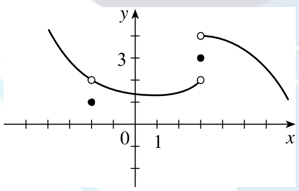
16. $\lim_{x \to 0} f(x) = 1, \ \lim_{x \to 3^-} f(x) = -2, \ \lim_{x \to 3^+} f(x) = 2,$

$f(0) = -1, \ f(3) = 1$



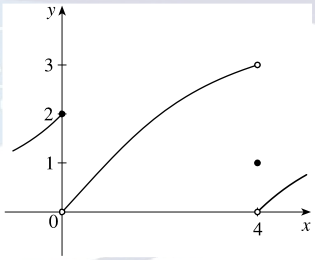
17. $\lim_{x \to 3^+} f(x) = 4, \quad \lim_{x \to 3^-} f(x) = 2, \ \lim_{x \to -2} f(x) = 2,$

$f(3) = 3, \quad f(-2) = 1$



18. $\lim_{x \to 0^-} f(x) = 2, \ \lim_{x \to 0^+} f(x) = 0, \ \lim_{x \to 4^-} f(x) = 3,$

$\lim_{x \to 4^+} f(x) = 0, \ f(0) = 2, \ f(4) = 1$



19. For $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$:

x	$f(x)$
3.1	0.508 197
3.05	0.504 132
3.01	0.500 832
3.001	0.500 083
3.0001	0.500 008

x	$f(x)$
2.9	0.491 525
2.95	0.495 798
2.99	0.499 165
2.999	0.499 917
2.9999	0.499 992

It appears that $\lim_{x \to 3} \dfrac{x^2 - 3x}{x^2 - 9} = \dfrac{1}{2}$.

20. For $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$:

x	$f(x)$
-2.5	-5
-2.9	-29
-2.95	-59
-2.99	-299
-2.999	-2999
-2.9999	$-29{,}999$

x	$f(x)$
-3.5	7
-3.1	31
-3.05	61
-3.01	301
-3.001	3001
-3.0001	$30{,}001$

It appears that $\lim\limits_{x \to -3^+} f(x) = -\infty$ and that $\lim\limits_{x \to -3^-} f(x) = \infty$, so $\lim\limits_{x \to -3} \dfrac{x^2 - 3x}{x^2 - 9}$ does not exist.

21. For $f(t) = \dfrac{e^{5t} - 1}{t}$:

t	$f(t)$
0.5	22.364 988
0.1	6.487 213
0.01	5.127 110
0.001	5.012 521
0.0001	5.001 250

t	$f(t)$
-0.5	1.835 830
-0.1	3.934 693
-0.01	4.877 058
-0.001	4.987 521
-0.0001	4.998 750

It appears that $\lim\limits_{t \to 0} \dfrac{e^{5t} - 1}{t} = 5$.

22. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

h	$f(h)$
0.5	131.312 500
0.1	88.410 100
0.01	80.804 010
0.001	80.080 040
0.0001	80.008 000

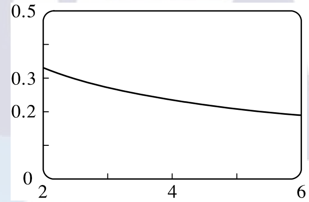
h	$f(h)$
-0.5	48.812 500
-0.1	72.390 100
-0.01	79.203 990
-0.001	79.920 040
-0.0001	79.992 000

It appears that $\lim\limits_{h \to 0} \dfrac{(2+h)^5 - 32}{h} = 80$.

23. For $f(x) = \dfrac{\ln x - \ln 4}{x - 4}$:

x	$f(x)$
3.9	0.253 178
3.99	0.250 313
3.999	0.250 031
3.9999	0.250 003

x	$f(x)$
4.1	0.246 926
4.01	0.249 688
4.001	0.249 969
4.0001	0.249 997

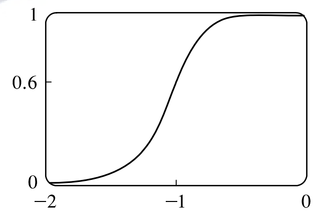


It appears that $\lim\limits_{x \to 4} f(x) = 0.25$. The graph confirms that result.

24. For $f(p) = \dfrac{1 + p^9}{1 + p^{15}}$:

p	$f(p)$
-1.1	0.427 397
-1.01	0.582 008
-1.001	0.598 200
-1.0001	0.599 820

p	$f(p)$
-0.9	0.771 405
-0.99	0.617 992
-0.999	0.601 800
-0.9999	0.600 180



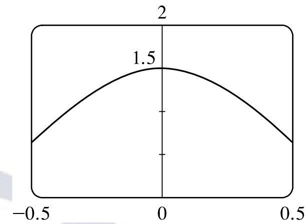
It appears that $\lim\limits_{p \to -1} f(p) = 0.6$. The graph confirms that result.

25. For $f(\theta) = \dfrac{\sin 3\theta}{\tan 2\theta}$:

θ	$f(\theta)$
± 0.1	1.457 847
± 0.01	1.499 575
± 0.001	1.499 996
± 0.0001	1.500 000

It appears that $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta}{\tan 2\theta} = 1.5$.

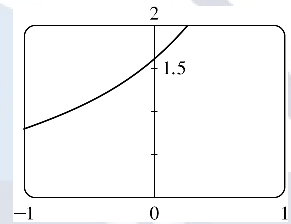
The graph confirms that result.



26. For $f(t) = \dfrac{5^t - 1}{t}$:

t	$f(t)$
0.1	1.746 189
0.01	1.622 459
0.001	1.610 734
0.0001	1.609 567

t	$f(t)$
−0.1	1.486 601
−0.01	1.596 556
−0.001	1.608 143
−0.0001	1.609 308



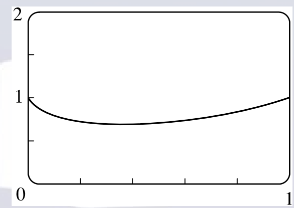
It appears that $\lim\limits_{t \to 0} f(t) \approx 1.6094$. The graph confirms that result.

27. For $f(x) = x^x$:

x	$f(x)$
0.1	0.794 328
0.01	0.954 993
0.001	0.993 116
0.0001	0.999 079

It appears that $\lim\limits_{x \to 0^+} f(x) = 1$.

The graph confirms that result.

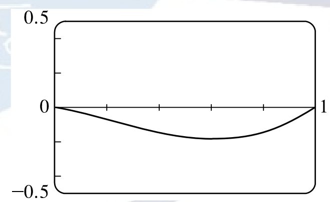


28. For $f(x) = x^2 \ln x$:

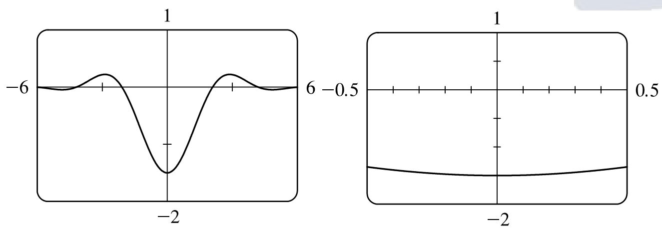
x	$f(x)$
0.1	−0.023 026
0.01	−0.000 461
0.001	−0.000 007
0.0001	−0.000 000

It appears that $\lim\limits_{x \to 0^+} f(x) = 0$.

The graph confirms that result.



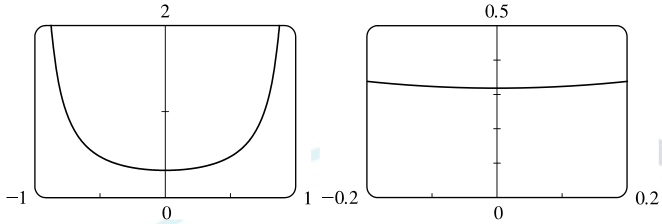
29. (a) From the graphs, it seems that $\lim\limits_{x \to 0} \dfrac{\cos 2x - \cos x}{x^2} = -1.5$.



(b)

x	$f(x)$
± 0.1	−1.493 759
± 0.01	−1.499 938
± 0.001	−1.499 999
± 0.0001	−1.500 000

30. (a) From the graphs, it seems that $\lim\limits_{x\to 0} \dfrac{\sin x}{\sin \pi x} = 0.32$.



(b)

x	$f(x)$
± 0.1	0.323 068
± 0.01	0.318 357
± 0.001	0.318 310
± 0.0001	0.318 310

Later we will be able to show that the exact value is $\dfrac{1}{\pi}$.

31. $\lim\limits_{x\to 5^+} \dfrac{x+1}{x-5} = \infty$ since the numerator is positive and the denominator approaches 0 from the positive side as $x \to 5^+$.

32. $\lim\limits_{x\to 5^-} \dfrac{x+1}{x-5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 5^-$.

33. $\lim\limits_{x\to 1} \dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

34. $\lim\limits_{x\to 3^-} \dfrac{\sqrt{x}}{(x-3)^5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 3^-$.

35. Let $t = x^2 - 9$. Then as $x \to 3^+$, $t \to 0^+$, and $\lim\limits_{x\to 3^+} \ln(x^2-9) = \lim\limits_{t\to 0^+} \ln t = -\infty$ by (5).

36. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

37. $\lim\limits_{x\to(\pi/2)^+} \dfrac{1}{x}\sec x = -\infty$ since $\dfrac{1}{x}$ is positive and $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

38. $\lim\limits_{x\to\pi^-} \cot x = \lim\limits_{x\to\pi^-} \dfrac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

39. $\lim\limits_{x\to 2\pi^-} x\csc x = \lim\limits_{x\to 2\pi^-} \dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

40. $\lim\limits_{x\to 2^-} \dfrac{x^2-2x}{x^2-4x+4} = \lim\limits_{x\to 2^-} \dfrac{x(x-2)}{(x-2)^2} = \lim\limits_{x\to 2^-} \dfrac{x}{x-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

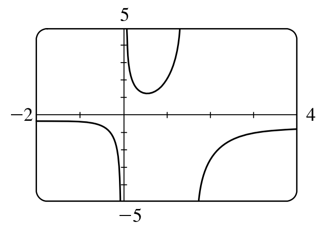
41. $\lim\limits_{x\to 2^+} \dfrac{x^2-2x-8}{x^2-5x+6} = \lim\limits_{x\to 2^+} \dfrac{(x-4)(x+2)}{(x-3)(x-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $x \to 2^+$.

42. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \ln x\right) = \infty$ since $\dfrac{1}{x} \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.

43. $\lim\limits_{x\to 0}(\ln x^2 - x^{-2}) = -\infty$ since $\ln x^2 \to -\infty$ and $x^{-2} \to \infty$ as $x \to 0$.

44. (a) The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when $x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are vertical asymptotes of the function.

(b)



45. (a) $f(x) = \dfrac{1}{x^3-1}$.

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33,333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

From these calculations, it seems that

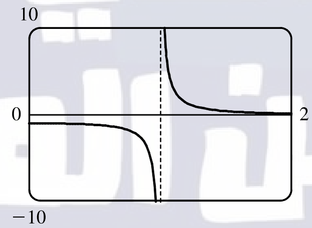
$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to 1^-} f(x) = -\infty$.

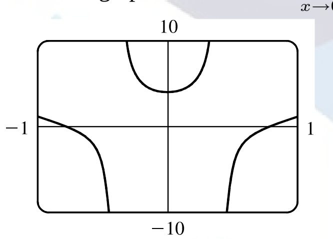
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to 1^+} f(x) = \infty$.

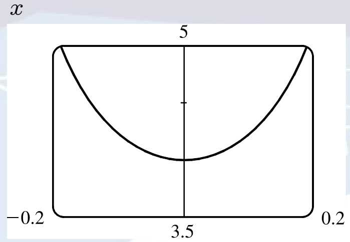
(c) It appears from the graph of f that

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.



46. (a) From the graphs, it seems that $\lim\limits_{x\to 0} \dfrac{\tan 4x}{x} = 4$.





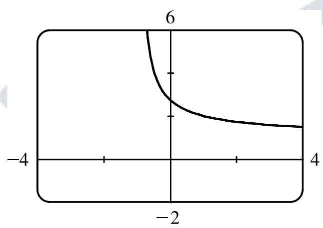
(b)

x	$f(x)$
±0.1	4.227 932
±0.01	4.002 135
±0.001	4.000 021
±0.0001	4.000 000

47. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
−0.001	2.71964
−0.0001	2.71842
−0.00001	2.71830
−0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

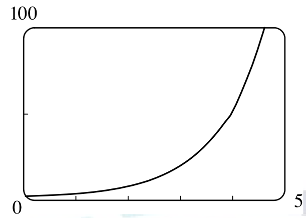
(b)

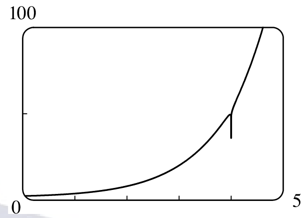


It appears that $\lim\limits_{x\to 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 3.6 we will see that the value of the limit is exactly e.

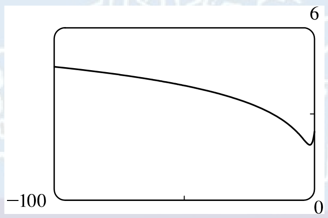
48. (a)

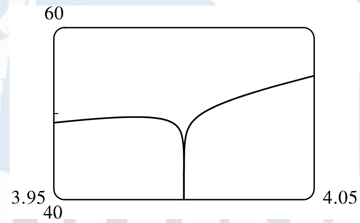




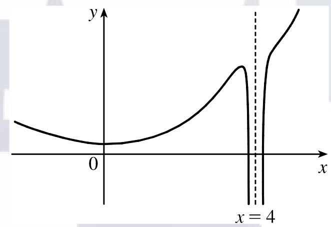
No, because the calculator-produced graph of $f(x) = e^x + \ln|x - 4|$ looks like an exponential function, but the graph of f has an infinite discontinuity at $x = 4$. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at $x = 4$.

(b) There isn't a single graph that shows all the features of f. Several graphs are needed since f looks like $\ln|x - 4|$ for large negative values of x and like e^x for $x > 5$, but yet has the infinite discontiuity at $x = 4$.





A hand-drawn graph, though distorted, might be better at revealing the main features of this function.



49. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998 000
0.8	0.638 259
0.6	0.358 484
0.4	0.158 680
0.2	0.038 851
0.1	0.008 928
0.05	0.001 465

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000 572
0.02	−0.000 614
0.01	−0.000 907
0.005	−0.000 978
0.003	−0.000 993
0.001	−0.001 000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

50. For $h(x) = \dfrac{\tan x - x}{x^3}$:

(a)

x	$h(x)$
1.0	0.557 407 73
0.5	0.370 419 92
0.1	0.334 672 09
0.05	0.333 667 00
0.01	0.333 346 67
0.005	0.333 336 67

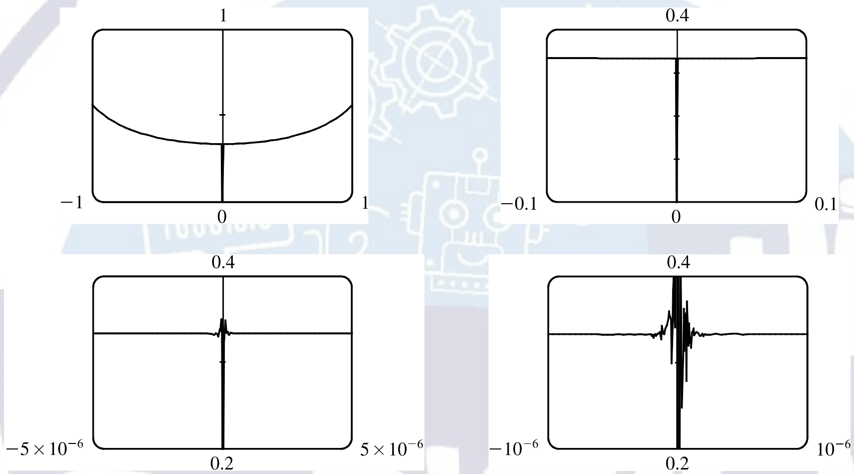
(b) It seems that $\lim_{x \to 0} h(x) = \frac{1}{3}$.

(c)

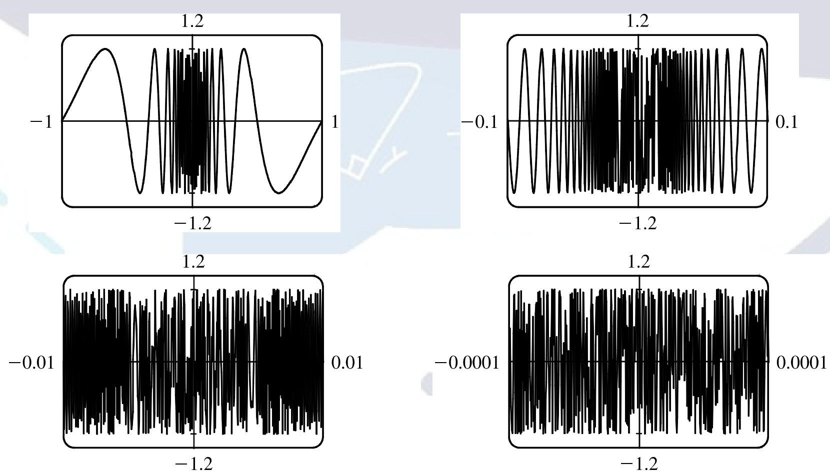
x	$h(x)$
0.001	0.333 333 50
0.0005	0.333 333 44
0.0001	0.333 330 00
0.00005	0.333 336 00
0.00001	0.333 000 00
0.000001	0.000 000 00

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



51. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.

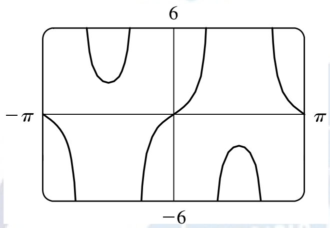


52. (a) For any positive integer n, if $x = \dfrac{1}{n\pi}$, then $f(x) = \tan \dfrac{1}{x} = \tan(n\pi) = 0$. (Remember that the tangent function has period π.)

(b) For any nonnegative number n, if $x = \dfrac{4}{(4n+1)\pi}$, then

$$f(x) = \tan\frac{1}{x} = \tan\frac{(4n+1)\pi}{4} = \tan\left(\frac{4n\pi}{4} + \frac{\pi}{4}\right) = \tan\left(n\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

(c) From part (a), $f(x) = 0$ infinitely often as $x \to 0$. From part (b), $f(x) = 1$ infinitely often as $x \to 0$. Thus, $\lim\limits_{x\to 0} \tan\dfrac{1}{x}$ does not exist since $f(x)$ does not get close to a fixed number as $x \to 0$.

53. 

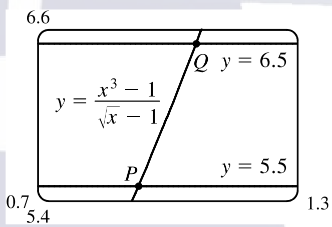
There appear to be vertical asymptotes of the curve $y = \tan(2\sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of vertical asymptotes (corresponding to $x \approx \pm 2.24$).

54. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

55. (a) Let $y = \dfrac{x^3 - 1}{\sqrt{x} - 1}$.

From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

x	y
0.99	5.925 31
0.999	5.992 50
0.9999	5.999 25
1.01	6.075 31
1.001	6.007 50
1.0001	6.000 75



(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P(0.9314, 5.5)$ and $Q(1.0649, 6.5)$. Now $1 - 0.9314 = 0.0686$ and $1.0649 - 1 = 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to 2} [f(x) + 5g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} [5g(x)]$ [Limit Law 1]

$= \lim\limits_{x\to 2} f(x) + 5\lim\limits_{x\to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim\limits_{x\to 2} [g(x)]^3 = \left[\lim\limits_{x\to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim_{x\to 2}\sqrt{f(x)} = \sqrt{\lim_{x\to 2} f(x)}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim_{x\to 2}\dfrac{3f(x)}{g(x)} = \dfrac{\lim_{x\to 2}[3f(x)]}{\lim_{x\to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim_{x\to 2}\dfrac{g(x)\,h(x)}{f(x)} = \dfrac{\lim_{x\to 2}[g(x)\,h(x)]}{\lim_{x\to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim_{x\to 2} g(x)\cdot\lim_{x\to 2} h(x)}{\lim_{x\to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2\cdot 0}{4} = 0$

2. (a) $\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)$ [Limit Law 1]

$= -1 + 2$

$= 1$

(b) $\lim_{x\to 0} f(x)$ exists, but $\lim_{x\to 0} g(x)$ does not exist, so we cannot apply Limit Law 2 to $\lim_{x\to 0}[f(x)-g(x)]$. The limit does not exist.

(c) $\lim_{x\to -1}[f(x)\,g(x)] = \lim_{x\to -1} f(x)\cdot\lim_{x\to -1} g(x)$ [Limit Law 4]

$= 1\cdot 2$

$= 2$

(d) $\lim_{x\to 3} f(x) = 1$, but $\lim_{x\to 3} g(x) = 0$, so we cannot apply Limit Law 5 to $\lim_{x\to 3}\dfrac{f(x)}{g(x)}$. The limit does not exist.

Note: $\lim_{x\to 3^-}\dfrac{f(x)}{g(x)} = \infty$ since $g(x)\to 0^+$ as $x\to 3^-$ and $\lim_{x\to 3^+}\dfrac{f(x)}{g(x)} = -\infty$ since $g(x)\to 0^-$ as $x\to 3^+$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim_{x\to 2}\left[x^2 f(x)\right] = \lim_{x\to 2} x^2\cdot\lim_{x\to 2} f(x)$ [Limit Law 4]

$= 2^2\cdot(-1)$

$= -4$

(f) $f(-1) + \lim_{x\to -1} g(x)$ is undefined since $f(-1)$ is not defined.

3. $\lim_{x\to 3}(5x^3 - 3x^2 + x - 6) = \lim_{x\to 3}(5x^3) - \lim_{x\to 3}(3x^2) + \lim_{x\to 3} x - \lim_{x\to 3} 6$ [Limit Laws 2 and 1]

$= 5\lim_{x\to 3} x^3 - 3\lim_{x\to 3} x^2 + \lim_{x\to 3} x - \lim_{x\to 3} 6$ [3]

$= 5(3^3) - 3(3^2) + 3 - 6$ [9, 8, and 7]

$= 105$

4. $\lim_{x\to -1}(x^4-3x)(x^2+5x+3) = \lim_{x\to -1}(x^4-3x)\lim_{x\to -1}(x^2+5x+3)$ [Limit Law 4]

$= \left(\lim_{x\to -1}x^4 - \lim_{x\to -1}3x\right)\left(\lim_{x\to -1}x^2+\lim_{x\to -1}5x+\lim_{x\to -1}3\right)$ [2, 1]

$= \left(\lim_{x\to -1}x^4 - 3\lim_{x\to -1}x\right)\left(\lim_{x\to -1}x^2+5\lim_{x\to -1}x+\lim_{x\to -1}3\right)$ [3]

$= (1+3)(1-5+3)$ [9, 8, and 7]

$= 4(-1) = -4$

5. $\lim_{t\to -2}\dfrac{t^4-2}{2t^2-3t+2} = \dfrac{\lim_{t\to -2}(t^4-2)}{\lim_{t\to -2}(2t^2-3t+2)}$ [Limit Law 5]

$= \dfrac{\lim_{t\to -2}t^4 - \lim_{t\to -2}2}{2\lim_{t\to -2}t^2 - 3\lim_{t\to -2}t + \lim_{t\to -2}2}$ [1, 2, and 3]

$= \dfrac{16-2}{2(4)-3(-2)+2}$ [9, 7, and 8]

$= \dfrac{14}{16} = \dfrac{7}{8}$

6. $\lim_{u\to -2}\sqrt{u^4+3u+6} = \sqrt{\lim_{u\to -2}(u^4+3u+6)}$ [11]

$= \sqrt{\lim_{u\to -2}u^4 + 3\lim_{u\to -2}u + \lim_{u\to -2}6}$ [1, 2, and 3]

$= \sqrt{(-2)^4+3(-2)+6}$ [9, 8, and 7]

$= \sqrt{16-6+6} = \sqrt{16} = 4$

7. $\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3) = \lim_{x\to 8}(1+\sqrt[3]{x})\cdot\lim_{x\to 8}(2-6x^2+x^3)$ [Limit Law 4]

$= \left(\lim_{x\to 8}1+\lim_{x\to 8}\sqrt[3]{x}\right)\cdot\left(\lim_{x\to 8}2-6\lim_{x\to 8}x^2+\lim_{x\to 8}x^3\right)$ [1, 2, and 3]

$= (1+\sqrt[3]{8})\cdot(2-6\cdot 8^2+8^3)$ [7, 10, 9]

$= (3)(130) = 390$

8. $\lim_{t\to 2}\left(\dfrac{t^2-2}{t^3-3t+5}\right)^2 = \left(\lim_{t\to 2}\dfrac{t^2-2}{t^3-3t+5}\right)^2$ [Limit Law 6]

$= \left(\dfrac{\lim_{t\to 2}(t^2-2)}{\lim_{t\to 2}(t^3-3t+5)}\right)^2$ [5]

$= \left(\dfrac{\lim_{t\to 2}t^2 - \lim_{t\to 2}2}{\lim_{t\to 2}t^3 - 3\lim_{t\to 2}t + \lim_{t\to 2}5}\right)^2$ [1, 2, and 3]

$= \left(\dfrac{4-2}{8-3(2)+5}\right)^2$ [9, 7, and 8]

$= \left(\dfrac{2}{7}\right)^2 = \dfrac{4}{49}$

9. $\lim\limits_{x\to 2}\sqrt{\dfrac{2x^2+1}{3x-2}} = \sqrt{\lim\limits_{x\to 2}\dfrac{2x^2+1}{3x-2}}$ [Limit Law 11]

$= \sqrt{\dfrac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(3x-2)}}$ [5]

$= \sqrt{\dfrac{2\lim\limits_{x\to 2}x^2+\lim\limits_{x\to 2}1}{3\lim\limits_{x\to 2}x-\lim\limits_{x\to 2}2}}$ [1, 2, and 3]

$= \sqrt{\dfrac{2(2)^2+1}{3(2)-2}} = \sqrt{\dfrac{9}{4}} = \dfrac{3}{2}$ [9, 8, and 7]

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x = a$.

11. $\lim\limits_{x\to 5}\dfrac{x^2-6x+5}{x-5} = \lim\limits_{x\to 5}\dfrac{(x-5)(x-1)}{x-5} = \lim\limits_{x\to 5}(x-1) = 5-1 = 4$

12. $\lim\limits_{x\to -3}\dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to -3}\dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to -3}\dfrac{x}{x-4} = \dfrac{-3}{-3-4} = \dfrac{3}{7}$

13. $\lim\limits_{x\to 5}\dfrac{x^2-5x+6}{x-5}$ does not exist since $x-5\to 0$, but $x^2-5x+6\to 6$ as $x\to 5$.

14. $\lim\limits_{x\to 4}\dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to 4}\dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to 4}\dfrac{x}{x-4}$. The last limit does not exist since $\lim\limits_{x\to 4^-}\dfrac{x}{x-4} = -\infty$ and $\lim\limits_{x\to 4^+}\dfrac{x}{x-4} = \infty$.

15. $\lim\limits_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3} = \lim\limits_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim\limits_{t\to -3}\dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

16. $\lim\limits_{x\to -1}\dfrac{2x^2+3x+1}{x^2-2x-3} = \lim\limits_{x\to -1}\dfrac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim\limits_{x\to -1}\dfrac{2x+1}{x-3} = \dfrac{2(-1)+1}{-1-3} = \dfrac{-1}{-4} = \dfrac{1}{4}$

17. $\lim\limits_{h\to 0}\dfrac{(-5+h)^2-25}{h} = \lim\limits_{h\to 0}\dfrac{(25-10h+h^2)-25}{h} = \lim\limits_{h\to 0}\dfrac{-10h+h^2}{h} = \lim\limits_{h\to 0}\dfrac{h(-10+h)}{h} = \lim\limits_{h\to 0}(-10+h) = -10$

18. $\lim\limits_{h\to 0}\dfrac{(2+h)^3-8}{h} = \lim\limits_{h\to 0}\dfrac{\left(8+12h+6h^2+h^3\right)-8}{h} = \lim\limits_{h\to 0}\dfrac{12h+6h^2+h^3}{h}$

$= \lim\limits_{h\to 0}\left(12+6h+h^2\right) = 12+0+0 = 12$

19. By the formula for the sum of cubes, we have

$$\lim_{x\to -2}\frac{x+2}{x^3+8} = \lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x\to -2}\frac{1}{x^2-2x+4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

20. We use the difference of squares in the numerator and the difference of cubes in the denominator.

$$\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\lim_{t\to 1}\frac{(t^2-1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t-1)(t+1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t+1)(t^2+1)}{t^2+t+1}=\frac{2(2)}{3}=\frac{4}{3}$$

21. $$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim_{h\to 0}\frac{\left(\sqrt{9+h}\right)^2-3^2}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$$
$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{3+3}=\frac{1}{6}$$

22. $$\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}=\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}\cdot\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}=\lim_{u\to 2}\frac{\left(\sqrt{4u+1}\right)^2-3^2}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4u+1-9}{(u-2)\left(\sqrt{4u+1}+3\right)}=\lim_{u\to 2}\frac{4(u-2)}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4}{\sqrt{4u+1}+3}=\frac{4}{\sqrt{9}+3}=\frac{2}{3}$$

23. $$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\cdot\frac{3x}{3x}=\lim_{x\to 3}\frac{3-x}{3x(x-3)}=\lim_{x\to 3}\frac{-1}{3x}=-\frac{1}{9}$$

24. $$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3}=\lim_{h\to 0}\frac{-h}{h(3+h)3}$$
$$=\lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right]=-\frac{1}{\lim_{h\to 0}[3(3+h)]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$$

25. $$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim_{t\to 0}\frac{\left(\sqrt{1+t}\,\right)^2-\left(\sqrt{1-t}\,\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\,\right)}$$
$$=\lim_{t\to 0}\frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\,\right)}=\lim_{t\to 0}\frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\,\right)}=\lim_{t\to 0}\frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$
$$=\frac{2}{\sqrt{1}+\sqrt{1}}=\frac{2}{2}=1$$

26. $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)=\lim_{t\to 0}\frac{t+1-1}{t(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{0+1}=1$$

27. $$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x}\,)(4+\sqrt{x}\,)}{(16x-x^2)(4+\sqrt{x}\,)}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x}\,)}$$
$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x}\,)}=\frac{1}{16\left(4+\sqrt{16}\,\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

28. $$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x^2-4)(x^2+1)}=\lim_{x\to 2}\frac{(x-2)^2}{(x+2)(x-2)(x^2+1)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x+2)(x^2+1)}=\frac{0}{4\cdot 5}=0$$

29. $$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\,\right)\left(1+\sqrt{1+t}\,\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\,\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}$$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\,\right)}=-\frac{1}{2}$$

30. $$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

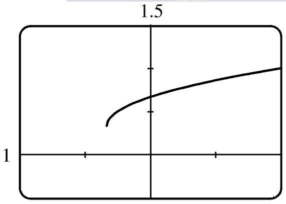
31. $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

32. $$\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$$

$$=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

33. (a)



$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

(b)

x	$f(x)$
−0.001	0.666 166 3
−0.000 1	0.666 616 7
−0.000 01	0.666 661 7
−0.000 001	0.666 666 2
0.000 001	0.666 667 2
0.000 01	0.666 671 7
0.000 1	0.666 716 7
0.001	0.667 166 3

The limit appears to be $\frac{2}{3}$.

(c) $$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$$

$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)\qquad\text{[Limit Law 3]}$$

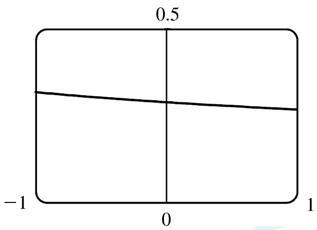
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]\qquad\text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)\qquad\text{[1, 3, and 7]}$$

$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)\qquad\text{[7 and 8]}$$

$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

34. (a)



$\lim_{x\to 0}\dfrac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$

(b)

x	$f(x)$
-0.001	$0.288\,699\,2$
$-0.000\,1$	$0.288\,677\,5$
$-0.000\,01$	$0.288\,675\,4$
$-0.000\,001$	$0.288\,675\,2$
$0.000\,001$	$0.288\,675\,1$
$0.000\,01$	$0.288\,674\,9$
$0.000\,1$	$0.288\,672\,7$
0.001	$0.288\,651\,1$

The limit appears to be approximately 0.2887.

(c) $\lim_{x\to 0}\left(\dfrac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\dfrac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\dfrac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\dfrac{1}{\sqrt{3+x}+\sqrt{3}}$

$$=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}}\qquad\text{[Limit Laws 5 and 1]}$$

$$=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}}\qquad\text{[7 and 11]}$$

$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}\qquad\text{[1, 7, and 8]}$$

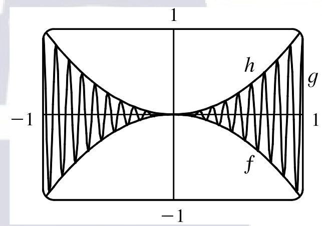
$$=\frac{1}{2\sqrt{3}}$$

35. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

$-1\leq\cos 20\pi x\leq 1\quad\Rightarrow\quad -x^2\leq x^2\cos 20\pi x\leq x^2\quad\Rightarrow\quad f(x)\leq g(x)\leq h(x)$.

So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have

$\lim_{x\to 0}g(x)=0$.

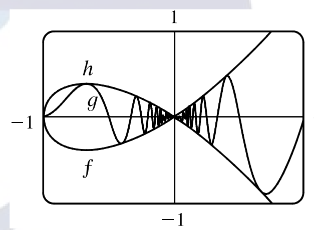


36. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then

$-1\leq\sin(\pi/x)\leq 1\quad\Rightarrow\quad -\sqrt{x^3+x^2}\leq\sqrt{x^3+x^2}\,\sin(\pi/x)\leq\sqrt{x^3+x^2}\quad\Rightarrow$

$f(x)\leq g(x)\leq h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem

we have $\lim_{x\to 0}g(x)=0$.



37. We have $\lim_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\leq f(x)\leq x^2-4x+7$ for $x\geq 0$, $\lim_{x\to 4}f(x)=7$ by the Squeeze Theorem.

38. We have $\lim_{x\to 1}(2x)=2(1)=2$ and $\lim_{x\to 1}(x^4-x^2+2)=1^4-1^2+2=2$. Since $2x\leq g(x)\leq x^4-x^2+2$ for all x, $\lim_{x\to 1}g(x)=2$ by the Squeeze Theorem.

39. $-1\leq\cos(2/x)\leq 1\quad\Rightarrow\quad -x^4\leq x^4\cos(2/x)\leq x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0}x^4=0$, we have $\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

40. $-1 \le \sin(\pi/x) \le 1 \quad \Rightarrow \quad e^{-1} \le e^{\sin(\pi/x)} \le e^1 \quad \Rightarrow \quad \sqrt{x}/e \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$. Since $\lim\limits_{x\to 0^+} (\sqrt{x}/e) = 0$ and $\lim\limits_{x\to 0^+} (\sqrt{x}\, e) = 0$, we have $\lim\limits_{x\to 0^+} \left[\sqrt{x}\, e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

41. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x\to 3^+} (2x + |x-3|) = \lim\limits_{x\to 3^+} (2x + x - 3) = \lim\limits_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$ and

$\lim\limits_{x\to 3^-} (2x + |x-3|) = \lim\limits_{x\to 3^-} (2x + 3 - x) = \lim\limits_{x\to 3^-} (x+3) = 3 + 3 = 6$. Since the left and right limits are equal, $\lim\limits_{x\to 3} (2x + |x-3|) = 6$.

42. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$\lim\limits_{x\to -6^+} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^+} \dfrac{2(x+6)}{x+6} = 2$ and $\lim\limits_{x\to -6^-} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^-} \dfrac{2(x+6)}{-(x+6)} = -2$

The left and right limits are different, so $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

43. $\left|2x^3 - x^2\right| = \left|x^2(2x-1)\right| = \left|x^2\right| \cdot |2x-1| = x^2\,|2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$

So $\left|2x^3 - x^2\right| = x^2[-(2x-1)]$ for $x < 0.5$.

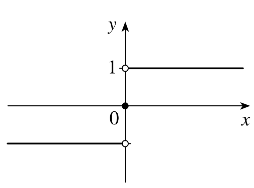
Thus, $\lim\limits_{x\to 0.5^-} \dfrac{2x-1}{|2x^3 - x^2|} = \lim\limits_{x\to 0.5^-} \dfrac{2x-1}{x^2[-(2x-1)]} = \lim\limits_{x\to 0.5^-} \dfrac{-1}{x^2} = \dfrac{-1}{(0.5)^2} = \dfrac{-1}{0.25} = -4$.

44. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x\to -2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x\to -2} \dfrac{2+x}{2+x} = \lim\limits_{x\to -2} 1 = 1$.

45. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

46. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x\to 0^+} 0 = 0$.

47. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x\to 0^+} \operatorname{sgn} x = \lim\limits_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x\to 0^-} \operatorname{sgn} x = \lim\limits_{x\to 0^-} -1 = -1$.

(iii) Since $\lim\limits_{x\to 0^-} \operatorname{sgn} x \ne \lim\limits_{x\to 0^+} \operatorname{sgn} x$, $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim\limits_{x\to 0} |\operatorname{sgn} x| = \lim\limits_{x\to 0} 1 = 1$.

48. (a) $g(x) = \operatorname{sgn}(\sin x) = \begin{cases} -1 & \text{if } \sin x < 0 \\ 0 & \text{if } \sin x = 0 \\ 1 & \text{if } \sin x > 0 \end{cases}$

(i) $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for small positive values of x.

(ii) $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for small negative values of x.

(iii) $\lim_{x \to 0} g(x)$ does not exist since $\lim_{x \to 0^+} g(x) \neq \lim_{x \to 0^-} g(x)$.

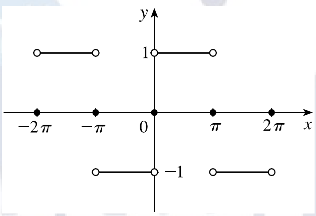
(iv) $\lim_{x \to \pi^+} g(x) = \lim_{x \to \pi^+} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for values of x slightly greater than π.

(v) $\lim_{x \to \pi^-} g(x) = \lim_{x \to \pi^-} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for values of x slightly less than π.

(vi) $\lim_{x \to \pi} g(x)$ does not exist since $\lim_{x \to \pi^+} g(x) \neq \lim_{x \to \pi^-} g(x)$.

(b) The sine function changes sign at every integer multiple of π, so the signum function equals 1 on one side and -1 on the other side of $n\pi$, n an integer. Thus, $\lim_{x \to a} g(x)$ does not exist for $a = n\pi$, n an integer.

(c)



49. (a) (i) $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} \dfrac{x^2 + x - 6}{|x - 2|} = \lim_{x \to 2^+} \dfrac{(x+3)(x-2)}{|x-2|}$

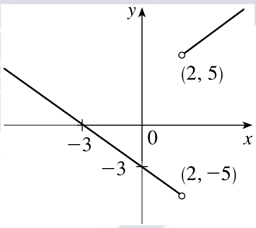
$$= \lim_{x \to 2^+} \frac{(x+3)(x-2)}{x-2} \quad [\text{since } x - 2 > 0 \text{ if } x \to 2^+]$$

$$= \lim_{x \to 2^+} (x+3) = 5$$

(ii) The solution is similar to the solution in part (i), but now $|x - 2| = 2 - x$ since $x - 2 < 0$ if $x \to 2^-$. Thus, $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of g at $x = 2$ are not equal, $\lim_{x \to 2} g(x)$ does not exist.

(c)

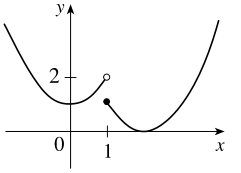


50. (a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$, $\quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of f at $x = 1$ are not equal, $\lim_{x \to 1} f(x)$ does not exist.

(c)



51. For the $\lim\limits_{t\to 2} B(t)$ to exist, the one-sided limits at $t = 2$ must be equal. $\lim\limits_{t\to 2^-} B(t) = \lim\limits_{t\to 2^-} \left(4 - \frac{1}{2}t\right) = 4 - 1 = 3$ and

$\lim\limits_{t\to 2^+} B(t) = \lim\limits_{t\to 2^+} \sqrt{t+c} = \sqrt{2+c}$. Now $3 = \sqrt{2+c} \;\Rightarrow\; 9 = 2 + c \;\Leftrightarrow\; c = 7$.

52. (a) (i) $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} x = 1$

(ii) $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim\limits_{x\to 1^-} g(x) = 1$ and $\lim\limits_{x\to 1^+} g(x) = 1$, we have $\lim\limits_{x\to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

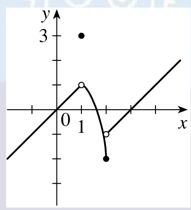
(iv) $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim\limits_{x\to 2} g(x)$ does not exist since $\lim\limits_{x\to 2^-} g(x) \neq \lim\limits_{x\to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$



53. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x\to -2^+} [\![x]\!] = \lim\limits_{x\to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2^-} [\![x]\!] = \lim\limits_{x\to -2^-} (-3) = -3$.

The right and left limits are different, so $\lim\limits_{x\to -2} [\![x]\!]$ does not exist.

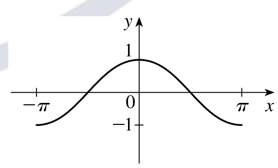
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2.4} [\![x]\!] = \lim\limits_{x\to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n - 1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\;\Leftrightarrow\;$ a is not an integer.

54. (a) See the graph of $y = \cos x$.

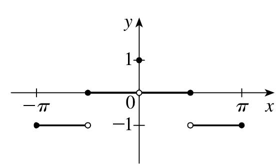


Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.

Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.



(b) (i) $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 0$, so $\lim_{x\to 0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim_{x\to(\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim_{x\to(\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim_{x\to\pi/2} f(x)$ does not exist.

(c) $\lim_{x\to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

55. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim_{x\to 2^-} f(x) = -1$ and $\lim_{x\to 2^+} f(x) = -1$, so $\lim_{x\to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim_{x\to 2} f(x) \neq f(2)$.

56. $\lim_{v\to c^-}\left(L_0\sqrt{1-\dfrac{v^2}{c^2}}\right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

57. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim_{x\to a} p(x) = p(a)$.

58. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Then

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 57]} \quad = r(a).$$

59. $\lim_{x\to 1}[f(x) - 8] = \lim_{x\to 1}\left[\dfrac{f(x)-8}{x-1}\cdot(x-1)\right] = \lim_{x\to 1}\dfrac{f(x)-8}{x-1}\cdot\lim_{x\to 1}(x-1) = 10\cdot 0 = 0$.

Thus, $\lim_{x\to 1} f(x) = \lim_{x\to 1}\{[f(x)-8]+8\} = \lim_{x\to 1}[f(x)-8] + \lim_{x\to 1} 8 = 0 + 8 = 8$.

Note: The value of $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ exists.

60. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x^2\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0} x^2 = 5\cdot 0 = 0$

(b) $\lim_{x\to 0}\dfrac{f(x)}{x} = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0} x = 5\cdot 0 = 0$

61. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

62. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist [Example 10]

but $\lim_{x\to 3}[f(x)+g(x)] = \lim_{x\to 3}([\![x]\!] - [\![x]\!]) = \lim_{x\to 3} 0 = 0$.

63. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.

Thus, either f or g is 0 for any value of x. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}[f(x)g(x)] = \lim_{x\to 0} 0 = 0$.

64. $$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$

$$= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$

$$= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}$$

65. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim_{x\to -2}\left(3x^2+ax+a+3\right) = 0 \;\Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \;\Leftrightarrow\; 12 - 2a + a + 3 = 0 \;\Leftrightarrow\; a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

66. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \;\Leftrightarrow\; r^2 = 1 + 2x - 1 \;\Leftrightarrow\; x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \;\Leftrightarrow\; y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \;\Leftrightarrow\; y = r\sqrt{1-\frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1-\frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

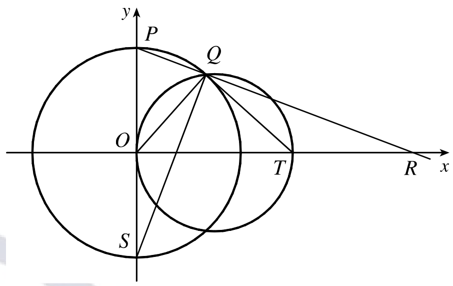
$$y - r = \frac{r\sqrt{1-\frac{1}{4}r^2}-r}{\frac{1}{2}r^2-0}(x-0).$$ We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)}{1-\frac{1}{4}r^2-1} = 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)$$

Now we take the limit as $r \to 0^+$: $\lim_{r\to 0^+} x = \lim_{r\to 0^+} 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) = \lim_{r\to 0^+} 2\left(\sqrt{1}+1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

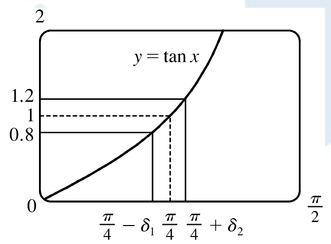
1. If $|f(x) - 1| < 0.2$, then $-0.2 < f(x) - 1 < 0.2 \quad\Rightarrow\quad 0.8 < f(x) < 1.2$. From the graph, we see that the last inequality is true if $0.7 < x < 1.1$, so we can choose $\delta = \min\{1 - 0.7, 1.1 - 1\} = \min\{0.3, 0.1\} = 0.1$ (or any smaller positive number).

2. If $|f(x) - 2| < 0.5$, then $-0.5 < f(x) - 2 < 0.5 \quad\Rightarrow\quad 1.5 < f(x) < 2.5$. From the graph, we see that the last inequality is true if $2.6 < x < 3.8$, so we can take $\delta = \min\{3 - 2.6, 3.8 - 3\} = \min\{0.4, 0.8\} = 0.4$ (or any smaller positive number). Note that $x \neq 3$.

3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

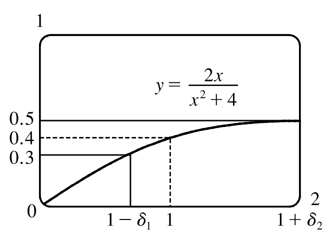
4. The leftmost question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

5.

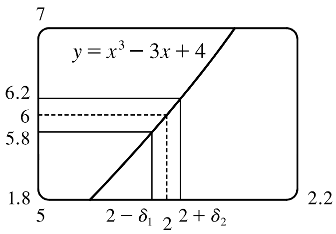


From the graph, we find that $y = \tan x = 0.8$ when $x \approx 0.675$, so $\frac{\pi}{4} - \delta_1 \approx 0.675 \quad\Rightarrow\quad \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106$. Also, $y = \tan x = 1.2$ when $x \approx 0.876$, so $\frac{\pi}{4} + \delta_2 \approx 0.876 \quad\Rightarrow\quad \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906$. Thus, we choose $\delta = 0.0906$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

6.

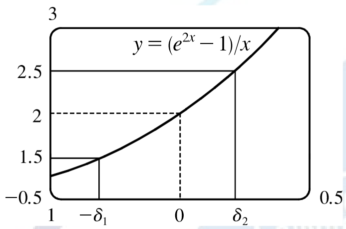


From the graph, we find that $y = 2x/(x^2 + 4) = 0.3$ when $x = \frac{2}{3}$, so $1 - \delta_1 = \frac{2}{3} \quad\Rightarrow\quad \delta_1 = \frac{1}{3}$. Also, $y = 2x/(x^2 + 4) = 0.5$ when $x = 2$, so $1 + \delta_2 = 2 \quad\Rightarrow\quad \delta_2 = 1$. Thus, we choose $\delta = \frac{1}{3}$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

7. 

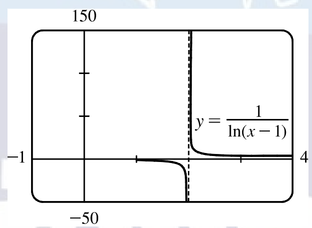
From the graph with $\varepsilon = 0.2$, we find that $y = x^3 - 3x + 4 = 5.8$ when $x \approx 1.9774$, so $2 - \delta_1 \approx 1.9774 \Rightarrow \delta_1 \approx 0.0226$. Also, $y = x^3 - 3x + 4 = 6.2$ when $x \approx 2.022$, so $2 + \delta_2 \approx 2.0219 \Rightarrow \delta_2 \approx 0.0219$. Thus, we choose $\delta = 0.0219$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

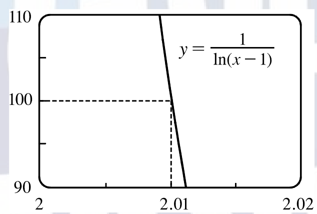
For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.0112$ and $\delta_2 \approx 0.0110$, so we choose $\delta = 0.011$ (or any smaller positive number).

8. 

From the graph with $\varepsilon = 0.5$, we find that $y = (e^{2x} - 1)/x = 1.5$ when $x \approx -0.303$, so $\delta_1 \approx 0.303$. Also, $y = (e^{2x} - 1)/x = 2.5$ when $x \approx 0.215$, so $\delta_2 \approx 0.215$. Thus, we choose $\delta = 0.215$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.052$ and $\delta_2 \approx 0.048$, so we choose $\delta = 0.048$ (or any smaller positive number).

9. (a) 

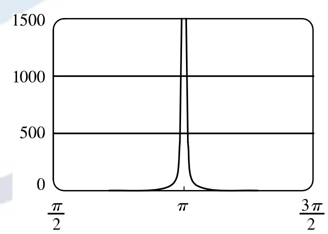


The first graph of $y = \dfrac{1}{\ln(x-1)}$ shows a vertical asymptote at $x = 2$. The second graph shows that $y = 100$ when $x \approx 2.01$ (more accurately, 2.01005). Thus, we choose $\delta = 0.01$ (or any smaller positive number).

(b) From part (a), we see that as x gets closer to 2 from the right, y increases without bound. In symbols,

$$\lim_{x \to 2^+} \frac{1}{\ln(x-1)} = \infty.$$

10. We graph $y = \csc^2 x$ and $y = 500$. The graphs intersect at $x \approx 3.186$, so we choose $\delta = 3.186 - \pi \approx 0.044$. Thus, if $0 < |x - \pi| < 0.044$, then $\csc^2 x > 500$. Similarly, for $M = 1000$, we get $\delta = 3.173 - \pi \approx 0.031$.



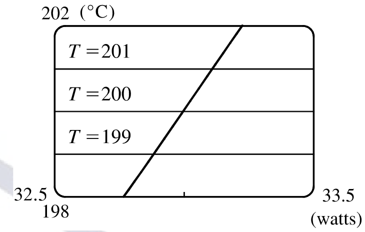
11. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow r = \sqrt{\frac{1000}{\pi}} \quad (r > 0) \quad \approx 17.8412$ cm.

(b) $|A - 1000| \le 5 \Rightarrow -5 \le \pi r^2 - 1000 \le 5 \Rightarrow 1000 - 5 \le \pi r^2 \le 1000 + 5 \Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

12. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$

$0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts ($w > 0$)



(b) From the graph, $199 \le T \le 201 \Rightarrow 32.89 < w < 33.11$.

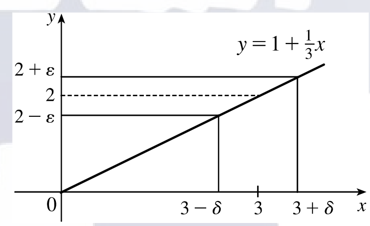
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

13. (a) $|4x-8| = 4\,|x-2| < 0.1 \Leftrightarrow |x-2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

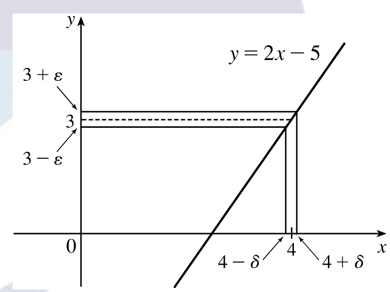
(b) $|4x-8| = 4\,|x-2| < 0.01 \Leftrightarrow |x-2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

14. $|(5x-7)-3| = |5x-10| = |5(x-2)| = 5\,|x-2|$. We must have $|f(x)-L| < \varepsilon$, so $5\,|x-2| < \varepsilon \Leftrightarrow |x-2| < \varepsilon/5$. Thus, choose $\delta = \varepsilon/5$. For $\varepsilon = 0.1$, $\delta = 0.02$; for $\varepsilon = 0.05$, $\delta = 0.01$; for $\varepsilon = 0.01$, $\delta = 0.002$.

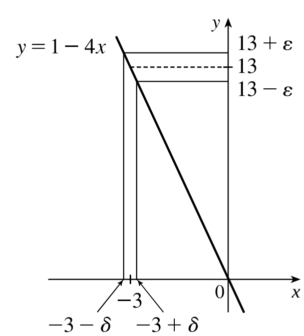
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|(1+\frac{1}{3}x)-2\right| < \varepsilon$. But $\left|(1+\frac{1}{3}x)-2\right| < \varepsilon \Leftrightarrow \left|\frac{1}{3}x - 1\right| < \varepsilon \Leftrightarrow \left|\frac{1}{3}\right|\,|x-3| < \varepsilon \Leftrightarrow |x-3| < 3\varepsilon$. So if we choose $\delta = 3\varepsilon$, then $0 < |x-3| < \delta \Rightarrow \left|(1+\frac{1}{3}x)-2\right| < \varepsilon$. Thus, $\lim\limits_{x\to 3}(1+\frac{1}{3}x) = 2$ by the definition of a limit.



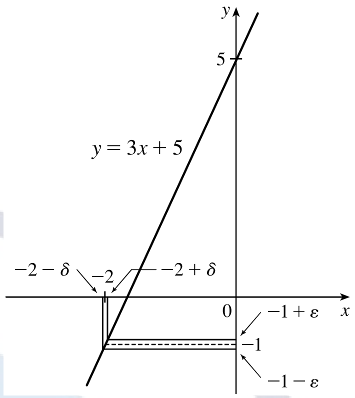
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-4| < \delta$, then $|(2x-5)-3| < \varepsilon$. But $|(2x-5)-3| < \varepsilon \Leftrightarrow |2x-8| < \varepsilon \Leftrightarrow |2|\,|x-4| < \varepsilon \Leftrightarrow |x-4| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x-4| < \delta \Rightarrow |(2x-5)-3| < \varepsilon$. Thus, $\lim\limits_{x\to 4}(2x-5) = 3$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$. But $|(1-4x)-13| < \varepsilon \Leftrightarrow |-4x-12| < \varepsilon \Leftrightarrow |-4|\;|x+3| < \varepsilon \Leftrightarrow |x-(-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x-(-3)| < \delta \Rightarrow |(1-4x)-13| < \varepsilon$. Thus, $\lim\limits_{x\to -3}(1-4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(3x + 5) - (-1)| < \varepsilon$. But $|(3x + 5) - (-1)| < \varepsilon \;\Leftrightarrow\; |3x + 6| < \varepsilon \;\Leftrightarrow\; |3|\,|x + 2| < \varepsilon \;\Leftrightarrow\; |x + 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x + 2| < \delta \;\Rightarrow\; |(3x + 5) - (-1)| < \varepsilon$. Thus, $\lim\limits_{x \to -2} (3x + 5) = -1$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. But $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon \;\Leftrightarrow\; \left|\dfrac{4x - 4}{3}\right| < \varepsilon \;\Leftrightarrow\; \left|\frac{4}{3}\right|\,|x - 1| < \varepsilon \;\Leftrightarrow\; |x - 1| < \frac{3}{4}\varepsilon$. So if we choose $\delta = \frac{3}{4}\varepsilon$, then $0 < |x - 1| < \delta \;\Rightarrow\; \left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 1} \dfrac{2 + 4x}{3} = 2$ by the definition of a limit.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 10| < \delta$, then $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. But $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon \;\Leftrightarrow\; \left|8 - \frac{4}{5}x\right| < \varepsilon \;\Leftrightarrow\; \left|-\frac{4}{5}\right|\,|x - 10| < \varepsilon \;\Leftrightarrow\; |x - 10| < \frac{5}{4}\varepsilon$. So if we choose $\delta = \frac{5}{4}\varepsilon$, then $0 < |x - 10| < \delta \;\Rightarrow\; \left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. Thus, $\lim\limits_{x \to 10} (3 - \frac{4}{5}x) = -5$ by the definition of a limit.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $\left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon \;\Leftrightarrow\; \left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \;\Leftrightarrow\; |x + 2 - 6| < \varepsilon \quad [x \neq 4] \;\Leftrightarrow\; |x - 4| < \varepsilon$. So choose $\delta = \varepsilon$. Then $0 < |x - 4| < \delta \;\Rightarrow\; |x - 4| < \varepsilon \;\Rightarrow\; |x + 2 - 6| < \varepsilon \;\Rightarrow\; \left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \quad [x \neq 4] \;\Rightarrow\; \left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4} = 6$.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x + 1.5| < \delta$, then $\left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon \;\Leftrightarrow\; \left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \;\Leftrightarrow\; |3 - 2x - 6| < \varepsilon \quad [x \neq -1.5] \;\Leftrightarrow\; |-2x - 3| < \varepsilon \;\Leftrightarrow\; |-2|\,|x + 1.5| < \varepsilon \;\Leftrightarrow\; |x + 1.5| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $0 < |x + 1.5| < \delta \;\Rightarrow\; |x + 1.5| < \varepsilon/2 \;\Rightarrow\; |-2|\,|x + 1.5| < \varepsilon \;\Rightarrow\; |-2x - 3| < \varepsilon \;\Rightarrow\; |3 - 2x - 6| < \varepsilon \;\Rightarrow\; \left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \quad [x \neq -1.5] \;\Rightarrow\; \left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon$.

By the definition of a limit, $\lim\limits_{x \to -1.5} \dfrac{9 - 4x^2}{3 + 2x} = 6$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \quad \Leftrightarrow \quad x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^2 - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \quad \Leftrightarrow \quad |x|^3 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x \to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\big||x| - 0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x \to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < x - (-6) < \delta$, then $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. But $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon \quad \Leftrightarrow$ $\sqrt[8]{6 + x} < \varepsilon \quad \Leftrightarrow \quad 6 + x < \varepsilon^8 \quad \Leftrightarrow \quad x - (-6) < \varepsilon^8$. So if we choose $\delta = \varepsilon^8$, then $0 < x - (-6) < \delta \quad \Rightarrow$ $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to -6^+} \sqrt[8]{6 + x} = 0$ by the definition of a right-hand limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\left(x^2 - 4x + 5\right) - 1\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2 - 4x + 4\right| < \varepsilon \quad \Leftrightarrow$ $\left|(x - 2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \quad \Leftrightarrow \quad |x - 2| < \sqrt{\varepsilon} \quad \Leftrightarrow \quad \left|(x - 2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 2} \left(x^2 - 4x + 5\right) = 1$ by the definition of a limit.

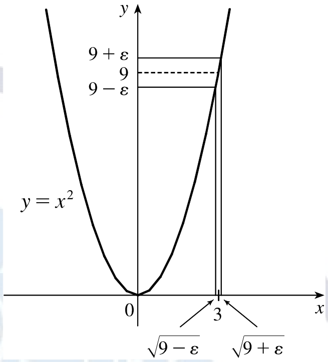
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon$. But $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon \quad \Leftrightarrow$ $\left|x^2 + 2x - 8\right| < \varepsilon \quad \Leftrightarrow \quad |x + 4|\,|x - 2| < \varepsilon$. Thus our goal is to make $|x - 2|$ small enough so that its product with $|x + 4|$ is less than ε. Suppose we first require that $|x - 2| < 1$. Then $-1 < x - 2 < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 5 < x + 4 < 7 \quad \Rightarrow$ $|x + 4| < 7$, and this gives us $7\,|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \varepsilon/7$. Choose $\delta = \min\{1, \varepsilon/7\}$. Then if $0 < |x - 2| < \delta$, we have $|x - 2| < \varepsilon/7$ and $|x + 4| < 7$, so $\left|(x^2 + 2x - 7) - 1\right| = |(x + 4)(x - 2)| = |x + 4|\,|x - 2| < 7(\varepsilon/7) = \varepsilon$, as desired. Thus, $\lim\limits_{x \to 2}(x^2 + 2x - 7) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(x^2 - 1\right) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \quad \Rightarrow \quad -5 < x - 2 < -3 \quad \Rightarrow$ $|x - 2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \quad \Rightarrow \quad |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $\left|\left(x^2 - 1\right) - 3\right| = |(x + 2)(x - 2)| = |x + 2|\,|x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to -2}(x^2 - 1) = 3$.

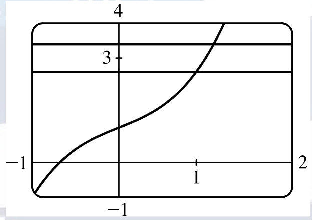
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x - 2)\left(x^2 + 2x + 4\right)\right|$. If $|x - 2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < 19\,|x - 2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x - 2| < \delta \quad \Rightarrow$ $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \Rightarrow -2 < x-3 < 2 \Rightarrow 4 < x+3 < 8 \Rightarrow |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2-9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9-\varepsilon}$ and $\delta_2 = \sqrt{9+\varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9+\varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then

$\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x-a < \frac{1}{2}a \quad \Rightarrow \quad \frac{1}{2}a < x < \frac{3}{2}a \quad \Rightarrow \quad \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \quad \Rightarrow \quad \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so

$\left|\sqrt{x}-\sqrt{a}\right| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon$. Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim\limits_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \quad \Rightarrow \quad |H(t)-L| < \frac{1}{2} \quad \Leftrightarrow$ $L-\frac{1}{2} < H(t) < L+\frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L+\frac{1}{2} \quad \Rightarrow \quad L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L-\frac{1}{2} < 0 \quad \Rightarrow \quad L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim\limits_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \quad \Rightarrow \quad |f(x)-L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0-L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1-L| < \frac{1}{2}$. Hence, $1-L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \quad \Rightarrow \quad |f(x)-L| < \varepsilon$. Then $a-\delta < x < a \quad \Rightarrow \quad 0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a+\delta \quad \Rightarrow$ $0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a-\delta_1 < x < a \quad \Rightarrow \quad |f(x)-L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a+\delta_2 \quad \Rightarrow$ $|f(x)-L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \quad \Rightarrow \quad a-\delta_1 < x < a$ or $a < x < a+\delta_2$ so $|f(x)-L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \quad \Leftrightarrow \quad (x+3)^4 < \dfrac{1}{10{,}000} \quad \Leftrightarrow \quad |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \quad \Leftrightarrow \quad |x-(-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \quad \Rightarrow \quad 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \quad \Leftrightarrow$ $(x+3)^4 < \dfrac{1}{M} \quad \Leftrightarrow \quad |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \quad \Rightarrow \quad \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim\limits_{x \to 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \;\Rightarrow\; f(x) > M + 1 - c$. Since $\lim\limits_{x \to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \;\Rightarrow\; |g(x) - c| < 1 \;\Rightarrow\; g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \;\Rightarrow\; f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim\limits_{x \to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x \to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \;\Rightarrow\; |g(x) - c| < c/2 \;\Rightarrow\; g(x) > c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \;\Rightarrow\; f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \;\Rightarrow\; f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x \to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x \to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \;\Rightarrow\; |g(x) - c| < -c/2 \;\Rightarrow\; g(x) < c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \;\Rightarrow\; f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \;\Rightarrow\; 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \;\Rightarrow\; f(x) > 2N/c \;\Rightarrow\; f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim\limits_{x \to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim\limits_{x \to 4} f(x) = f(4)$.

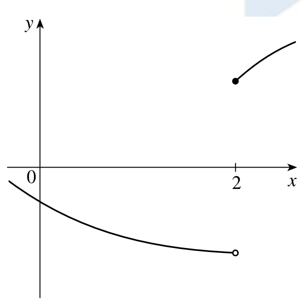
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

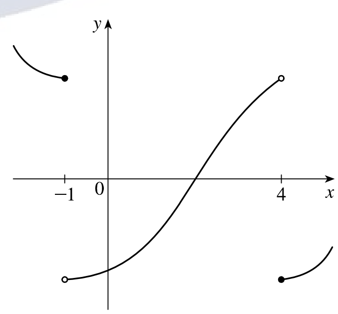
(b) f is continuous from the left at -2 since $\lim\limits_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x \to 2^+} f(x) = f(2)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. From the graph of g, we see that g is continuous on the intervals $[-3, -2)$, $(-2, -1)$, $(-1, 0]$, $(0, 1)$, and $(1, 3]$.

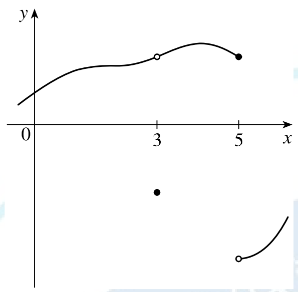
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim\limits_{x \to 2^+} f(x) = f(2)$.



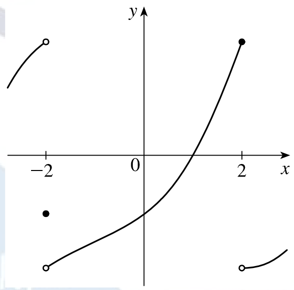
6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim\limits_{x \to -1^-} f(x) = f(-1)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$.



7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.

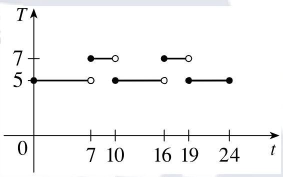


8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim\limits_{x \to -2^-} f(x) \neq f(-2)$ and $\lim\limits_{x \to -2^+} f(x) \neq f(-2)$. It must also show that $\lim\limits_{x \to 2^-} f(x) = f(2)$ and $\lim\limits_{x \to 2^+} f(x) \neq f(2)$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function T has jump discontinuities at $t = 7$, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.



10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \left(x + 2x^3\right)^4 = \left(\lim\limits_{x \to -1} x + 2 \lim\limits_{x \to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1).$

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim\limits_{t \to 2} g(t) = \lim\limits_{t \to 2} \dfrac{t^2 + 5t}{2t + 1} = \dfrac{\lim\limits_{t \to 2}(t^2 + 5t)}{\lim\limits_{t \to 2}(2t + 1)} = \dfrac{\lim\limits_{t \to 2} t^2 + 5 \lim\limits_{t \to 2} t}{2 \lim\limits_{t \to 2} t + \lim\limits_{t \to 2} 1} = \dfrac{2^2 + 5(2)}{2(2) + 1} = \dfrac{14}{5} = g(2).$

By the definition of continuity, g is continuous at $a = 2$.

13. $\lim\limits_{v\to 1} p(v) = \lim\limits_{v\to 1} 2\sqrt{3v^2+1} = 2\lim\limits_{v\to 1}\sqrt{3v^2+1} = 2\sqrt{\lim\limits_{v\to 1}(3v^2+1)} = 2\sqrt{3\lim\limits_{v\to 1} v^2 + \lim\limits_{v\to 1} 1}$

$$= 2\sqrt{3(1)^2+1} = 2\sqrt{4} = 4 = p(1)$$

By the definition of continuity, p is continuous at $a = 1$.

14. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) = 3\lim\limits_{x\to 2} x^4 - 5\lim\limits_{x\to 2} x + \sqrt[3]{\lim\limits_{x\to 2}(x^2+4)}$

$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48 - 10 + 2 = 40 = f(2)$$

By the definition of continuity, f is continuous at $a = 2$.

15. For $a > 4$, we have

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a}(x + \sqrt{x-4}) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4} && \text{[Limit Law 1]} \\ &= a + \sqrt{\lim_{x\to a} x - \lim_{x\to a} 4} && \text{[8, 11, and 2]} \\ &= a + \sqrt{a-4} && \text{[8 and 7]} \\ &= f(a) \end{aligned}$$

So f is continuous at $x = a$ for every a in $(4, \infty)$. Also, $\lim\limits_{x\to 4^+} f(x) = 4 = f(4)$, so f is continuous from the right at 4.

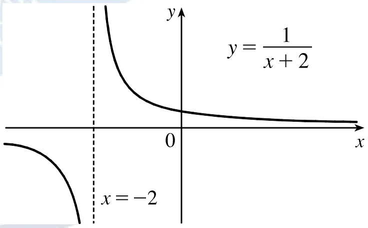
Thus, f is continuous on $[4, \infty)$.

16. For $a < -2$, we have

$$\begin{aligned} \lim_{x\to a} g(x) &= \lim_{x\to a}\frac{x-1}{3x+6} = \frac{\lim\limits_{x\to a}(x-1)}{\lim\limits_{x\to a}(3x+6)} && \text{[Limit Law 5]} \\ &= \frac{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 1}{3\lim\limits_{x\to a} x + \lim\limits_{x\to a} 6} && \text{[2, 1, and 3]} \\ &= \frac{a-1}{3a+6} && \text{[8 and 7]} \end{aligned}$$

Thus, g is continuous at $x = a$ for every a in $(-\infty, -2)$; that is, g is continuous on $(-\infty, -2)$.

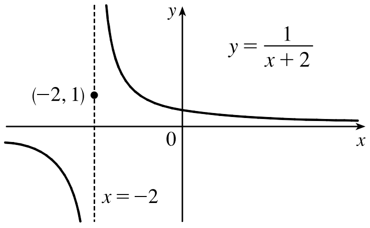
17. $f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.



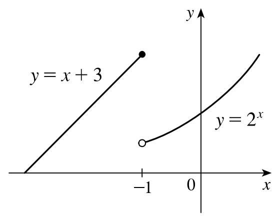
18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

Here $f(-2) = 1$, but $\lim\limits_{x\to -2^-} f(x) = -\infty$ and $\lim\limits_{x\to -2^+} f(x) = \infty$,

so $\lim\limits_{x\to -2} f(x)$ does not exist and f is discontinuous at -2.



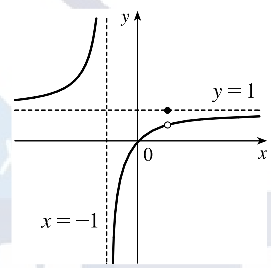
19. $f(x) = \begin{cases} x+3 & \text{if } x \le -1 \\ 2^x & \text{if } x > -1 \end{cases}$



$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (x+3) = -1+3 = 2$ and

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} 2^x = 2^{-1} = \frac{1}{2}$. Since the left-hand and the right-hand limits of f at -1 are not equal, $\lim\limits_{x \to -1} f(x)$ does not exist, and f is discontinuous at -1.

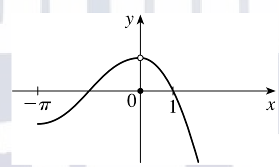
20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \ne 1 \\ 1 & \text{if } x = 1 \end{cases}$



$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2},$$

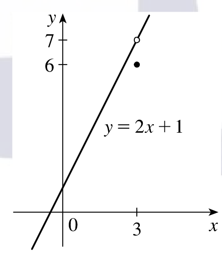
but $f(1) = 1$, so f is discontinous at 1.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim\limits_{x \to 0} f(x) = 1$, but $f(0) = 0 \ne 1$, so f is discontinuous at 0.

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$



$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x+1)(x-3)}{x-3} = \lim_{x \to 3} (2x+1) = 7,$$

but $f(3) = 6$, so f is discontinuous at 3.

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2} = x + 1$ for $x \ne 2$. Since $\lim\limits_{x \to 2} f(x) = 2 + 1 = 3$, define $f(2) = 3$. Then f is continuous at 2.

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4} = \dfrac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} = \dfrac{x^2 + 2x + 4}{x + 2}$ for $x \ne 2$. Since $\lim\limits_{x \to 2} f(x) = \dfrac{4+4+4}{2+2} = 3$, define $f(2) = 3$. Then f is continuous at 2.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$ is a rational function, so it is continuous on its domain, $(-\infty, \infty)$, by Theorem 5(b).

26. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1} = \dfrac{x^2 + 1}{(2x+1)(x-1)}$ is a rational function, so it is continuous on its domain, $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$, by Theorem 5(b).

27. $x^3 - 2 = 0 \quad \Rightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2}$, so $Q(x) = \dfrac{\sqrt[3]{x-2}}{x^3-2}$ has domain $\left(-\infty, \sqrt[3]{2}\right) \cup \left(\sqrt[3]{2}, \infty\right)$. Now $x^3 - 2$ is continuous everywhere by Theorem 5(a) and $\sqrt[3]{x-2}$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, Q is continuous on its domain by part 5 of Theorem 4.

28. The domain of $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$ is $(-\infty, \infty)$ since the denominator is never 0 $[\cos \pi t \geq -1 \quad \Rightarrow \quad 2 + \cos \pi t \geq 1]$. By Theorems 7 and 9, $e^{\sin t}$ and $\cos \pi t$ are continuous on $\mathbb{R}$. By part 1 of Theorem 4, $2 + \cos \pi t$ is continuous on $\mathbb{R}$ and by part 5 of Theorem 4, R is continuous on $\mathbb{R}$.

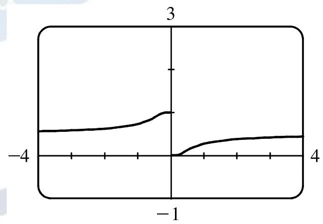
29. By Theorem 5(a), the polynomial $1 + 2t$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\arcsin x$ is continuous on its domain, $[-1, 1]$. By Theorem 9, $A(t) = \arcsin(1 + 2t)$ is continuous on its domain, which is $\{t \mid -1 \leq 1 + 2t \leq 1\} = \{t \mid -2 \leq 2t \leq 0\} = \{t \mid -1 \leq t \leq 0\} = [-1, 0]$.

30. By Theorem 7, the trigonometric function $\tan x$ is continuous on its domain, $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - x^2}$ is continuous on its domain $[-2, 2]$. By part 5 of Theorem 4, $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$ is continuous on its domain, $(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$.

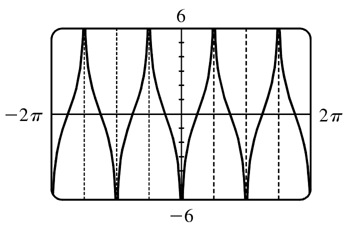
31. $M(x) = \sqrt{1 + \dfrac{1}{x}} = \sqrt{\dfrac{x+1}{x}}$ is defined when $\dfrac{x+1}{x} \geq 0 \quad \Rightarrow \quad x + 1 \geq 0$ and $x > 0$ or $x + 1 \leq 0$ and $x < 0 \quad \Rightarrow \quad x > 0$ or $x \leq -1$, so M has domain $(-\infty, -1] \cup (0, \infty)$. M is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

32. By Theorems 7 and 9, the composite function e^{-r^2} is continuous on $\mathbb{R}$. By part 1 of Theorem 4, $1 + e^{-r^2}$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\tan^{-1}$ is continuous on its domain, $\mathbb{R}$. By Theorem 9, the composite function $N(r) = \tan^{-1}\left(1 + e^{-r^2}\right)$ is continuous on its domain, $\mathbb{R}$.

33. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



34. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where k is any integer. The function $y = \ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln\left(\tan^2 x\right)$ is discontinuous at $x = \frac{\pi}{2} n$, n any integer.



35. Because x is continuous on $\mathbb{R}$ and $\sqrt{20-x^2}$ is continuous on its domain, $-\sqrt{20} \le x \le \sqrt{20}$, the product $f(x) = x\sqrt{20-x^2}$ is continuous on $-\sqrt{20} \le x \le \sqrt{20}$. The number 2 is in that domain, so f is continuous at 2, and $\lim\limits_{x\to 2} f(x) = f(2) = 2\sqrt{16} = 8$.

36. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

37. The function $f(x) = \ln\left(\dfrac{5-x^2}{1+x}\right)$ is continuous throughout its domain because it is the composite of a logarithm function and a rational function. For the domain of f, we must have $\dfrac{5-x^2}{1+x} > 0$, so the numerator and denominator must have the same sign, that is, the domain is $(-\infty, -\sqrt{5}\,] \cup (-1, \sqrt{5}\,]$. The number 1 is in that domain, so f is continuous at 1, and $\lim\limits_{x\to 1} f(x) = f(1) = \ln\dfrac{5-1}{1+1} = \ln 2$.

38. The function $f(x) = 3^{\sqrt{x^2-2x-4}}$ is continuous throughout its domain because it is the composite of an exponential function, a root function, and a polynomial. Its domain is

$$\begin{aligned}\left\{x \mid x^2-2x-4 \ge 0\right\} &= \left\{x \mid x^2-2x+1 \ge 5\right\} = \left\{x \mid (x-1)^2 \ge 5\right\} \\ &= \left\{x \,\middle|\, |x-1| \ge \sqrt{5}\,\right\} = (-\infty, 1-\sqrt{5}\,] \cup [1+\sqrt{5}, \infty)\end{aligned}$$

The number 4 is in that domain, so f is continuous at 4, and $\lim\limits_{x\to 4} f(x) = f(4) = 3^{\sqrt{16-8-4}} = 3^2 = 9$.

39. $f(x) = \begin{cases} 1-x^2 & \text{if } x \le 1 \\ \ln x & \text{if } x > 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial $1-x^2$ on $(-\infty, 1]$, f is continuous on $(-\infty, 1]$.

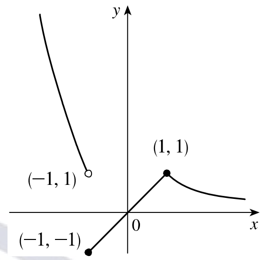
By Theorem 7, since $f(x)$ equals the logarithm function $\ln x$ on $(1, \infty)$, f is continuous on $(1, \infty)$.

At $x = 1$, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (1-x^2) = 1-1^2 = 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \ln x = \ln 1 = 0$. Thus, $\lim\limits_{x\to 1} f(x)$ exists and equals 0. Also, $f(1) = 1-1^2 = 0$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \ge \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

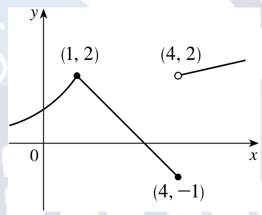
41. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$



f is continuous on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, where it is a polynomial, a polynomial, and a rational function, respectively.

Now $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} x^2 = 1$ and $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} x = -1$, so f is discontinuous at -1. Since $f(-1) = -1$, f is continuous from the right at -1. Also, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{x} = 1 = f(1)$, so f is continuous at 1.

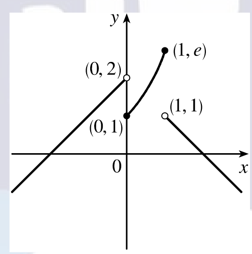
42. $f(x) = \begin{cases} 2^x & \text{if } x \le 1 \\ 3 - x & \text{if } 1 < x \le 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, where it is an exponential, a polynomial, and a root function, respectively.

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2^x = 2$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - x) = 2$. Since $f(1) = 2$ we have continuity at 1. Also, $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (3 - x) = -1 = f(4)$ and $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \sqrt{x} = 2$, so f is discontinuous at 4, but it is continuous from the left at 4.

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} e^x = e$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

44. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

$\lim\limits_{r \to R^-} F(r) = \lim\limits_{r \to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r \to R^+} F(r) = \lim\limits_{r \to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r \to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

45. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^3 - cx) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

46. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim\limits_{x\to 2^-} (x+2) = 2 + 2 = 4$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - 2b + 3 = 4$, or $\boldsymbol{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\boldsymbol{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rcr} -8a + 4b & = & -2 \\ 10a - 4b & = & 3 \\ \hline 2a & = & 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

47. If f and g are continuous and $g(2) = 6$, then $\lim\limits_{x\to 2} [3f(x) + f(x)\,g(x)] = 36$ $\Rightarrow$

$3\lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} f(x) \cdot \lim\limits_{x\to 2} g(x) = 36$ $\Rightarrow$ $3f(2) + f(2)\cdot 6 = 36$ $\Rightarrow$ $9f(2) = 36$ $\Rightarrow$ $f(2) = 4$.

48. (a) $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$, so $(f \circ g)(x) = f(g(x)) = f(1/x^2) = 1\,/(1/x^2) = x^2$.

(b) The domain of $f \circ g$ is the set of numbers x in the domain of g (all nonzero reals) such that $g(x)$ is in the domain of f (also all nonzero reals). Thus, the domain is $\left\{ x \;\middle|\; x \ne 0 \text{ and } \dfrac{1}{x^2} \ne 0 \right\} = \{x \mid x \ne 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $f \circ g$ is the composite of two rational functions, it is continuous throughout its domain; that is, everywhere except $x = 0$.

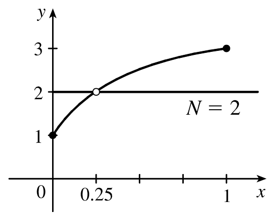
49. (a) $f(x) = \dfrac{x^4 - 1}{x - 1} = \dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)(x-1)}{x-1} = (x^2+1)(x+1)$ [or $x^3 + x^2 + x + 1$]

for $x \ne 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \ne 1$ and is continuous on $\mathbb{R}$.

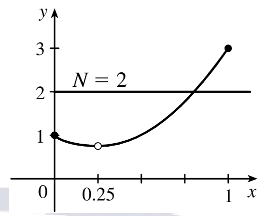
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2} = \dfrac{x(x^2 - x - 2)}{x - 2} = \dfrac{x(x-2)(x+1)}{x-2} = x(x+1)$ [or $x^2 + x$] for $x \ne 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \ne 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} [\![\sin x]\!] = \lim\limits_{x\to\pi^-} 0 = 0$ and $\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+} [\![\sin x]\!] = \lim\limits_{x\to\pi^+} (-1) = -1$, so $\lim\limits_{x\to\pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

50.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

51. $f(x) = x^2 + 10 \sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

52. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

53. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

54. The equation $\ln x = x - \sqrt{x}$ is equivalent to the equation $\ln x - x + \sqrt{x} = 0$. $f(x) = \ln x - x + \sqrt{x}$ is continuous on the interval $[2, 3]$, $f(2) = \ln 2 - 2 + \sqrt{2} \approx 0.107$, and $f(3) = \ln 3 - 3 + \sqrt{3} \approx -0.169$. Since $f(2) > 0 > f(3)$, there is a number c in $(2, 3)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - x + \sqrt{x} = 0$, or $\ln x = x - \sqrt{x}$, in the interval $(2, 3)$.

55. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

56. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

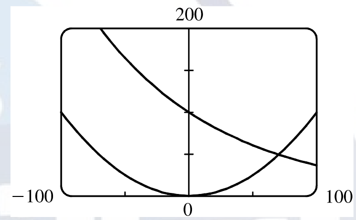
57. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

58. (a) $f(x) = \ln x - 3 + 2x$ is continuous on the interval $[1, 2]$, $f(1) = -1 < 0$, and $f(2) = \ln 2 + 1 \approx 1.7 > 0$. Since $-1 < 0 < 1.7$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - 3 + 2x = 0$, or $\ln x = 3 - 2x$, in the interval $(1, 2)$.

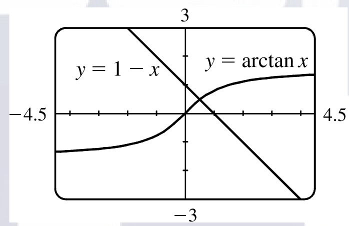
(b) $f(1.34) \approx -0.03 < 0$ and $f(1.35) \approx 0.0001 > 0$, so there is a root between 1.34 and 1.35, that is, in the interval $(1.34, 1.35)$.

59. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.

60. (a) Let $f(x) = \arctan x + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = \frac{\pi}{4} > 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. This implies that $\arctan c = 1 - c$.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 0.520$, correct to three decimal places.

61. Let $f(x) = \sin x^3$. Then f is continuous on $[1, 2]$ since f is the composite of the sine function and the cubing function, both of which are continuous on $\mathbb{R}$. The zeros of the sine are at $n\pi$, so we note that $0 < 1 < \pi < \frac{3}{2}\pi < 2\pi < 8 < 3\pi$, and that the pertinent cube roots are related by $1 < \sqrt[3]{\frac{3}{2}\pi}$ [call this value A] < 2. [By observation, we might notice that $x = \sqrt[3]{\pi}$ and $x = \sqrt[3]{2\pi}$ are zeros of f.]

Now $f(1) = \sin 1 > 0$, $f(A) = \sin \frac{3}{2}\pi = -1 < 0$, and $f(2) = \sin 8 > 0$. Applying the Intermediate Value Theorem on $[1, A]$ and then on $[A, 2]$, we see there are numbers c and d in $(1, A)$ and $(A, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(1, 2)$.

62. Let $f(x) = x^2 - 3 + 1/x$. Then f is continuous on $(0, 2]$ since f is a rational function whose domain is $(0, \infty)$. By inspection, we see that $f\left(\frac{1}{4}\right) = \frac{17}{16} > 0$, $f(1) = -1 < 0$, and $f(2) = \frac{3}{2} > 0$. Appling the Intermediate Value Theorem on $\left[\frac{1}{4}, 1\right]$ and then on $[1, 2]$, we see there are numbers c and d in $\left(\frac{1}{4}, 1\right)$ and $(1, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(0, 2)$.

63. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\Big(\lim_{h\to 0}(a+h)\Big) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \quad\Rightarrow$

$|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x-a| < \delta$, then $|f(x) - f(a)| = |f(a+(x-a)) - f(a)| < \varepsilon$.

Thus, $\lim_{x\to a} f(x) = f(a)$ and so f is continuous at a.

64. $\lim_{h\to 0} \sin(a+h) = \lim_{h\to 0} (\sin a\cos h + \cos a\sin h) = \lim_{h\to 0} (\sin a\cos h) + \lim_{h\to 0} (\cos a\sin h)$

$$= \Big(\lim_{h\to 0}\sin a\Big)\Big(\lim_{h\to 0}\cos h\Big) + \Big(\lim_{h\to 0}\cos a\Big)\Big(\lim_{h\to 0}\sin h\Big) = (\sin a)(1) + (\cos a)(0) = \sin a$$

65. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h\to 0}\cos(a+h) = \lim_{h\to 0}(\cos a\cos h - \sin a\sin h) = \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)$$

$$= \Big(\lim_{h\to 0}\cos a\Big)\Big(\lim_{h\to 0}\cos h\Big) - \Big(\lim_{h\to 0}\sin a\Big)\Big(\lim_{h\to 0}\sin h\Big) = (\cos a)(1) - (\sin a)(0) = \cos a$$

66. (a) Since f is continuous at a, $\lim_{x\to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have

$\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law

of Limits: $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

67. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x\to a} f(x) \neq f(a)$. [In fact, $\lim_{x\to a} f(x)$ does not even exist.]

68. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0. To see why, note that $-|x| \le g(x) \le |x|$, so by the Squeeze Theorem $\lim_{x\to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x\to a} g(x) \neq g(a)$.

69. If there is such a number, it satisfies the equation $x^3 + 1 = x \quad\Leftrightarrow\quad x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

70. $\dfrac{a}{x^3 + 2x^2 - 1} + \dfrac{b}{x^3 + x - 2} = 0 \quad\Rightarrow\quad a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that

$p(c) = 0$ by the Intermediate Value Theorem. Note that the only root of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a root of either denominator, so $p(c) = 0 \quad \Rightarrow$ $x = c$ is a root of the given equation.

71. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \leq \sin(1/x) \leq 1$, we have $-x^4 \leq x^4 \sin(1/x) \leq x^4$. Because $\lim\limits_{x\to 0}(-x^4) = 0$ and $\lim\limits_{x\to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim\limits_{x\to 0}(x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

72. (a) $\lim\limits_{x\to 0^+} F(x) = 0$ and $\lim\limits_{x\to 0^-} F(x) = 0$, so $\lim\limits_{x\to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim\limits_{x\to a} F(x) = \lim\limits_{x\to a} x = a = F(a)$. For $a < 0$, $\lim\limits_{x\to a} F(x) = \lim\limits_{x\to a}(-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim\limits_{x\to a} |f(x)| = \left|\lim\limits_{x\to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

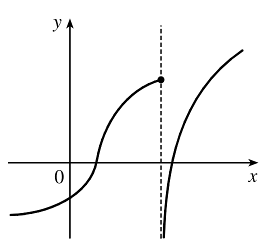
73. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow$ $u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

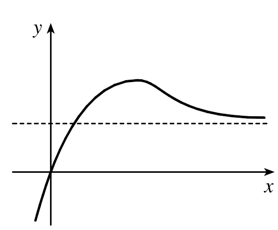
1. (a) As x becomes large, the values of $f(x)$ approach 5.

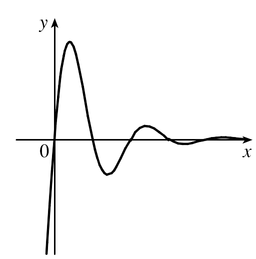
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

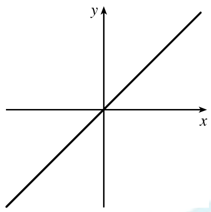


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

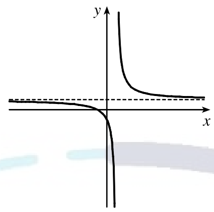




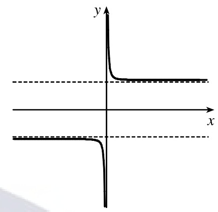
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

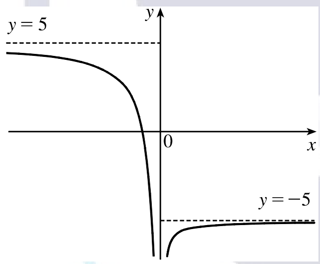
3. (a) $\lim_{x\to\infty} f(x) = -2$ (b) $\lim_{x\to-\infty} f(x) = 2$ (c) $\lim_{x\to 1} f(x) = \infty$

(d) $\lim_{x\to 3} f(x) = -\infty$ (e) Vertical: $x = 1$, $x = 3$; horizontal: $y = -2$, $y = 2$

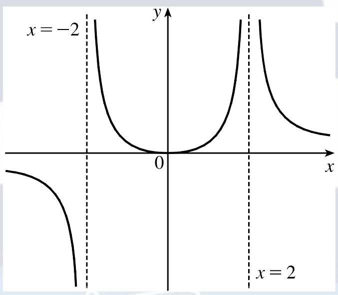
4. (a) $\lim_{x\to\infty} g(x) = 2$ (b) $\lim_{x\to-\infty} g(x) = -1$ (c) $\lim_{x\to 0} g(x) = -\infty$

(d) $\lim_{x\to 2^-} g(x) = -\infty$ (e) $\lim_{x\to 2^+} g(x) = \infty$ (f) Vertical: $x = 0$, $x = 2$; horizontal: $y = -1$, $y = 2$

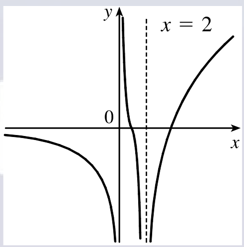
5. $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = 5$, $\lim_{x\to\infty} f(x) = -5$



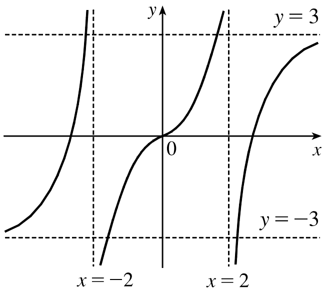
6. $\lim_{x\to 2} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = \infty$, $\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$, $f(0) = 0$



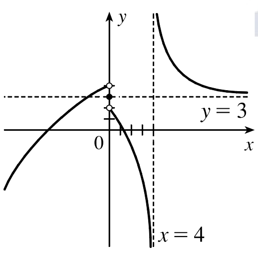
7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$



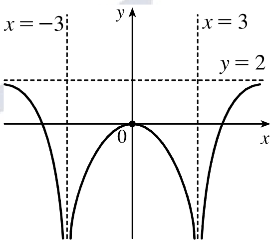
8. $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$, f is odd



9. $f(0) = 3$, $\lim_{x\to 0^-} f(x) = 4$, $\lim_{x\to 0^+} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 3$



10. $\lim_{x\to 3} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 2$, $f(0) = 0$, f is even



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$. It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

12. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135 308
100,000	0.135 333
1,000,000	0.135 335

From the table, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

13.
$$\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3} = \lim_{x\to\infty} \frac{(2x^2 - 7)/x^2}{(5x^2 + x - 3)/x^2}$$
[Divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]
$$= \frac{\lim\limits_{x\to\infty} (2 - 7/x^2)}{\lim\limits_{x\to\infty} (5 + 1/x - 3/x^2)}$$
[Limit Law 5]
$$= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} (7/x^2)}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} (1/x) - \lim\limits_{x\to\infty} (3/x^2)}$$
[Limit Laws 1 and 2]
$$= \frac{2 - 7\lim\limits_{x\to\infty} (1/x^2)}{5 + \lim\limits_{x\to\infty} (1/x) - 3\lim\limits_{x\to\infty} (1/x^2)}$$
[Limit Laws 7 and 3]
$$= \frac{2 - 7(0)}{5 + 0 + 3(0)}$$
[Theorem 2.6.5]
$$= \frac{2}{5}$$

14.
$$\lim_{x\to\infty} \sqrt{\frac{9x^3 + 8x - 4}{3 - 5x + x^3}} = \sqrt{\lim_{x\to\infty} \frac{9x^3 + 8x - 4}{3 - 5x + x^3}}$$
[Limit Law 11]
$$= \sqrt{\lim_{x\to\infty} \frac{9 + 8/x^2 - 4/x^3}{3/x^3 - 5/x^2 + 1}}$$
[Divide by x^3]
$$= \sqrt{\frac{\lim\limits_{x\to\infty} (9 + 8/x^2 - 4/x^3)}{\lim\limits_{x\to\infty} (3/x^3 - 5/x^2 + 1)}}$$
[Limit Law 5]
$$= \sqrt{\frac{\lim\limits_{x\to\infty} 9 + \lim\limits_{x\to\infty} (8/x^2) - \lim\limits_{x\to\infty} (4/x^3)}{\lim\limits_{x\to\infty} (3/x^3) - \lim\limits_{x\to\infty} (5/x^2) + \lim\limits_{x\to\infty} 1}}$$
[Limit Laws 1 and 2]
$$= \sqrt{\frac{9 + 8\lim\limits_{x\to\infty} (1/x^2) - 4\lim\limits_{x\to\infty} (1/x^3)}{3\lim\limits_{x\to\infty} (1/x^3) - 5\lim\limits_{x\to\infty} (1/x^2) + 1}}$$
[Limit Laws 7 and 3]
$$= \sqrt{\frac{9 + 8(0) - 4(0)}{3(0) - 5(0) + 1}}$$
[Theorem 2.6.5]
$$= \sqrt{\frac{9}{1}} = \sqrt{9} = 3$$

15. $\lim\limits_{x\to\infty}\dfrac{3x-2}{2x+1}=\lim\limits_{x\to\infty}\dfrac{(3x-2)/x}{(2x+1)/x}=\lim\limits_{x\to\infty}\dfrac{3-2/x}{2+1/x}=\dfrac{\lim\limits_{x\to\infty}3-2\lim\limits_{x\to\infty}1/x}{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}1/x}=\dfrac{3-2(0)}{2+0}=\dfrac{3}{2}$

16. $\lim\limits_{x\to\infty}\dfrac{1-x^2}{x^3-x+1}=\lim\limits_{x\to\infty}\dfrac{(1-x^2)/x^3}{(x^3-x+1)/x^3}=\lim\limits_{x\to\infty}\dfrac{1/x^3-1/x}{1-1/x^2+1/x^3}$

$=\dfrac{\lim\limits_{x\to\infty}1/x^3-\lim\limits_{x\to\infty}1/x}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}1/x^2+\lim\limits_{x\to\infty}1/x^3}=\dfrac{0-0}{1-0+0}=0$

17. $\lim\limits_{x\to-\infty}\dfrac{x-2}{x^2+1}=\lim\limits_{x\to-\infty}\dfrac{(x-2)/x^2}{(x^2+1)/x^2}=\lim\limits_{x\to-\infty}\dfrac{1/x-2/x^2}{1+1/x^2}=\dfrac{\lim\limits_{x\to-\infty}1/x-2\lim\limits_{x\to-\infty}1/x^2}{\lim\limits_{x\to-\infty}1+\lim\limits_{x\to-\infty}1/x^2}=\dfrac{0-2(0)}{1+0}=0$

18. $\lim\limits_{x\to-\infty}\dfrac{4x^3+6x^2-2}{2x^3-4x+5}=\lim\limits_{x\to-\infty}\dfrac{(4x^3+6x^2-2)/x^3}{(2x^3-4x+5)/x^3}=\lim\limits_{x\to-\infty}\dfrac{4+6/x-2/x^3}{2-4/x^2+5/x^3}=\dfrac{4+0-0}{2-0+0}=2$

19. $\lim\limits_{t\to\infty}\dfrac{\sqrt{t}+t^2}{2t-t^2}=\lim\limits_{t\to\infty}\dfrac{(\sqrt{t}+t^2)/t^2}{(2t-t^2)/t^2}=\lim\limits_{t\to\infty}\dfrac{1/t^{3/2}+1}{2/t-1}=\dfrac{0+1}{0-1}=-1$

20. $\lim\limits_{t\to\infty}\dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}=\lim\limits_{t\to\infty}\dfrac{\left(t-t\sqrt{t}\right)/t^{3/2}}{\left(2t^{3/2}+3t-5\right)/t^{3/2}}=\lim\limits_{t\to\infty}\dfrac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}}=\dfrac{0-1}{2+0-0}=-\dfrac{1}{2}$

21. $\lim\limits_{x\to\infty}\dfrac{(2x^2+1)^2}{(x-1)^2(x^2+x)}=\lim\limits_{x\to\infty}\dfrac{(2x^2+1)^2/x^4}{[(x-1)^2(x^2+x)]/x^4}=\lim\limits_{x\to\infty}\dfrac{[(2x^2+1)/x^2]^2}{[(x^2-2x+1)/x^2][(x^2+x)/x^2]}$

$=\lim\limits_{x\to\infty}\dfrac{(2+1/x^2)^2}{(1-2/x+1/x^2)(1+1/x)}=\dfrac{(2+0)^2}{(1-0+0)(1+0)}=4$

22. $\lim\limits_{x\to\infty}\dfrac{x^2}{\sqrt{x^4+1}}=\lim\limits_{x\to\infty}\dfrac{x^2/x^2}{\sqrt{x^4+1}/x^2}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{(x^4+1)/x^4}}$ [since $x^2=\sqrt{x^4}$ for $x>0$]

$=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1+1/x^4}}=\dfrac{1}{\sqrt{1+0}}=1$

23. $\lim\limits_{x\to\infty}\dfrac{\sqrt{1+4x^6}}{2-x^3}=\lim\limits_{x\to\infty}\dfrac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\dfrac{\lim\limits_{x\to\infty}\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to\infty}(2/x^3-1)}$ [since $x^3=\sqrt{x^6}$ for $x>0$]

$=\dfrac{\lim\limits_{x\to\infty}\sqrt{1/x^6+4}}{\lim\limits_{x\to\infty}(2/x^3)-\lim\limits_{x\to\infty}1}=\dfrac{\sqrt{\lim\limits_{x\to\infty}(1/x^6)+\lim\limits_{x\to\infty}4}}{0-1}$

$=\dfrac{\sqrt{0+4}}{-1}=\dfrac{2}{-1}=-2$

24. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{1+4x^6}}{2-x^3}=\lim\limits_{x\to-\infty}\dfrac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\dfrac{\lim\limits_{x\to-\infty}-\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty}(2/x^3-1)}$ [since $x^3=-\sqrt{x^6}$ for $x<0$]

$=\dfrac{\lim\limits_{x\to-\infty}-\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty}(1/x^3)-\lim\limits_{x\to-\infty}1}=\dfrac{-\sqrt{\lim\limits_{x\to-\infty}(1/x^6)+\lim\limits_{x\to-\infty}4}}{2(0)-1}$

$=\dfrac{-\sqrt{0+4}}{-1}=\dfrac{-2}{-1}=2$

25. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x+3x^2}}{4x-1}=\lim_{x\to\infty}\frac{\sqrt{x+3x^2}/x}{(4x-1)/x}=\frac{\lim_{x\to\infty}\sqrt{(x+3x^2)/x^2}}{\lim_{x\to\infty}(4-1/x)}$ [since $x=\sqrt{x^2}$ for $x>0$]

$$=\frac{\lim_{x\to\infty}\sqrt{1/x+3}}{\lim_{x\to\infty}4-\lim_{x\to\infty}(1/x)}=\frac{\sqrt{\lim_{x\to\infty}(1/x)+\lim_{x\to\infty}3}}{4-0}=\frac{\sqrt{0+3}}{4}=\frac{\sqrt{3}}{4}$$

26. $\displaystyle\lim_{x\to\infty}\frac{x+3x^2}{4x-1}=\lim_{x\to\infty}\frac{(x+3x^2)/x}{(4x-1)/x}=\lim_{x\to\infty}\frac{1+3x}{4-1/x}$

$=\infty$ since $1+3x\to\infty$ and $4-1/x\to 4$ as $x\to\infty$.

27. $\displaystyle\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$$=\lim_{x\to\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

28. $\displaystyle\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]$

$$=\lim_{x\to-\infty}\frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x}=\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}$$

$$=\lim_{x\to-\infty}\frac{3x/x}{\left(\sqrt{4x^2+3x}-2x\right)/x}=\lim_{x\to-\infty}\frac{3}{-\sqrt{4+3/x}-2}\quad\text{[since } x=-\sqrt{x^2}\text{ for } x<0]$$

$$=\frac{3}{-\sqrt{4+0}-2}=-\frac{3}{4}$$

29. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

30. For $x>0$, $\sqrt{x^2+1}>\sqrt{x^2}=x$. So as $x\to\infty$, we have $\sqrt{x^2+1}\to\infty$, that is, $\lim_{x\to\infty}\sqrt{x^2+1}=\infty$.

31. $\displaystyle\lim_{x\to\infty}\frac{x^4-3x^2+x}{x^3-x+2}=\lim_{x\to\infty}\frac{(x^4-3x^2+x)/x^3}{(x^3-x+2)/x^3}$ [divide by the highest power of x in the denominator] $\displaystyle=\lim_{x\to\infty}\frac{x-3/x+1/x^2}{1-1/x^2+2/x^3}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to\infty$.

32. $\lim_{x\to\infty}(e^{-x}+2\cos 3x)$ does not exist. $\lim_{x\to\infty}e^{-x}=0$, but $\lim_{x\to\infty}(2\cos 3x)$ does not exist because the values of $2\cos 3x$ oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

33. $\lim\limits_{x\to-\infty}(x^2+2x^7)=\lim\limits_{x\to-\infty}x^7\left(\dfrac{1}{x^5}+2\right)$ [factor out the largest power of x] $=-\infty$ because $x^7\to-\infty$ and $1/x^5+2\to2$ as $x\to-\infty$.

Or: $\lim\limits_{x\to-\infty}\left(x^2+2x^7\right)=\lim\limits_{x\to-\infty}x^2\left(1+2x^5\right)=-\infty$.

34. $\lim\limits_{x\to-\infty}\dfrac{1+x^6}{x^4+1}=\lim\limits_{x\to-\infty}\dfrac{(1+x^6)/x^4}{(x^4+1)/x^4}$ [divide by the highest power of x in the denominator] $=\lim\limits_{x\to-\infty}\dfrac{1/x^4+x^2}{1+1/x^4}=\infty$ since the numerator increases without bound and the denominator approaches 1 as $x\to-\infty$.

35. Let $t=e^x$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\arctan(e^x)=\lim\limits_{t\to\infty}\arctan t=\frac{\pi}{2}$ by (3).

36. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim\limits_{x\to\infty}\dfrac{1-e^{-6x}}{1+e^{-6x}}=\dfrac{1-0}{1+0}=1$

37. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+2e^x}=\lim\limits_{x\to\infty}\dfrac{(1-e^x)/e^x}{(1+2e^x)/e^x}=\lim\limits_{x\to\infty}\dfrac{1/e^x-1}{1/e^x+2}=\dfrac{0-1}{0+2}=-\dfrac{1}{2}$

38. Since $0\le\sin^2x\le1$, we have $0\le\dfrac{\sin^2x}{x^2+1}\le\dfrac{1}{x^2+1}$. We know that $\lim\limits_{x\to\infty}0=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^2+1}=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\sin^2x}{x^2+1}=0$.

39. Since $-1\le\cos x\le1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x})=0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right)=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x)=0$.

40. Let $t=\ln x$. As $x\to0^+$, $t\to-\infty$. $\lim\limits_{x\to0^+}\tan^{-1}(\ln x)=\lim\limits_{t\to-\infty}\tan^{-1}t=-\frac{\pi}{2}$ by (4).

41. $\lim\limits_{x\to\infty}[\ln(1+x^2)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\dfrac{1+x^2}{1+x}=\ln\left(\lim\limits_{x\to\infty}\dfrac{1+x^2}{1+x}\right)=\ln\left(\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+x}{\frac{1}{x}+1}\right)=\infty$, since the limit in parentheses is ∞.

42. $\lim\limits_{x\to\infty}[\ln(2+x)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\left(\dfrac{2+x}{1+x}\right)=\lim\limits_{x\to\infty}\ln\left(\dfrac{2/x+1}{1/x+1}\right)=\ln\dfrac{1}{1}=\ln1=0$

43. (a) (i) $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\dfrac{x}{\ln x}=0$ since $x\to0^+$ and $\ln x\to-\infty$ as $x\to0^+$.

(ii) $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}\dfrac{x}{\ln x}=-\infty$ since $x\to1$ and $\ln x\to0^-$ as $x\to1^-$.

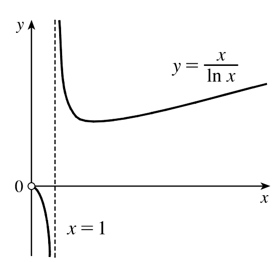
(iii) $\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}\dfrac{x}{\ln x}=\infty$ since $x\to1$ and $\ln x\to0^+$ as $x\to1^+$.

(b)

x	$f(x)$
10,000	1085.7
100,000	8685.9
1,000,000	72,382.4

It appears that $\lim\limits_{x\to\infty}f(x)=\infty$.

(c)



44. (a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{2}{x} - \frac{1}{\ln x}\right) = 0$

since $\frac{2}{x} \to 0$ and $\frac{1}{\ln x} \to 0$ as $x \to \infty$.

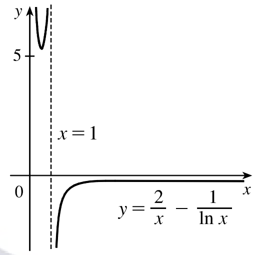
(b) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{2}{x} - \frac{1}{\ln x}\right) = \infty$

since $\frac{2}{x} \to \infty$ and $\frac{1}{\ln x} \to 0$ as $x \to 0^+$.

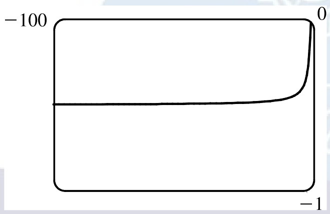
(c) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(\frac{2}{x} - \frac{1}{\ln x}\right) = \infty$ since $\frac{2}{x} \to 2$ and $\frac{1}{\ln x} \to -\infty$ as $x \to 1^-$.

(d) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\frac{2}{x} - \frac{1}{\ln x}\right) = -\infty$ since $\frac{2}{x} \to 2$ and $\frac{1}{\ln x} \to \infty$ as $x \to 1^+$.

(e)



45. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x\to -\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	$-0.499\,962\,5$
$-100{,}000$	$-0.499\,996\,2$
$-1{,}000{,}000$	$-0.499\,999\,6$

From the table, we estimate the limit to be -0.5.

(c) $\lim_{x\to -\infty}\left(\sqrt{x^2+x+1} + x\right) = \lim_{x\to -\infty}\left(\sqrt{x^2+x+1} + x\right)\left[\frac{\sqrt{x^2+x+1} - x}{\sqrt{x^2+x+1} - x}\right] = \lim_{x\to -\infty}\frac{(x^2+x+1) - x^2}{\sqrt{x^2+x+1} - x}$

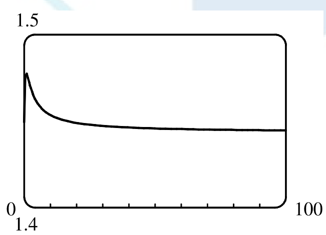
$$= \lim_{x\to -\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1} - x\right)(1/x)} = \lim_{x\to -\infty}\frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$

$$= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get

$\frac{1}{x}\sqrt{x^2+x+1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1 + (1/x) + (1/x^2)}$.

46. (a)



From the graph of

$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$, we estimate

(to one decimal place) the value of $\lim_{x\to\infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.443 39
100,000	1.443 38
1,000,000	1.443 38

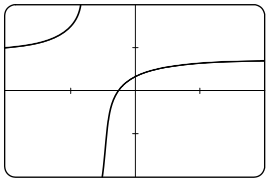
From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$

$$= \lim_{x\to\infty} \frac{\left(3x^2+8x+6\right)-\left(3x^2+3x+1\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}} = \lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

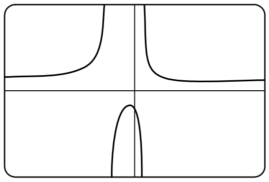
$$= \lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

47. $\lim\limits_{x\to\pm\infty} \dfrac{5+4x}{x+3} = \lim\limits_{x\to\pm\infty} \dfrac{(5+4x)/x}{(x+3)/x} = \lim\limits_{x\to\pm\infty} \dfrac{5/x+4}{1+3/x} = \dfrac{0+4}{1+0} = 4$, so



$y = 4$ is a horizontal asymptote. $y = f(x) = \dfrac{5+4x}{x+3}$, so $\lim\limits_{x\to-3^+} f(x) = -\infty$ since $5+4x \to -7$ and $x+3 \to 0^+$ as $x \to -3^+$. Thus, $x = -3$ is a vertical asymptote. The graph confirms our work.

48. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+1}{3x^2+2x-1} = \lim\limits_{x\to\pm\infty} \dfrac{(2x^2+1)/x^2}{(3x^2+2x-1)/x^2}$



$$= \lim_{x\to\pm\infty} \frac{2+1/x^2}{3+2/x-1/x^2} = \frac{2}{3}$$

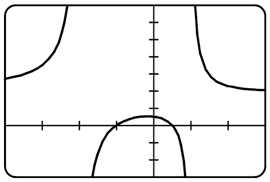
so $y = \dfrac{2}{3}$ is a horizontal asymptote. $y = f(x) = \dfrac{2x^2+1}{3x^2+2x-1} = \dfrac{2x^2+1}{(3x-1)(x+1)}$.

The denominator is zero when $x = \frac{1}{3}$ and -1, but the numerator is nonzero, so $x = \frac{1}{3}$ and $x = -1$ are vertical asymptotes. The graph confirms our work.

49. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+x-1}{x^2+x-2} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{2x^2+x-1}{x^2}}{\dfrac{x^2+x-2}{x^2}} = \lim\limits_{x\to\pm\infty} \dfrac{2+\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{1}{x}-\dfrac{2}{x^2}} = \dfrac{\lim\limits_{x\to\pm\infty}\left(2+\dfrac{1}{x}-\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}-\dfrac{2}{x^2}\right)}$

$$= \frac{\lim\limits_{x\to\pm\infty} 2 + \lim\limits_{x\to\pm\infty}\dfrac{1}{x} - \lim\limits_{x\to\pm\infty}\dfrac{1}{x^2}}{\lim\limits_{x\to\pm\infty} 1 + \lim\limits_{x\to\pm\infty}\dfrac{1}{x} - 2\lim\limits_{x\to\pm\infty}\dfrac{1}{x^2}} = \frac{2+0-0}{1+0-2(0)} = 2,$$ so $y = 2$ is a horizontal asymptote.

$y = f(x) = \dfrac{2x^2+x-1}{x^2+x-2} = \dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\lim\limits_{x\to-2^-} f(x) = \infty$,

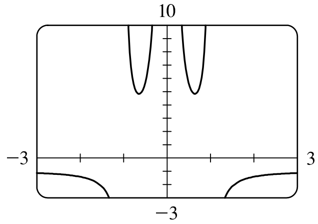


$\lim\limits_{x\to-2^+} f(x) = -\infty$, $\lim\limits_{x\to1^-} f(x) = -\infty$, and $\lim\limits_{x\to1^+} f(x) = \infty$. Thus, $x = -2$ and $x = 1$ are vertical asymptotes. The graph confirms our work.

50. $\lim\limits_{x\to\pm\infty} \dfrac{1+x^4}{x^2-x^4} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1+x^4}{x^4}}{\dfrac{x^2-x^4}{x^4}} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1}{x^4}+1}{\dfrac{1}{x^2}-1} = \dfrac{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^4}+1\right)}{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^2}-1\right)} = \dfrac{\lim\limits_{x\to\pm\infty}\dfrac{1}{x^4}+\lim\limits_{x\to\pm\infty}1}{\lim\limits_{x\to\pm\infty}\dfrac{1}{x^2}-\lim\limits_{x\to\pm\infty}1}$

$$= \frac{0+1}{0-1} = -1,$$ so $y = -1$ is a horizontal asymptote.

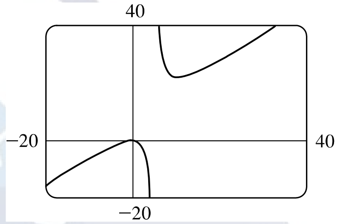
$y = f(x) = \dfrac{1+x^4}{x^2-x^4} = \dfrac{1+x^4}{x^2(1-x^2)} = \dfrac{1+x^4}{x^2(1+x)(1-x)}$. The denominator is zero when $x = 0, -1$, and 1, but the numerator is nonzero, so $x = 0$, $x = -1$, and $x = 1$ are vertical asymptotes. Notice that as $x \to 0$, the numerator and denominator are both positive, so $\lim\limits_{x\to 0} f(x) = \infty$. The graph confirms our work.



51. $y = f(x) = \dfrac{x^3-x}{x^2-6x+5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2+x}{x-5} = x + 6 + \dfrac{30}{x-5}$.

As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x\to 5^-} g(x) = -\infty$ and $\lim\limits_{x\to 5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.

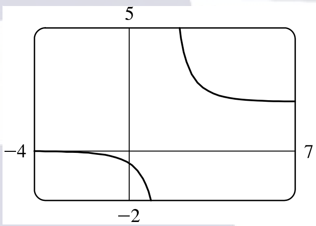


52. $\lim\limits_{x\to\infty} \dfrac{2e^x}{e^x-5} = \lim\limits_{x\to\infty} \dfrac{2e^x}{e^x-5} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{2}{1-(5/e^x)} = \dfrac{2}{1-0} = 2$, so $y = 2$ is a horizontal asymptote.

$\lim\limits_{x\to -\infty} \dfrac{2e^x}{e^x-5} = \dfrac{2(0)}{0-5} = 0$, so $y = 0$ is a horizontal asymptote. The denominator is zero (and the numerator isn't) when $e^x - 5 = 0 \quad\Rightarrow\quad e^x = 5 \quad\Rightarrow\quad x = \ln 5$.

$\lim\limits_{x\to(\ln 5)^+} \dfrac{2e^x}{e^x-5} = \infty$ since the numerator approaches 10 and the denominator approaches 0 through positive values as $x \to (\ln 5)^+$. Similarly,

$\lim\limits_{x\to(\ln 5)^-} \dfrac{2e^x}{e^x-5} = -\infty$. Thus, $x = \ln 5$ is a vertical asymptote. The graph confirms our work.



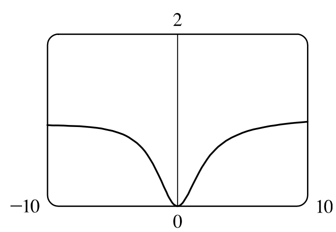
53. From the graph, it appears $y = 1$ is a horizontal asymptote.

$$\lim_{x\to\pm\infty} \frac{3x^3+500x^2}{x^3+500x^2+100x+2000} = \lim_{x\to\pm\infty} \frac{\dfrac{3x^3+500x^2}{x^3}}{\dfrac{x^3+500x^2+100x+2000}{x^3}}$$

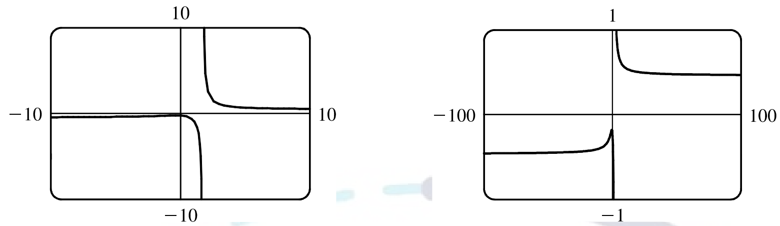
$$= \lim_{x\to\pm\infty} \frac{3+(500/x)}{1+(500/x)+(100/x^2)+(2000/x^3)}$$

$$= \frac{3+0}{1+0+0+0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000, 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.



54. (a)



From the graph, it appears at first that there is only one horizontal asymptote, at $y \approx 0$, and a vertical asymptote at $x \approx 1.7$. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y \approx 0.5$ and one at $y \approx -0.5$. So we estimate that

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx 0.5 \quad \text{and} \quad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx -0.5$$

(b) $f(1000) \approx 0.4722$ and $f(10{,}000) \approx 0.4715$, so we estimate that $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx 0.47$.

$f(-1000) \approx -0.4706$ and $f(-10{,}000) \approx -0.4713$, so we estimate that $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx -0.47$.

(c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2+1/x^2}}{3-5/x}$ $\quad$ [since $\sqrt{x^2} = x$ for $x > 0$] $\quad = \dfrac{\sqrt{2}}{3} \approx 0.471404$.

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{2x^2+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+1/x^2}$. Therefore,

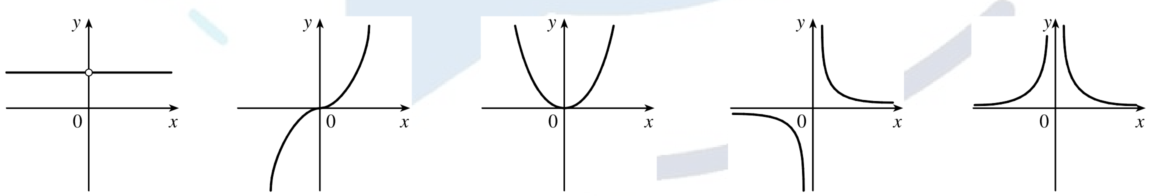
$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

55. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

56.



(i) $n = 0$ (ii) $n > 0$ (n odd) (iii) $n > 0$ (n even) (iv) $n < 0$ (n odd) (v) $n < 0$ (n even)

From these sketches we see that

(a) $\lim\limits_{x\to 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$ $\qquad$ (b) $\lim\limits_{x\to 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim\limits_{x \to \infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$ (d) $\lim\limits_{x \to -\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

57. Let's look for a rational function.

(1) $\lim\limits_{x \to \pm\infty} f(x) = 0 \quad \Rightarrow$ degree of numerator < degree of denominator

(2) $\lim\limits_{x \to 0} f(x) = -\infty \quad \Rightarrow$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x \to 3^-} f(x) = \infty$ and $\lim\limits_{x \to 3^+} f(x) = -\infty \quad \Rightarrow$ vertical asymptote at $x = 3$; there is a factor of $(x - 3)$ in the denominator.

(4) $f(2) = 0 \quad \Rightarrow$ 2 is an x-intercept; there is at least one factor of $(x - 2)$ in the numerator.

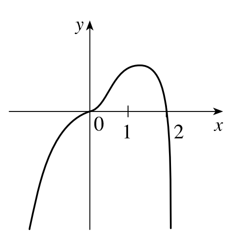
Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2 - x}{x^2(x - 3)}$ as one possibility.

58. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x - 1)$ and $(x - 3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x - 1)(x - 3)}$.

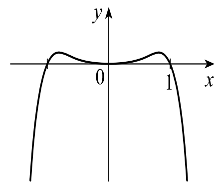
59. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x - 1)(x + 1)}{(x - 4)(x + 1)}$, where a is still to be determined. Then $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \dfrac{a(x - 1)(x + 1)}{(x - 4)(x + 1)} = \lim\limits_{x \to -1} \dfrac{a(x - 1)}{x - 4} = \dfrac{a(-1 - 1)}{(-1 - 4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x - 1)(x + 1)}{(x - 4)(x + 1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x \to \infty} f(x) = 5 \lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x^2 - 3x - 4} = 5 \lim\limits_{x \to \infty} \dfrac{(x^2/x^2) - (1/x^2)}{(x^2/x^2) - (3x/x^2) - (4/x^2)} = 5\dfrac{1 - 0}{1 - 0 - 0} = 5(1) = 5$

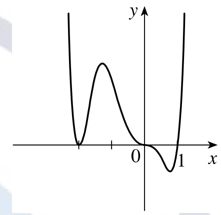
60. $y = f(x) = 2x^3 - x^4 = x^3(2 - x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0 and 2. There are sign changes at 0 and 2 (odd exponents on x and $2 - x$). As $x \to \infty$, $f(x) \to -\infty$ because $x^3 \to \infty$ and $2 - x \to -\infty$. As $x \to -\infty$, $f(x) \to -\infty$ because $x^3 \to -\infty$ and $2 - x \to \infty$. Note that the graph of f near $x = 0$ flattens out (looks like $y = x^3$).



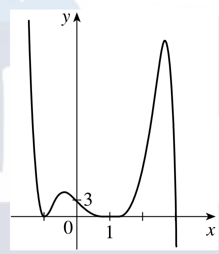
61. $y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1 + x)(1 - x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1 - x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1 - x^2) \to -\infty$.



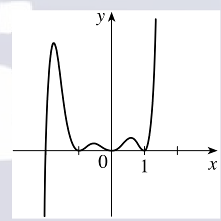
62. $y = f(x) = x^3(x + 2)^2(x - 1)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -2, and 1. There are sign changes at 0 and 1 (odd exponents on x and $x - 1$). There is no sign change at -2. Also, $f(x) \to \infty$ as $x \to \infty$ because all three factors are large. And $f(x) \to \infty$ as $x \to -\infty$ because $x^3 \to -\infty$, $(x + 2)^2 \to \infty$, and $(x - 1) \to -\infty$. Note that the graph of f at $x = 0$ flattens out (looks like $y = -x^3$).



63. $y = f(x) = (3 - x)(1 + x)^2(1 - x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3 - x$ is negative and the other factors are positive, so $\lim\limits_{x \to \infty} f(x) = -\infty$. When x is large negative, $3 - x$ is positive, so $\lim\limits_{x \to -\infty} f(x) = \infty$.

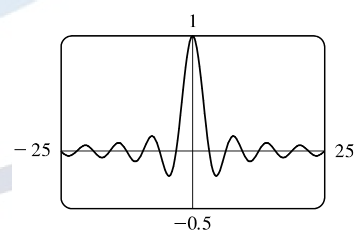


64. $y = f(x) = x^2(x^2 - 1)^2(x + 2) = x^2(x + 1)^2(x - 1)^2(x + 2)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, 1, and -2. There is a sign change at -2, but not at 0, -1, and 1. When x is large positive, all the factors are positive, so $\lim\limits_{x \to \infty} f(x) = \infty$. When x is large negative, only $x + 2$ is negative, so $\lim\limits_{x \to -\infty} f(x) = -\infty$.



65. (a) Since $-1 \leq \sin x \leq 1$ for all x, $-\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*

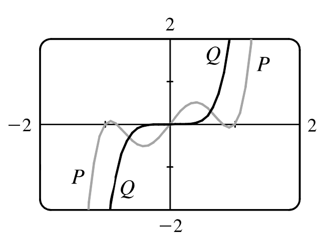


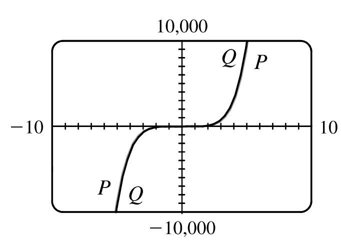
66. (a) In both viewing rectangles,

$\lim\limits_{x \to \infty} P(x) = \lim\limits_{x \to \infty} Q(x) = \infty$ and

$\lim\limits_{x \to -\infty} P(x) = \lim\limits_{x \to -\infty} Q(x) = -\infty$.

In the larger viewing rectangle, P and Q become less distinguishable.





(b) $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim\limits_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \quad \Rightarrow$

P and Q have the same end behavior.

67. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10 - (21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x - 21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

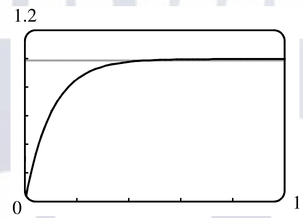
68. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be

$C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t} \, \dfrac{\text{g}}{\text{L}}$.

(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200+t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

69. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

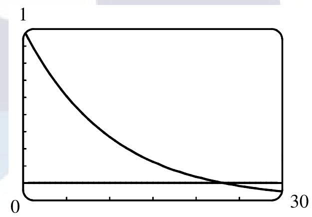


70. (a) $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$.

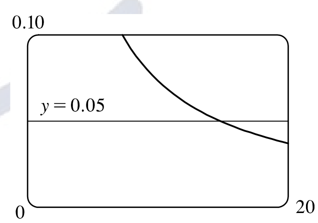
If $x > x_1$, then $e^{-x/10} < 0.1$.

(b) $e^{-x/10} < 0.1 \quad \Rightarrow \quad -x/10 < \ln 0.1 \quad \Rightarrow$

$x > -10 \ln \frac{1}{10} = -10 \ln 10^{-1} = 10 \ln 10 \approx 23.03$

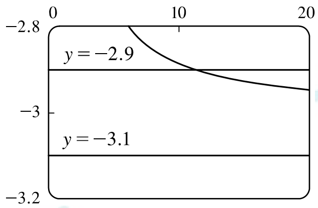


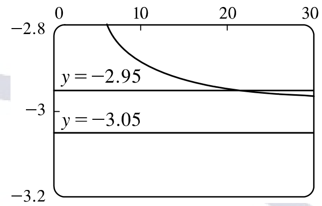
71. Let $g(x) = \dfrac{3x^2 + 1}{2x^2 + x + 1}$ and $f(x) = |g(x) - 1.5|$. Note that $\lim\limits_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim\limits_{x\to\infty} f(x) = 0$. We are interested in finding the x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$, so we choose $N = 15$ (or any larger number).



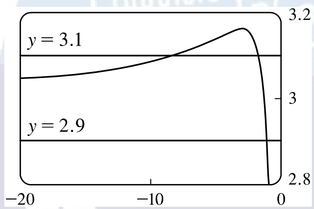
72. We want to find a value of N such that $x > N \quad \Rightarrow \quad \left| \dfrac{1-3x}{\sqrt{x^2+1}} - (-3) \right| < \varepsilon$, or equivalently,

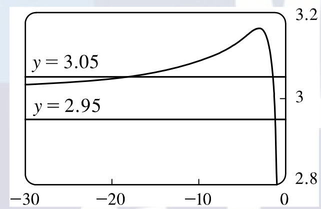
$-3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < -3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = -3.1$, and $y = -2.9$. From the graph, we find that $f(x) = -2.9$ at about $x = 11.283$, so we choose $N = 12$ (or any larger number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = -2.95$ at about $x = 21.379$, so we choose $N = 22$ (or any larger number).



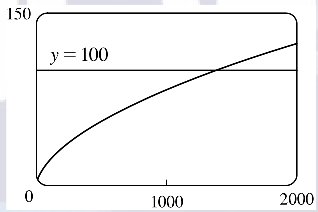


73. We want a value of N such that $x < N \Rightarrow \left|\dfrac{1-3x}{\sqrt{x^2+1}} - 3\right| < \varepsilon$, or equivalently, $3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < 3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = 3.1$, and $y = 2.9$. From the graph, we find that $f(x) = 3.1$ at about $x = -8.092$, so we choose $N = -9$ (or any lesser number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = 3.05$ at about $x = -18.338$, so we choose $N = -19$ (or any lesser number).





74. We want to find a value of N such that $x > N \quad \Rightarrow \quad \sqrt{x \ln x} > 100$. We graph $y = f(x) = \sqrt{x \ln x}$ and $y = 100$. From the graph, we find that $f(x) = 100$ at about $x = 1382.773$, so we choose $N = 1383$ (or any larger number).



75. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10\,000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$.

76. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$.

77. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\lim\limits_{x\to-\infty} (1/x) = 0$.

78. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\lim\limits_{x\to\infty} x^3 = \infty$.

79. Given $M > 0$, we need $N > 0$ such that $x > N \;\Rightarrow\; e^x > M$. Now $e^x > M \;\Leftrightarrow\; x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \;\Rightarrow\; e^x > \max(e, M) \geq M$, so $\lim\limits_{x\to\infty} e^x = \infty$.

80. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim\limits_{x\to-\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \;\Rightarrow\; 1 + x^3 < M$. Now $1 + x^3 < M \;\Leftrightarrow\; x^3 < M - 1 \;\Leftrightarrow\; x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \;\Rightarrow\; 1 + x^3 < M$. This proves that $\lim\limits_{x\to-\infty} \left(1 + x^3\right) = -\infty$.

81. (a) Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \;\Leftrightarrow\; 0 < 1/x < 1/N \;\Leftrightarrow\; 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim\limits_{t\to 0^+} f(1/t) = L = \lim\limits_{x\to\infty} f(x)$.

Now suppose that $\lim\limits_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \;\Leftrightarrow\; 1/N < 1/x < 0 \;\Leftrightarrow\; 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t\to 0^-} f(1/t) = L = \lim\limits_{x\to-\infty} f(x)$.

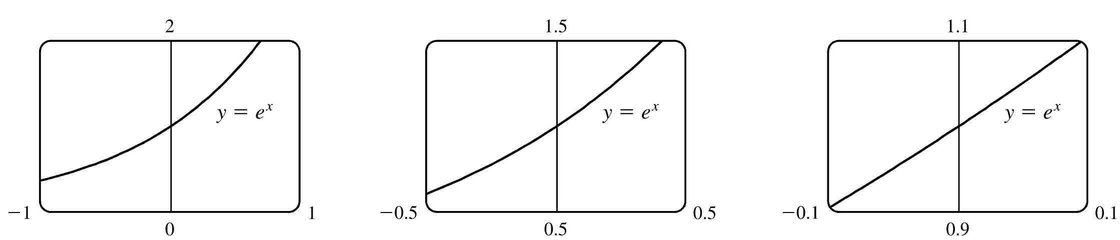
(b)
$$\begin{aligned}
\lim_{x\to 0^+} x \sin\frac{1}{x} &= \lim_{t\to 0^+} t \sin\frac{1}{t} && [\text{let } x = t] \\
&= \lim_{y\to\infty} \frac{1}{y}\sin y && [\text{part (a) with } y = 1/t] \\
&= \lim_{x\to\infty} \frac{\sin x}{x} && [\text{let } y = x] \\
&= 0 && [\text{by Exercise 65}]
\end{aligned}$$

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

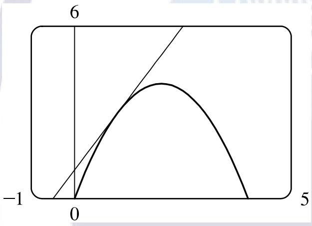
$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x\to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x\to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$
$$= \lim_{x\to 1}(3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h\to 0} \frac{-h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(-h + 2)}{h} = \lim_{h\to 0}(-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

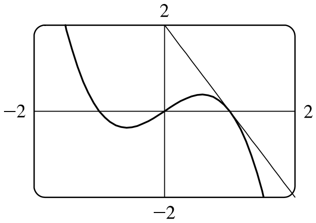
$$m = \lim_{x\to 1} \frac{f(x) - 0}{x - 1} = \lim_{x\to 1} \frac{x - x^3}{x - 1} = \lim_{x\to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x\to 1} \frac{x(1 + x)(1 - x)}{x - 1}$$
$$= \lim_{x\to 1}\left[-x(1 + x)\right] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0} \frac{\left[(1 + h) - (1 + h)^3\right] - 0}{h}$$
$$= \lim_{h\to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h\to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h\to 0} \frac{h(-h^2 - 3h - 2)}{h}$$
$$= \lim_{h\to 0}(-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c)



The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x \to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x \to 2} (-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \quad \Leftrightarrow \quad y + 4 = -8x + 16 \quad \Leftrightarrow \quad y = -8x + 12.$

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^3 - 3(2 + h) + 1 - 3}{h}$$

$$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(9 + 6h + h^2)}{h}$$

$$= \lim_{h \to 0} (9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \quad \Leftrightarrow \quad y - 3 = 9x - 18 \quad \Leftrightarrow \quad y = 9x - 15$

7. Using (1), $m = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x + 1}{x + 2}$ and $P(1, 1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x + 1}{x + 2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x + 1 - (x + 2)}{x + 2}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 2)}$$

$$= \lim_{x \to 1} \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$

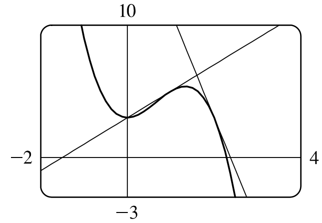
$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$

$$= \lim_{h \to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \quad \Leftrightarrow \quad y = 2x + 3.$

At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \quad \Leftrightarrow \quad y = -8x + 19.$

(c)



10. (a) Using (1),

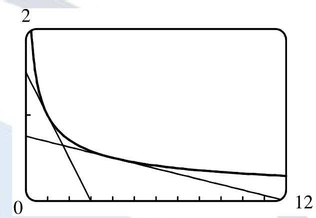
$$m = \lim_{x\to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x-a} = \lim_{x\to a} \frac{\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a} = \lim_{x\to a} \frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})} = \lim_{x\to a} \frac{a-x}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})}$$

$$= \lim_{x\to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a}+\sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \quad [a>0]$$

(b) At $(1,1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

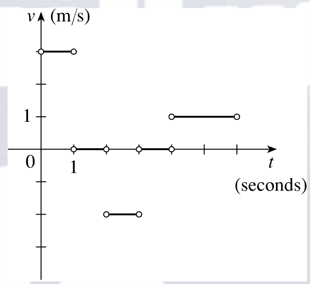
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0,1)$ and $(4,6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2,3)$. The particle is standing still when s is constant; that is, on the intervals $(1,2)$ and $(3,4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0,1)$, the slope is $\dfrac{3-0}{1-0} = 3$. On the interval $(2,3)$, the slope is $\dfrac{1-3}{3-2} = -2$. On the interval $(4,6)$, the slope is $\dfrac{3-1}{6-4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant. **Runner B** starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t)-s(2)}{t-2} = \lim_{t\to 2} \frac{(40t-16t^2)-16}{t-2} = \lim_{t\to 2} \frac{-16t^2+40t-16}{t-2} = \lim_{t\to 2} \frac{-8(2t^2-5t+2)}{t-2}$$

$$= \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8\lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$v(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h\to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h}$$
$$= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h}$$
$$= \lim_{h\to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h}$$
$$= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0} (6.28 - 1.86h) = 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$v(a) = \lim_{h\to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h\to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h}$$
$$= \lim_{h\to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h}$$
$$= \lim_{h\to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h\to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h\to 0} (10 - 3.72a - 1.86h) = 10 - 3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \Leftrightarrow 10t - 1.86t^2 = 0 \Leftrightarrow t(10 - 1.86t) = 0 \Leftrightarrow t = 0$ or $1.86t = 10$.

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$v(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$
$$= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s}$$

So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is

$$\frac{s(t+h) - s(t)}{(t+h) - t} = \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - \left(\frac{1}{2}t^2 - 6t + 23\right)}{h}$$
$$= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h}$$
$$= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h\left(t + \frac{1}{2}h - 6\right)}{h} = \left(t + \tfrac{1}{2}h - 6\right) \text{ ft/s}$$

(i) $[4, 8]$: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft/s.

(ii) $[6, 8]$: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft/s.

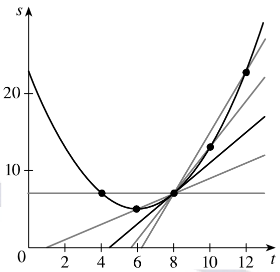
(iii) $[8, 10]$: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft/s.

(iv) $[8, 12]$: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft/s.

(b) $v(t) = \lim_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \left(t + \frac{1}{2}h - 6\right)$

$= t - 6$, so $v(8) = 2$ ft/s.

(c)



17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\dfrac{f(60) - f(20)}{60 - 20} = \dfrac{700 - 300}{40} = \dfrac{400}{40} = 10$

(b) Pick any interval that has the same y-value at its endpoints. $[0, 57]$ is such an interval since $f(0) = 600$ and $f(57) = 600$.

(c) On $[40, 60]$: $\dfrac{f(60) - f(40)}{60 - 40} = \dfrac{700 - 200}{20} = \dfrac{500}{20} = 25$

On $[40, 70]$: $\dfrac{f(70) - f(40)}{70 - 40} = \dfrac{900 - 200}{30} = \dfrac{700}{30} = 23\frac{1}{3}$

Since $25 > 23\frac{1}{3}$, the average rate of change on $[40, 60]$ is larger.

(d) $\dfrac{f(40) - f(10)}{40 - 10} = \dfrac{200 - 400}{30} = \dfrac{-200}{30} = -6\frac{2}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

19. (a) The tangent line at $x = 50$ appears to pass through the points $(43, 200)$ and $(60, 640)$, so

$f'(50) \approx \dfrac{640 - 200}{60 - 43} = \dfrac{440}{17} \approx 26$.

(b) The tangent line at $x = 10$ is steeper than the tangent line at $x = 30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10) < f'(30)$.

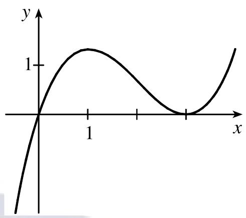
(c) The slope of the tangent line at $x = 60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$. So yes, $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$.

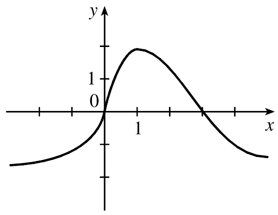
20. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g. Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4. Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

21. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

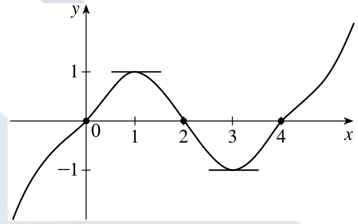
22. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.

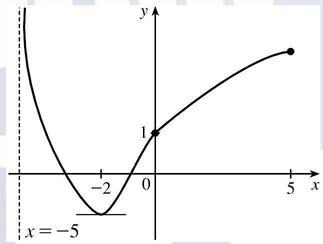




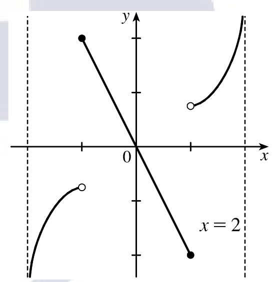
24. We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2, 0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4, 0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.



25. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \to -5^+$, $y \to \infty$, so we draw a vertical asymptote at $x = -5$. As $x \to 5^-$, $y \to 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



26. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \to 2^-$ since $\lim_{x\to 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since f is an odd function.



27. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h\to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h\to 0} \frac{3h - h^3}{h} = \lim_{h\to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h\to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

28. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1}$$

$$= \lim_{x\to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x\to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4$$

Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

29. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

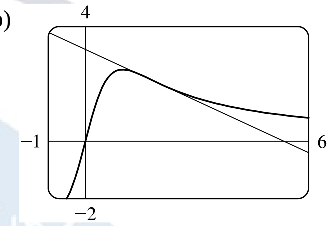
$$F'(2) = \lim_{h\to 0} \frac{F(2 + h) - F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2 + h)}{1 + (2 + h)^2} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h}$$

$$= \lim_{h\to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)



30. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$G'(a) = \lim_{h\to 0} \frac{G(a + h) - G(a)}{h} = \lim_{h\to 0} \frac{[4(a + h)^2 - (a + h)^3] - (4a^2 - a^3)}{h}$$

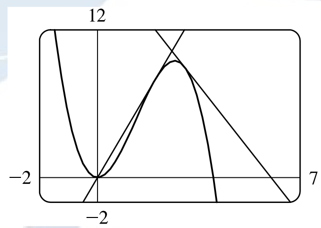
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h}$$

$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h}$$

$$= \lim_{h\to 0} (8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



31. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a + h)^2 - 4(a + h) + 1] - (3a^2 - 4a + 1)]}{h}$$

$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$

$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

32. Use (4) with $f(t) = 2t^3 + t$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{[2(a+h)^3+(a+h)]-(2a^3+a)}{h} \\ &= \lim_{h\to 0}\frac{2a^3+6a^2h+6ah^2+2h^3+a+h-2a^3-a}{h} = \lim_{h\to 0}\frac{6a^2h+6ah^2+2h^3+h}{h} \\ &= \lim_{h\to 0}\frac{h(6a^2+6ah+2h^2+1)}{h} = \lim_{h\to 0}(6a^2+6ah+2h^2+1) = 6a^2+1 \end{aligned}$$

33. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\dfrac{2(a+h)+1}{(a+h)+3}-\dfrac{2a+1}{a+3}}{h} \\ &= \lim_{h\to 0}\frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0}\frac{(2a^2+6a+2ah+6h+a+3)-(2a^2+2ah+6a+a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0}\frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0}\frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2} \end{aligned}$$

34. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{(a+h)^2}-\dfrac{1}{a^2}}{h} = \lim_{h\to 0}\frac{\dfrac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} \\ &= \lim_{h\to 0}\frac{a^2-(a^2+2ah+h^2)}{ha^2(a+h)^2} = \lim_{h\to 0}\frac{-2ah-h^2}{ha^2(a+h)^2} = \lim_{h\to 0}\frac{h(-2a-h)}{ha^2(a+h)^2} \\ &= \lim_{h\to 0}\frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3} \end{aligned}$$

35. Use (4) with $f(x) = \sqrt{1-2x}$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\sqrt{1-2(a+h)}-\sqrt{1-2a}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{1-2(a+h)}-\sqrt{1-2a}}{h}\cdot\frac{\sqrt{1-2(a+h)}+\sqrt{1-2a}}{\sqrt{1-2(a+h)}+\sqrt{1-2a}} \\ &= \lim_{h\to 0}\frac{\left(\sqrt{1-2(a+h)}\right)^2-\left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)} = \lim_{h\to 0}\frac{(1-2a-2h)-(1-2a)}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)} \\ &= \lim_{h\to 0}\frac{-2h}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)} = \lim_{h\to 0}\frac{-2}{\sqrt{1-2(a+h)}+\sqrt{1-2a}} \\ &= \frac{-2}{\sqrt{1-2a}+\sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}} \end{aligned}$$

36. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\dfrac{4}{\sqrt{1-(a+h)}}-\dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4\lim_{h\to 0}\frac{\dfrac{\sqrt{1-a}-\sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4\lim_{h\to 0}\frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4\lim_{h\to 0}\frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}\cdot\frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}} = 4\lim_{h\to 0}\frac{(\sqrt{1-a})^2-(\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0}\frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\lim_{h\to 0}\frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0}\frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\cdot\frac{1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a})}$$

$$= \frac{4}{(1-a)(2\sqrt{1-a})} = \frac{2}{(1-a)^1(1-a)^{1/2}} = \frac{2}{(1-a)^{3/2}}$$

37. By (4), $\displaystyle\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x)=\sqrt{x}$ and $a=9$.

38. By (4), $\displaystyle\lim_{h\to 0}\frac{e^{-2+h}-e^{-2}}{h} = f'(-2)$, where $f(x)=e^x$ and $a=-2$.

39. By Equation 5, $\displaystyle\lim_{x\to 2}\frac{x^6-64}{x-2} = f'(2)$, where $f(x)=x^6$ and $a=2$.

40. By Equation 5, $\displaystyle\lim_{x\to 1/4}\frac{\dfrac{1}{x}-4}{x-\dfrac{1}{4}} = f'(4)$, where $f(x)=\dfrac{1}{x}$ and $a=\dfrac{1}{4}$.

41. By (4), $\displaystyle\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x)=\cos x$ and $a=\pi$.

Or: By (4), $\displaystyle\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x)=\cos(\pi+x)$ and $a=0$.

42. By Equation 5, $\displaystyle\lim_{\theta\to\pi/6}\frac{\sin\theta-\frac{1}{2}}{\theta-\dfrac{\pi}{6}} = f'\left(\frac{\pi}{6}\right)$, where $f(\theta)=\sin\theta$ and $a=\dfrac{\pi}{6}$.

43. $\displaystyle v(4) = f'(4) = \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0}\frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$

$$= \lim_{h\to 0}\frac{(320+80h-96-48h-6h^2)-(320-96)}{h} = \lim_{h\to 0}\frac{32h-6h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(32-6h)}{h} = \lim_{h\to 0}(32-6h) = 32 \text{ m/s}$$

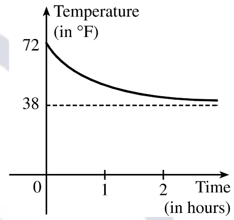
The speed when $t=4$ is $|32| = 32$ m/s.

44. $v(4) = f'(4) = \lim_{h\to 0} \dfrac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \dfrac{\left(10 + \dfrac{45}{4+h+1}\right) - \left(10 + \dfrac{45}{4+1}\right)}{h} = \lim_{h\to 0} \dfrac{\dfrac{45}{5+h} - 9}{h}$

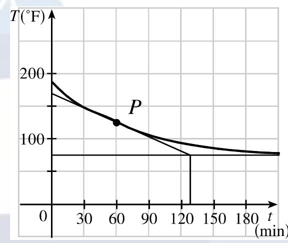
$$= \lim_{h\to 0} \frac{45 - 9(5+h)}{h(5+h)} = \lim_{h\to 0} \frac{-9h}{h(5+h)} = \lim_{h\to 0} \frac{-9}{5+h} = -\frac{9}{5} \text{ m/s.}$$

The speed when $t = 4$ is $\left|-\frac{9}{5}\right| = \frac{9}{5}$ m/s.

45. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,^\circ\text{F/min}$.



47. (a) (i) $[1.0, 2.0]$: $\dfrac{C(2) - C(1)}{2 - 1} = \dfrac{0.18 - 0.33}{1} = -0.15 \dfrac{\text{mg/mL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\dfrac{C(2) - C(1.5)}{2 - 1.5} = \dfrac{0.18 - 0.24}{0.5} = \dfrac{-0.06}{0.5} = -0.12 \dfrac{\text{mg/mL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\dfrac{C(2.5) - C(2)}{2.5 - 2} = \dfrac{0.12 - 0.18}{0.5} = \dfrac{-0.06}{0.5} = -0.12 \dfrac{\text{mg/mL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\dfrac{C(3) - C(2)}{3 - 2} = \dfrac{0.07 - 0.18}{1} = -0.11 \dfrac{\text{mg/mL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$:

$\dfrac{-0.12 + (-0.12)}{2} = -0.12 \dfrac{\text{mg/mL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of 0.12 (mg/mL)/h.

48. (a) (i) $[2006, 2008]$: $\dfrac{N(2008) - N(2006)}{2008 - 2006} = \dfrac{16{,}680 - 12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations/year

(ii) $[2008, 2010]$: $\dfrac{N(2010) - N(2008)}{2010 - 2008} = \dfrac{16{,}858 - 16{,}680}{2} = \dfrac{178}{2} = 89$ locations/year.

The rate of growth decreased over the period from 2006 to 2010.

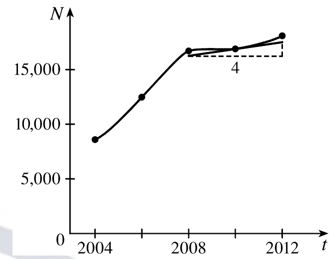
(b) $[2010, 2012]$: $\dfrac{N(2012) - N(2010)}{2012 - 2010} = \dfrac{18{,}066 - 16{,}858}{2} = \dfrac{1208}{2} = 604$ locations/year.

Using that value and the value from part (a)(ii), we have $\dfrac{89 + 604}{2} = \dfrac{693}{2} = 346.5$ locations/year.

(c) The tangent segment has endpoints $(2008, 16{,}250)$ and $(2012, 17{,}500)$.

An estimate of the instantaneous rate of growth in 2010 is

$$\frac{17{,}500 - 16{,}250}{2012 - 2008} = \frac{1250}{4} = 312.5 \text{ locations/year.}$$



49. (a) $[1990, 2005]$: $\dfrac{84{,}077 - 66{,}533}{2005 - 1990} = \dfrac{17{,}544}{15} = 1169.6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169.6 thousands of barrels per day each year from 1990 to 2005.

(b) $[1995, 2000]$: $\dfrac{76{,}784 - 70{,}099}{2000 - 1995} = \dfrac{6685}{5} = 1337$

$[2000, 2005]$: $\dfrac{84{,}077 - 76{,}784}{2005 - 2000} = \dfrac{7293}{5} = 1458.6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}(1337 + 1458.6) = 1397.8$ thousands of barrels per day per year.

50. (a) (i) $[4, 11]$: $\dfrac{V(11) - V(4)}{11 - 4} = \dfrac{9.4 - 53}{7} = \dfrac{-43.6}{7} \approx -6.23 \dfrac{\text{RNA copies/mL}}{\text{day}}$

(ii) $[8, 11]$: $\dfrac{V(11) - V(8)}{11 - 8} = \dfrac{9.4 - 18}{3} = \dfrac{-8.6}{3} \approx -2.87 \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iii) $[11, 15]$: $\dfrac{V(15) - V(11)}{15 - 11} = \dfrac{5.2 - 9.4}{4} = \dfrac{-4.2}{4} = -1.05 \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iv) $[11, 22]$: $\dfrac{V(22) - V(11)}{22 - 11} = \dfrac{3.6 - 9.4}{11} = \dfrac{-5.8}{11} \approx -0.53 \dfrac{\text{RNA copies/mL}}{\text{day}}$

(b) An estimate of $V'(11)$ is the average of the answers from part (a)(ii) and (iii).

$V'(11) \approx \frac{1}{2}[-2.87 + (-1.05)] = -1.96 \dfrac{\text{RNA copies/mL}}{\text{day}}$.

$V'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $$\frac{C(100 + h) - C(100)}{h} = \frac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$
$$= 20 + 0.05h,\ h \neq 0$$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

52. $\Delta V = V(t+h) - V(t) = 100{,}000\left(1 - \dfrac{t+h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$$= 100{,}000\left[\left(1 - \frac{t+h}{30} + \frac{(t+h)^2}{3600}\right) - \left(1 - \frac{t}{30} + \frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30} + \frac{2th}{3600} + \frac{h^2}{3600}\right)$$

$$= \frac{100{,}000}{3600}h\,(-120 + 2t + h) = \frac{250}{9}h\,(-120 + 2t + h)$$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

53. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

54. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

55. (a) $H'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is $58\,^\circ\mathrm{F}$. The units are dollars$/\,^\circ\mathrm{F}$.

(b) If the outside temperature increases, the building should require less heating, so we would expect $H'(58)$ to be negative.

56. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

57. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\mathrm{mg/L})/^\circ\mathrm{C}$.

(b) For $T = 16^\circ\mathrm{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6-14}{32-0} = -\dfrac{8}{32} = -0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$. This means that as the temperature increases past $16^\circ\mathrm{C}$, the oxygen solubility is decreasing at a rate of $0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$.

58. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/^\circ\text{C}$.

(b) For $T = 15^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7\ (\text{cm/s})/^\circ\text{C}$. This tells us that at $T = 15^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 $(\text{cm/s})/^\circ\text{C}$. In a similar fashion for $T = 25^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2\ (\text{cm/s})/^\circ\text{C}$. As it gets warmer than 20°C, the maximum sustainable speed decreases rapidly.

59. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

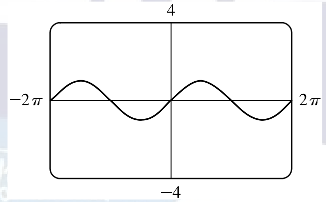
$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 2.2.4.)

60. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

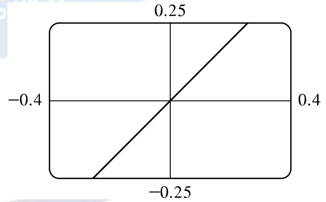
$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h|\sin\dfrac{1}{h} \leq |h| \quad\Rightarrow\quad -|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0}(-|h|) = 0$ and $\lim\limits_{h\to 0}|h| = 0$, we know that $\lim\limits_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

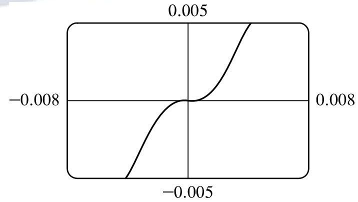
61. (a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.

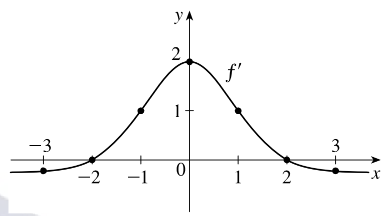


(c) Yes, the slope at the origin now appears to be 0.



2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

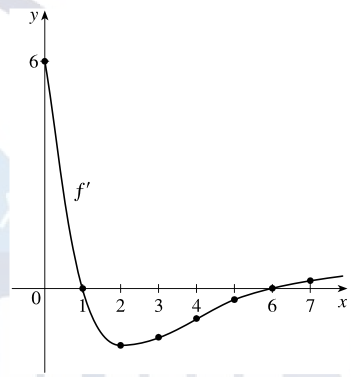


(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$

2. Your answers may vary depending on your estimates.



(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.

(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$ (d) $f'(3) \approx -1.3$ (e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$ (g) $f'(6) \approx 0$ (h) $f'(7) \approx 0.2$

3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

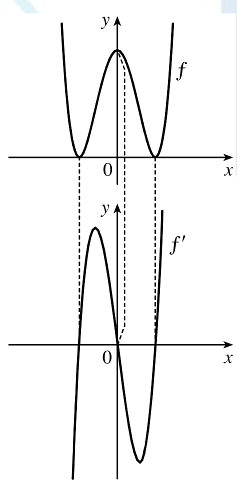
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

4.



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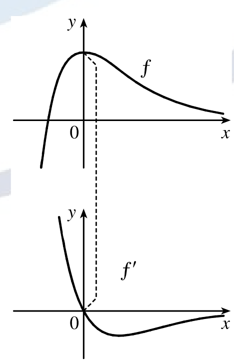

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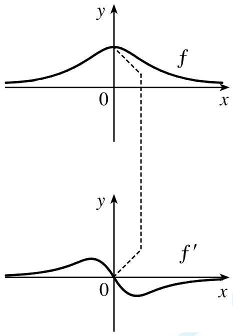

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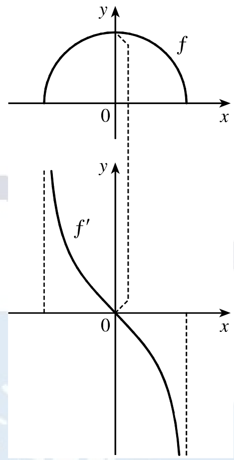
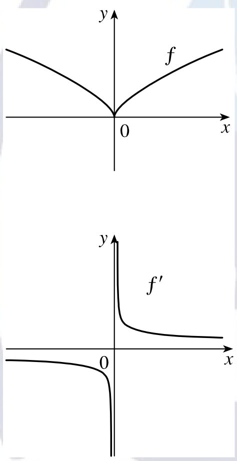


9.



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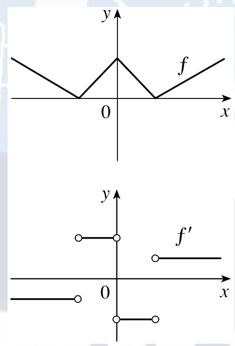
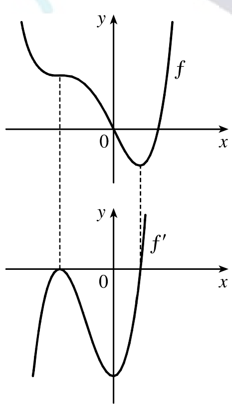
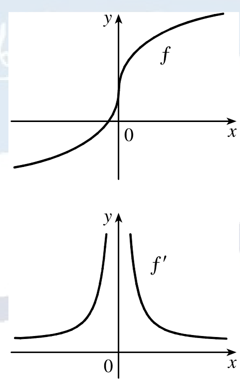


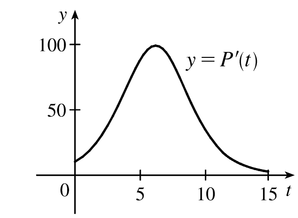
10.



11.

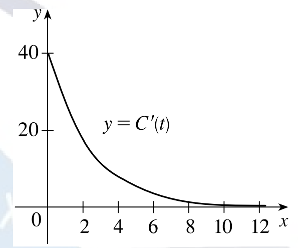


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

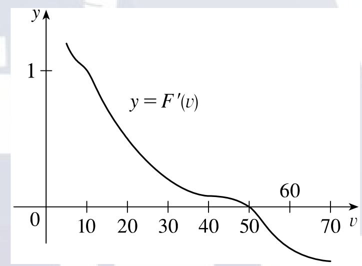
(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



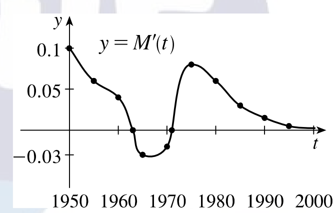
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.

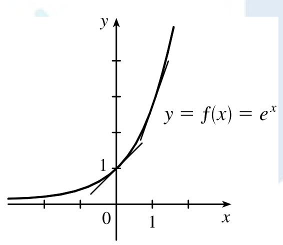


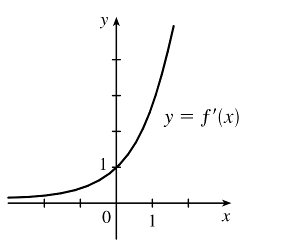
15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



16. See Figure 3.3.1.

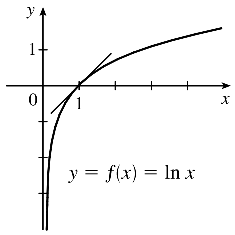
17.

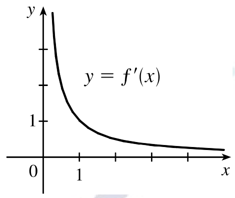




The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

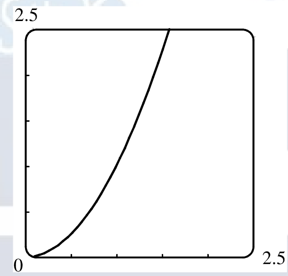
18.





As x increases toward 1, $f'(x)$ decreases from very large numbers to 1. As x becomes large, $f'(x)$ gets closer to 0. As a guess, $f'(x) = 1/x^2$ or $f'(x) = 1/x$ makes sense.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

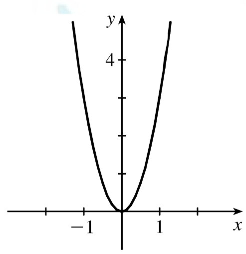
(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

20. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$$

21. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)-8]-(3x-8)}{h} = \lim_{h\to 0} \frac{3x+3h-8-3x+8}{h}$

$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[m(x+h)+b]-(mx+b)}{h} = \lim_{h\to 0} \frac{mx+mh+b-mx-b}{h}$

$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{\left[2.5(t+h)^2+6(t+h)\right]-\left(2.5t^2+6t\right)}{h}$

$= \lim_{h\to 0} \frac{2.5(t^2+2th+h^2)+6t+6h-2.5t^2-6t}{h} = \lim_{h\to 0} \frac{2.5t^2+5th+2.5h^2+6h-2.5t^2}{h}$

$= \lim_{h\to 0} \frac{5th+2.5h^2+6h}{h} = \lim_{h\to 0} \frac{h\,(5t+2.5h+6)}{h} = \lim_{h\to 0} (5t+2.5h+6)$

$= 5t+6$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[4+8(x+h)-5(x+h)^2\right]-(4+8x-5x^2)}{h}$

$= \lim_{h\to 0} \frac{4+8x+8h-5(x^2+2xh+h^2)-4-8x+5x^2}{h} = \lim_{h\to 0} \frac{8h-5x^2-10xh-5h^2+5x^2}{h}$

$= \lim_{h\to 0} \frac{8h-10xh-5h^2}{h} = \lim_{h\to 0} \frac{h(8-10x-5h)}{h} = \lim_{h\to 0} (8-10x-5h)$

$= 8-10x$

Domain of f = domain of $f' = \mathbb{R}$.

25. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}$

$= \lim_{h\to 0} \frac{x^2+2xh+h^2-2x^3-6x^2h-6xh^2-2h^3-x^2+2x^3}{h}$

$= \lim_{h\to 0} \frac{2xh+h^2-6x^2h-6xh^2-2h^3}{h} = \lim_{h\to 0} \frac{h(2x+h-6x^2-6xh-2h^2)}{h}$

$= \lim_{h\to 0} (2x+h-6x^2-6xh-2h^2) = 2x-6x^2$

Domain of f = domain of $f' = \mathbb{R}$.

26. $g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \frac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}} \right)$

$= \lim_{h\to 0} \frac{t-(t+h)}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \frac{-h}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}$

$= \frac{-1}{\sqrt{t}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)} = \frac{-1}{t\left(2\sqrt{t}\right)} = -\frac{1}{2t^{3/2}}$

Domain of g = domain of $g' = (0, \infty)$.

27. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\dfrac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$

$$= \lim_{h\to 0} \frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

28. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[(x+h)^2-1](2x-3)-[2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)-(2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$$

$$= \lim_{h\to 0} \frac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3)-(2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim_{h\to 0} \frac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Domain of f = domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

29. $G'(t) = \lim\limits_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1-2(t+h)}{3+(t+h)} - \dfrac{1-2t}{3+t}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$$

$$= \lim_{h\to 0} \frac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0} \frac{-6h-h}{h(3+t+h)(3+t)}$$

$$= \lim_{h\to 0} \frac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0} \frac{-7}{(3+t+h)(3+t)} = \frac{-7}{(3+t)^2}$$

Domain of G = domain of $G' = (-\infty, -3) \cup (-3, \infty)$.

30. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^{3/2}-x^{3/2}}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^{3/2}-x^{3/2}][(x+h)^{3/2}+x^{3/2}]}{h\,[(x+h)^{3/2}+x^{3/2}]}$

$$= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h[(x+h)^{3/2}+x^{3/2}]}$$

$$= \lim_{h\to 0} \frac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \tfrac{3}{2}x^{1/2}$$

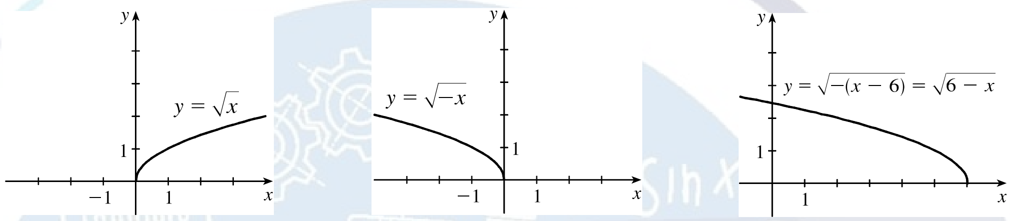
Domain of f = domain of $f' = [0, \infty)$. Strictly speaking, the domain of f' is $(0, \infty)$ because the limit that defines $f'(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 64) does exist at 0, so in that sense one could regard the domain of f' to be $[0, \infty)$.

31. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x+h)^4 - x^4}{h} = \lim\limits_{h \to 0} \dfrac{\left(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4\right) - x^4}{h}$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

32. (a)



(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

(c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

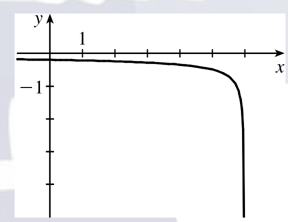
$$= \lim_{h \to 0} \frac{\sqrt{6 - (x+h)} - \sqrt{6 - x}}{h} \left[\frac{\sqrt{6 - (x+h)} + \sqrt{6 - x}}{\sqrt{6 - (x+h)} + \sqrt{6 - x}}\right]$$

$$= \lim_{h \to 0} \frac{[6 - (x+h)] - (6 - x)}{h\left[\sqrt{6 - (x+h)} + \sqrt{6 - x}\right]} = \lim_{h \to 0} \frac{-h}{h\left(\sqrt{6 - x - h} + \sqrt{6 - x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{6 - x - h} + \sqrt{6 - x}} = \frac{-1}{2\sqrt{6 - x}}$$

Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



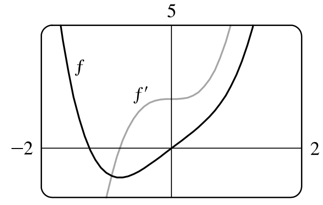
33. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h}$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h}$$

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.

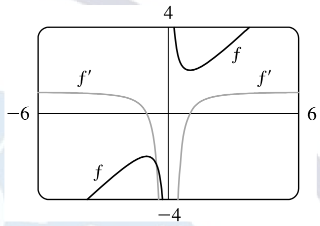


34. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)+1/(x+h)]-(x+1/x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(x+h)^2+1}{x+h} - \dfrac{x^2+1}{x}}{h}$

$$= \lim_{h\to 0} \frac{x[(x+h)^2+1]-(x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0} \frac{(x^3+2hx^2+xh^2+x)-(x^3+x+hx^2+h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{hx^2+xh^2-h}{h(x+h)x} = \lim_{h\to 0} \frac{h(x^2+xh-1)}{h(x+h)x} = \lim_{h\to 0} \frac{x^2+xh-1}{(x+h)x} = \frac{x^2-1}{x^2}, \text{ or } 1-\frac{1}{x^2}$$

(b) Notice that $f'(x)=0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x=0$.



35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $U'(t)$, we use $\lim_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 2003: $U'(2003) \approx \dfrac{U(2004)-U(2003)}{2004-2003} = \dfrac{5.5-6.0}{1} = -0.5$

For 2004: We estimate $U'(2004)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \;\Rightarrow\; U'(2004) \approx \dfrac{U(2003)-U(2004)}{2003-2004} = \dfrac{6.0-5.5}{-1} = -0.5;$

$h=1 \;\Rightarrow\; U'(2004) \approx \dfrac{U(2005)-U(2004)}{2005-2004} = \dfrac{5.1-5.5}{1} = -0.4.$

So we estimate that $U'(2004) \approx \frac{1}{2}[-0.5+(-0.4)] = -0.45$.

t	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$U'(t)$	−0.50	−0.45	−0.45	−0.25	0.60	2.35	1.90	−0.20	−0.75	−0.80

36. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim_{h\to 0} \dfrac{N(t+h)-N(t)}{h} \approx \dfrac{N(t+h)-N(t)}{h}$ for small values of h.

For 2000: $N'(2000) \approx \dfrac{N(2002)-N(2000)}{2002-2000} = \dfrac{4897-5500}{2} = -301.5$

For 2002: We estimate $N'(2002)$ by using $h=-2$ and $h=2$, and then average the two results to obtain a final estimate.

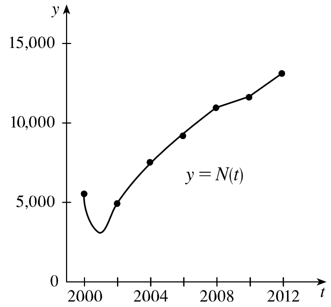
$h=-2 \;\Rightarrow\; N'(2002) \approx \dfrac{N(2000)-N(2002)}{2000-2002} = \dfrac{5500-4897}{-2} = -301.5$

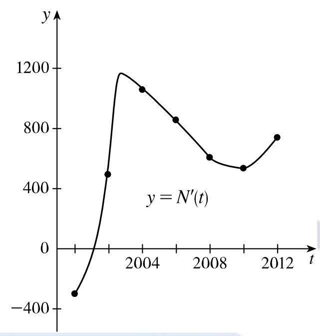
$h=2 \;\Rightarrow\; N'(2002) \approx \dfrac{N(2004)-N(2002)}{2004-2002} = \dfrac{7470-4897}{2} = 1286.5$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5+1286.5] = 492.5$.

t	2000	2002	2004	2006	2008	2010	2012
$N'(t)$	−301.5	492.5	1060.25	856.75	605.75	534.5	737

(c)

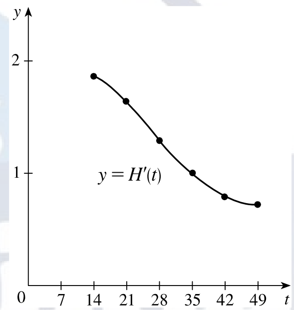




(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t.

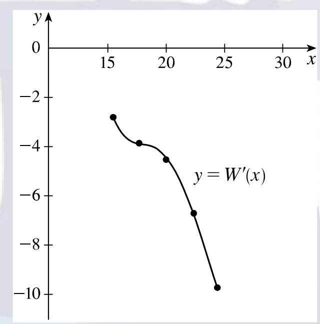
37. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	41	54	64	72	78	83
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



38. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree $(\text{g}/^\circ\text{C})$.

x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	−9.8
$W'(x)$	−2.82	−3.87	−4.53	−6.73	−9.75



39. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t, measured in percentage points per year.

(b) 2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

40. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.

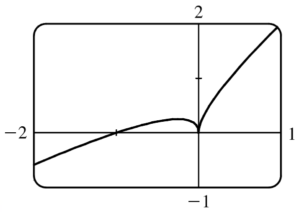
41. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

42. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

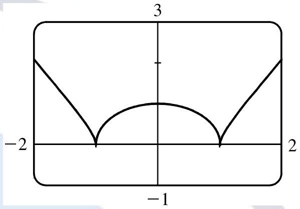
43. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

44. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

45. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



46. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



47. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

48. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

49. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

50. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

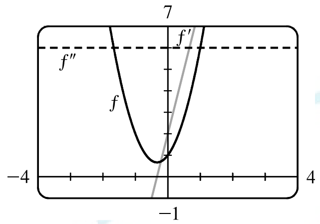
51. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

52. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

53. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$
$$= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[6(x+h)+2] - (6x+2)}{h} = \lim_{h \to 0} \frac{(6x+6h+2) - (6x+2)}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

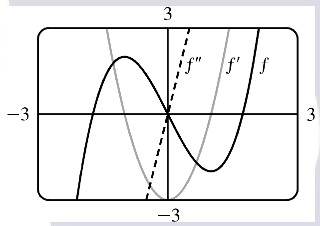
54. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{h(6x + 3h)}{h} = \lim_{h \to 0} (6x + 3h) = 6x$$



We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

55. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[2(x+h)^2 - (x+h)^3\right] - (2x^2 - x^3)}{h}$$

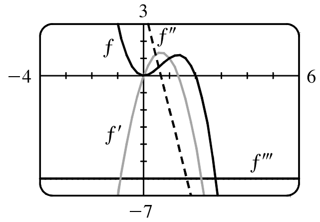
$$= \lim_{h \to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{\left[4(x+h) - 3(x+h)^2\right] - (4x - 3x^2)}{h} = \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h}$$

$$= \lim_{h \to 0} (4 - 6x - 3h) = 4 - 6x$$

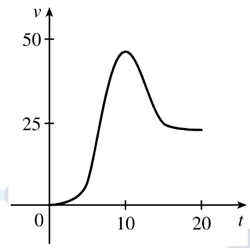
$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \to 0} \frac{-6h}{h} = \lim_{h \to 0} (-6) = -6$$

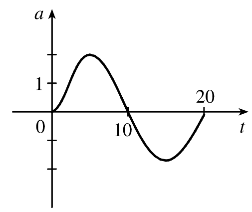
$$f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \to 0} \frac{-6 - (-6)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

56. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by 10 ft/s^2 over a period of 20 s.

So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

57. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$= \lim_{x\to a}\frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{\sqrt[3]{h}-0}{h} = \lim\limits_{h\to 0}\dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x\to 0}|f'(x)| = \lim\limits_{x\to 0}\dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

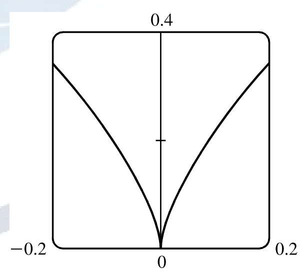
58. (a) $g'(0) = \lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^{2/3}-0}{x} = \lim\limits_{x\to 0}\dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a} = \lim_{x\to a}\frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$= \lim_{x\to a}\frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim\limits_{x\to 0}|g'(x)| = \lim\limits_{x\to 0}\dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)



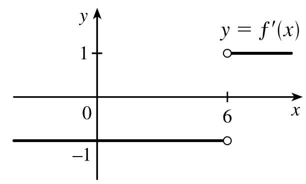
59. $f(x) = |x-6| = \begin{cases} x-6 & \text{if } x-6 \geq 6 \\ -(x-6) & \text{if } x-6 < 0 \end{cases} = \begin{cases} x-6 & \text{if } x \geq 6 \\ 6-x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim\limits_{x\to 6^+}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^+}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^+}\dfrac{x-6}{x-6} = \lim\limits_{x\to 6^+}1 = 1$, and the left-hand limit is $\lim\limits_{x\to 6^-}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^-}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^-}\dfrac{6-x}{x-6} = \lim\limits_{x\to 6^-}(-1) = -1$. Since these limits are not equal,

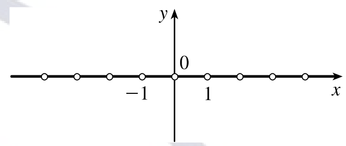
$f'(6) = \lim\limits_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

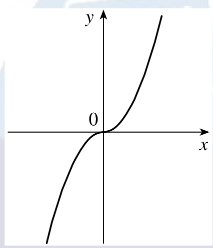
Another way of writing the formula is $f'(x) = \dfrac{x-6}{|x-6|}$.



60. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



61. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 19(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x\,|x|}{x} = \lim_{x \to 0} |x| = 0.$$

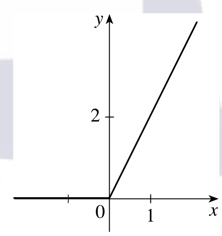
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{ll} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2\,|x|$.

62. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

so $f(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

63. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

64. (a) $f'_-(4) = \lim_{h\to 0^-}\dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-}\dfrac{5-(4+h)-1}{h}$

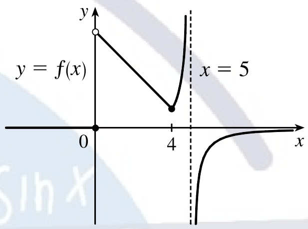
$= \lim_{h\to 0^-}\dfrac{-h}{h} = -1$

and

$f'_+(4) = \lim_{h\to 0^+}\dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+}\dfrac{\dfrac{1}{5-(4+h)}-1}{h}$

$= \lim_{h\to 0^+}\dfrac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+}\dfrac{1}{1-h} = 1$

(b)



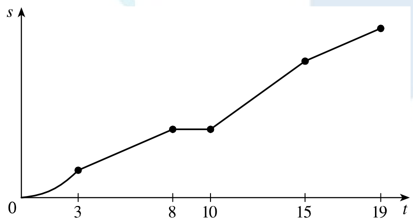
(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

At 4 we have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5-x) = 1$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\dfrac{1}{5-x} = 1$, so $\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5. These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5-x) = 5 \ne 0 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

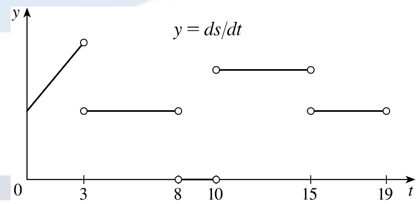
(d) From (a), f is not differentiable at 4 since $f'_-(4) \ne f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

65. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

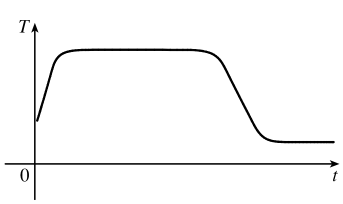
(a)



(b)

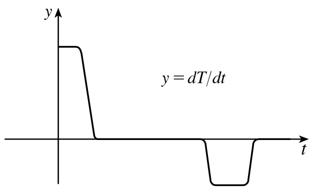


66. (a)

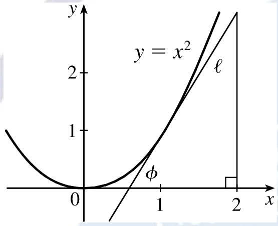


(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



67.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

2 Review

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. False. $\dfrac{x^2 - 9}{x - 3}$ is not defined when $x = 3$, but $x + 3$ is.

5. True. $\displaystyle\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x\to 3}(x+3)$

6. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

7. False. Consider $\displaystyle\lim_{x\to 5} \frac{x(x-5)}{x-5}$ or $\displaystyle\lim_{x\to 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 2.2.3, we know that the latter limit exists (and it is equal to 1).

8. False. If $f(x) = 1/x$, $g(x) = -1/x$, and $a = 0$, then $\displaystyle\lim_{x\to 0} f(x)$ does not exist, $\displaystyle\lim_{x\to 0} g(x)$ does not exist, but $\displaystyle\lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} 0 = 0$ exists.

9. True. Suppose that $\displaystyle\lim_{x\to a} [f(x) + g(x)]$ exists. Now $\displaystyle\lim_{x\to a} f(x)$ exists and $\displaystyle\lim_{x\to a} g(x)$ does not exist, but $\displaystyle\lim_{x\to a} g(x) = \lim_{x\to a} \{[f(x) + g(x)] - f(x)\} = \lim_{x\to a} [f(x) + g(x)] - \lim_{x\to a} f(x)$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\displaystyle\lim_{x\to a} [f(x) + g(x)]$ does not exist.

10. False. Consider $\lim_{x\to 6}[f(x)g(x)] = \lim_{x\to 6}\left[(x-6)\,\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

11. True. A polynomial is continuous everywhere, so $\lim_{x\to b} p(x)$ exists and is equal to $p(b)$.

12. False. Consider $\lim_{x\to 0}[f(x) - g(x)] = \lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

13. True. See Figure 2.6.8.

14. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x\to\infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

15. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

16. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

17. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

18. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

19. True, by the definition of a limit with $\varepsilon = 1$.

20. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 + 1\right) = 1$.

21. False. See the note after Theorem 2.8.4.

22. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x) = f(r)$.

23. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

24. True. $f(x) = x^{10} - 10x^2 + 5$ is continuous on the interval $[0, 2]$, $f(0) = 5$, $f(1) = -4$, and $f(2) = 989$. Since $-4 < 0 < 5$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^{10} - 10x^2 + 5 = 0$ in the interval $(0, 1)$. Similarly, there is a root in $(1, 2)$.

25. True. See Exercise 2.5.72(b).

26. False See Exercise 2.5.72(b).

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x) = 3$ (ii) $\lim_{x\to -3^+} f(x) = 0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x\to 4} f(x) = 2$

(v) $\lim_{x\to 0} f(x) = \infty$ (vi) $\lim_{x\to 2^-} f(x) = -\infty$

(vii) $\lim_{x\to \infty} f(x) = 4$ (viii) $\lim_{x\to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

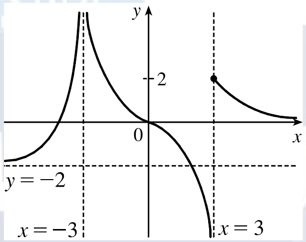
(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. $\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to \infty} f(x) = 0$, $\lim_{x\to -3} f(x) = \infty$,

$\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = 2$,

f is continuous from the right at 3



3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

4. Since rational functions are continuous, $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0$.

5. $\lim_{x\to -3} \dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0^+$ as $x \to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0} \dfrac{(h-1)^3+1}{h} = \lim_{h\to 0} \dfrac{\left(h^3-3h^2+3h-1\right)+1}{h} = \lim_{h\to 0} \dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0} \left(h^2-3h+3\right) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to 0} \frac{(h-1)^3+1}{h} &= \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h} \\ &= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3\end{aligned}$$

8. $\lim_{t\to 2} \dfrac{t^2-4}{t^3-8} = \lim_{t\to 2} \dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0^+$ as $r \to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\displaystyle\lim_{v\to 4^+} \frac{4-v}{|4-v|} = \lim_{v\to 4^+} \frac{4-v}{-(4-v)} = \lim_{v\to 4^+} \frac{1}{-1} = -1$

11. $\displaystyle\lim_{u\to 1} \frac{u^4-1}{u^3+5u^2-6u} = \lim_{u\to 1} \frac{(u^2+1)(u^2-1)}{u(u^2+5u-6)} = \lim_{u\to 1} \frac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)} = \lim_{u\to 1} \frac{(u^2+1)(u+1)}{u(u+6)} = \frac{2(2)}{1(7)} = \frac{4}{7}$

12. $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} = \lim_{x\to 3} \left[\frac{\sqrt{x+6}-x}{x^2(x-3)} \cdot \frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right] = \lim_{x\to 3} \frac{(\sqrt{x+6})^2-x^2}{x^2(x-3)(\sqrt{x+6}+x)}$

$$= \lim_{x\to 3} \frac{x+6-x^2}{x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x^2-x-6)}{x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6}+x)}$$

$$= \lim_{x\to 3} \frac{-(x+2)}{x^2(\sqrt{x+6}+x)} = -\frac{5}{9(3+3)} = -\frac{5}{54}$$

13. Since x is positive, $\sqrt{x^2} = |x| = x$. Thus,

$$\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to\infty} \frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x} = \lim_{x\to\infty} \frac{\sqrt{1-9/x^2}}{2-6/x} = \frac{\sqrt{1-0}}{2-0} = \frac{1}{2}$$

14. Since x is negative, $\sqrt{x^2} = |x| = -x$. Thus,

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to-\infty} \frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/(-x)} = \lim_{x\to-\infty} \frac{\sqrt{1-9/x^2}}{-2+6/x} = \frac{\sqrt{1-0}}{-2+0} = -\frac{1}{2}$$

15. Let $t = \sin x$. Then as $x \to \pi^-$, $\sin x \to 0^+$, so $t \to 0^+$. Thus, $\displaystyle\lim_{x\to\pi^-} \ln(\sin x) = \lim_{t\to 0^+} \ln t = -\infty$.

16. $\displaystyle\lim_{x\to-\infty} \frac{1-2x^2-x^4}{5+x-3x^4} = \lim_{x\to-\infty} \frac{(1-2x^2-x^4)/x^4}{(5+x-3x^4)/x^4} = \lim_{x\to-\infty} \frac{1/x^4-2/x^2-1}{5/x^4+1/x^3-3} = \frac{0-0-1}{0+0-3} = \frac{-1}{-3} = \frac{1}{3}$

17. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+4x+1}-x\right) = \lim_{x\to\infty} \left[\frac{\sqrt{x^2+4x+1}-x}{1} \cdot \frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right] = \lim_{x\to\infty} \frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}$

$$= \lim_{x\to\infty} \frac{(4x+1)/x}{(\sqrt{x^2+4x+1}+x)/x} \qquad \left[\text{divide by } x = \sqrt{x^2} \text{ for } x > 0\right]$$

$$= \lim_{x\to\infty} \frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1} = \frac{4+0}{\sqrt{1+0+0}+1} = \frac{4}{2} = 2$$

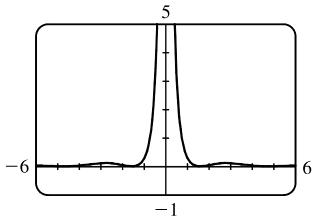
18. Let $t = x - x^2 = x(1-x)$. Then as $x \to \infty$, $t \to -\infty$, and $\displaystyle\lim_{x\to\infty} e^{x-x^2} = \lim_{t\to-\infty} e^t = 0$.

19. Let $t = 1/x$. Then as $x \to 0^+$, $t \to \infty$, and $\displaystyle\lim_{x\to 0^+} \tan^{-1}(1/x) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$.

20. $\displaystyle\lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right) = \lim_{x\to 1} \left[\frac{1}{x-1} + \frac{1}{(x-1)(x-2)}\right] = \lim_{x\to 1} \left[\frac{x-2}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}\right]$

$$= \lim_{x\to 1} \left[\frac{x-1}{(x-1)(x-2)}\right] = \lim_{x\to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1$$

21. From the graph of $y = (\cos^2 x)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1 \quad\Rightarrow$ $\dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2} \quad\Rightarrow\quad 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x\to\pm\infty} 0 = 0$ and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \dfrac{\cos^2 x}{x^2} = 0$.



Thus, $y = 0$ is the horizontal asymptote. $\lim\limits_{x\to 0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2 \to 0^+$ as $x \to 0$, so $x = 0$ is the vertical asymptote.

22. From the graph of $y = f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$\begin{aligned} f_1(x) &= \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\ &= \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \end{aligned}$$

Now

$$\begin{aligned} \lim_{x\to\infty} f_1(x) &= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\ &= \lim_{x\to\infty} \frac{2+(1/x)}{\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}} \qquad \left[\text{since } \sqrt{x^2} = x \text{ for } x > 0\right] \\ &= \frac{2}{1+1} = 1, \end{aligned}$$

so $y = 1$ is a horizontal asymptote. For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the denominator by x, with $x < 0$, we get

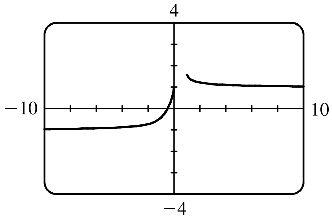
$$\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{x} = -\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2}} = -\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}\right]$$

Therefore,

$$\begin{aligned} \lim_{x\to-\infty} f_1(x) &= \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2+(1/x)}{-\left[\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}\right]} \\ &= \frac{2}{-(1+1)} = -1, \end{aligned}$$

so $y = -1$ is a horizontal asymptote.

The domain of f is $(-\infty, 0] \cup [1, \infty)$. As $x \to 0^-$, $f(x) \to 1$, so $x = 0$ is *not* a vertical asymptote. As $x \to 1^+$, $f(x) \to \sqrt{3}$, so $x = 1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



23. Since $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim\limits_{x \to 1}(2x-1) = 1 = \lim\limits_{x \to 1} x^2$, we have $\lim\limits_{x \to 1} f(x) = 1$ by the Squeeze Theorem.

24. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have

$f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} h(x) = 0 \;\Rightarrow\; \lim\limits_{x \to 0} g(x) = 0$ by the Squeeze Theorem.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|(14-5x) - 4| < \varepsilon$. But $|(14-5x)-4| < \varepsilon \;\Leftrightarrow$ $|-5x+10| < \varepsilon \;\Leftrightarrow\; |-5|\,|x-2| < \varepsilon \;\Leftrightarrow\; |x-2| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $0 < |x-2| < \delta \;\Rightarrow$ $|(14-5x)-4| < \varepsilon$. Thus, $\lim\limits_{x \to 2}(14-5x) = 4$ by the definition of a limit.

26. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x-0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \;\Rightarrow$ $|x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x-0| = |x| < \varepsilon^3 \;\Rightarrow\; |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$.
Therefore, by the definition of a limit, $\lim\limits_{x \to 0} \sqrt[3]{x} = 0$.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-2| < \delta$, then $\left|x^2 - 3x - (-2)\right| < \varepsilon$. First, note that if $|x-2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \;\Rightarrow\; |x-1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x-2| < \delta \;\Rightarrow$
$\left|x^2 - 3x - (-2)\right| = |(x-2)(x-1)| = |x-2|\,|x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim\limits_{x \to 2}(x^2 - 3x) = -2$ by the definition of a limit.

28. Given $M > 0$, we need $\delta > 0$ such that if $0 < x - 4 < \delta$, then $2/\sqrt{x-4} > M$. This is true $\;\Leftrightarrow\; \sqrt{x-4} < 2/M \;\Leftrightarrow$ $x - 4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x - 4 < \delta \;\Rightarrow\; 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim\limits_{x \to 4^+}\left(2/\sqrt{x-4}\right) = \infty$.

29. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \le x < 3$, $f(x) = (x-3)^2$ if $x > 3$.

(i) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+}(3-x) = 3$

(ii) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim\limits_{x \to 0} f(x)$ does not exist.

(iv) $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-}(3-x) = 0$

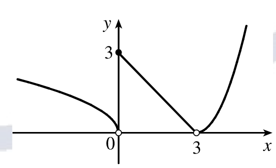
(v) $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim\limits_{x \to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim\limits_{x \to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

(c)



30. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$.

Therefore, $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-}\left(2x - x^2\right) = 0$ and $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+}(2-x) = 0$. Thus, $\lim\limits_{x \to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim\limits_{x \to 3^-} g(x) = \lim\limits_{x \to 3^-}(2-x) = -1$ and $\lim\limits_{x \to 3^+} g(x) = \lim\limits_{x \to 3^+}(x-4) = -1$. Thus,

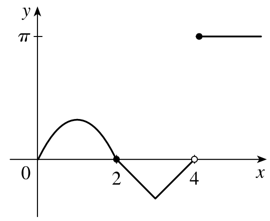
$\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3.

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x-4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \pi = \pi$.

Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But

$\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



31. $\sin x$ and e^x are continuous on $\mathbb{R}$ by Theorem 2.5.7. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 2.5.9. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 2.5.4.

32. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$ by Theorem 2.5.7, so the composition $\sqrt{x^2-9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.9. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.4.

33. $f(x) = x^5 - x^3 + 3x - 5$ is continuous on the interval $[1, 2]$, $f(1) = -2$, and $f(2) = 25$. Since $-2 < 0 < 25$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^3 + 3x - 5 = 0$ in the interval $(1, 2)$.

34. $f(x) = \cos\sqrt{x} - e^x + 2$ is continuous on the interval $[0, 1]$, $f(0) = 2$, and $f(1) \approx -0.2$. Since $-0.2 < 0 < 2$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos\sqrt{x} - e^x + 2 = 0$, or $\cos\sqrt{x} = e^x - 2$, in the interval $(0, 1)$.

35. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{9 - 2x^2 - 1}{x-2} = \lim_{x\to 2} \frac{8 - 2x^2}{x-2} = \lim_{x\to 2} \frac{-2(x^2-4)}{x-2} = \lim_{x\to 2} \frac{-2(x-2)(x+2)}{x-2}$$
$$= \lim_{x\to 2} [-2(x+2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x-2)$ or $y = -8x + 17$.

36. For a general point with x-coordinate a, we have

$$m = \lim_{x\to a} \frac{2/(1-3x) - 2/(1-3a)}{x-a} = \lim_{x\to a} \frac{2(1-3a) - 2(1-3x)}{(1-3a)(1-3x)(x-a)} = \lim_{x\to a} \frac{6(x-a)}{(1-3a)(1-3x)(x-a)}$$
$$= \lim_{x\to a} \frac{6}{(1-3a)(1-3x)} = \frac{6}{(1-3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x-0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x+1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

37. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10+h}{4}$$

[continued]

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s (ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s (iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim\limits_{h \to 0} \dfrac{s(1 + h) - s(1)}{h} = \lim\limits_{h \to 0} \dfrac{10 + h}{4} = \dfrac{10}{4} = 2.5$ m/s.

38. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change

is $\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016 \ \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

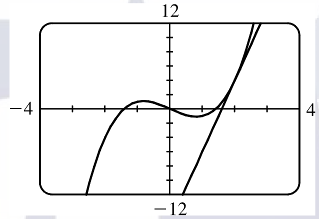
$$\begin{aligned}\lim_{h \to 0} \frac{\Delta V}{\Delta P} &= \lim_{h \to 0} \frac{V(P + h) - V(P)}{h} = \lim_{h \to 0} \frac{800/(P + h) - 800/P}{h} = \lim_{h \to 0} \frac{800\,[P - (P + h)]}{h(P + h)P} \\ &= \lim_{h \to 0} \frac{-800}{(P + h)P} = -\frac{800}{P^2}\end{aligned}$$

which is inversely proportional to the square of P.

39. (a) $$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 2) = 10 \end{aligned}$$

(b) $y - 4 = 10(x - 2)$ or $y = 10x - 16$

(c)



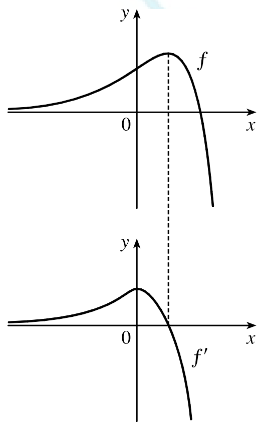
40. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

41. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

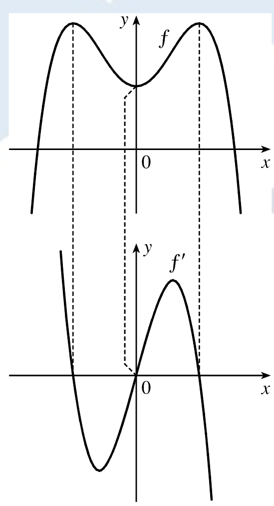
(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

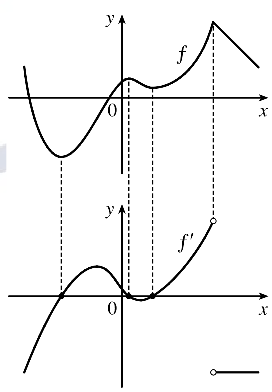
42.



43.



44.



45. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

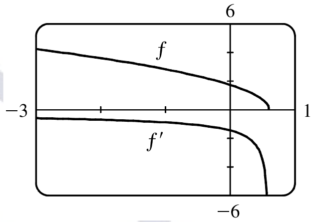
$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \quad \Rightarrow$

$5x \leq 3 \quad \Rightarrow \quad x \in \left(-\infty, \frac{3}{5}\right]$

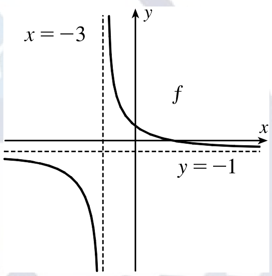
Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$

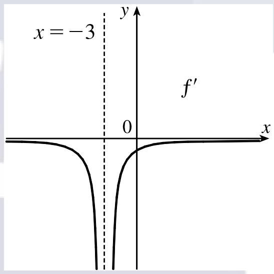


(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

46. (a) As $x \to \pm\infty$, $f(x) = (4-x)/(3+x) \to -1$, so there is a horizontal asymptote at $y=-1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x=-3$.



(b) Note that f is decreasing on $(-\infty,-3)$ and $(-3,\infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim_{h\to 0} \dfrac{(3+x)\,[4-(x+h)]-(4-x)\,[3+(x+h)]}{h\,[3+(x+h)]\,(3+x)}$

$$= \lim_{h\to 0} \frac{(12-3x-3h+4x-x^2-hx)-(12+4x+4h-3x-x^2-hx)}{h[3+(x+h)](3+x)}$$

$$= \lim_{h\to 0} \frac{-7h}{h\,[3+(x+h)]\,(3+x)} = \lim_{h\to 0} \frac{-7}{[3+(x+h)]\,(3+x)} = -\frac{7}{(3+x)^2}$$

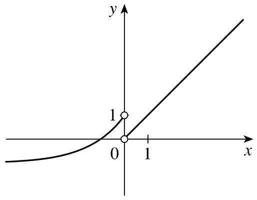
(d) The graphing device confirms our graph in part (b).

47. f is not differentiable: at $x=-4$ because f is not continuous, at $x=-1$ because f has a corner, at $x=2$ because f is not continuous, and at $x=5$ because f has a vertical tangent.

48. The graph of a has tangent lines with positive slope for $x<0$ and negative slope for $x>0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

49. Domain: $(-\infty, 0) \cup (0, \infty)$; $\lim\limits_{x \to 0^-} f(x) = 1$; $\lim\limits_{x \to 0^+} f(x) = 0$;

$f'(x) > 0$ for all x in the domain; $\lim\limits_{x \to -\infty} f'(x) = 0$; $\lim\limits_{x \to \infty} f'(x) = 1$



50. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim\limits_{h \to 0} \dfrac{P(t+h) - P(t)}{h} \approx \dfrac{P(t+h) - P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \dfrac{P(1960) - P(1950)}{1960 - 1950} = \dfrac{35.7 - 31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h = -10$ and $h = 10$, and then average the two results to obtain a final estimate.

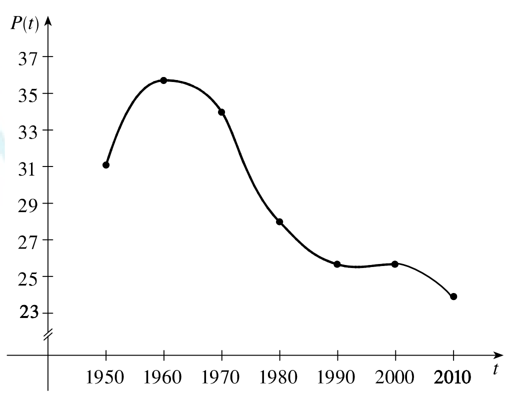
$h = -10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1950) - P(1960)}{1950 - 1960} = \dfrac{31.1 - 35.7}{-10} = 0.46$

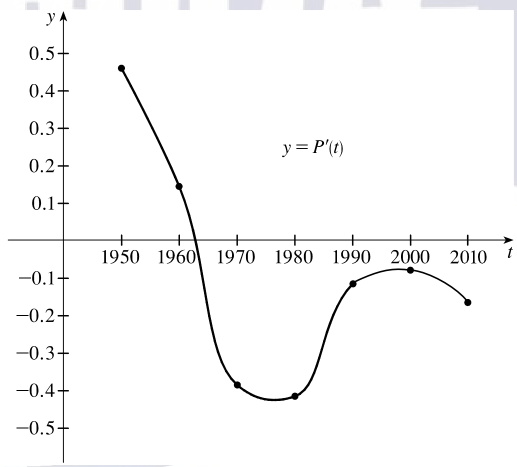
$h = 10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1970) - P(1960)}{1970 - 1960} = \dfrac{34.0 - 35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	−0.085	−0.170

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

51. $B'(t)$ is the rate at which the number of US \$20 bills in circulation is changing with respect to time. Its units are billions of bills per year. We use a symmetric difference quotient to estimate $B'(2000)$.

$B'(2000) \approx \dfrac{B(2005) - B(1995)}{2005 - 1995} = \dfrac{5.77 - 4.21}{10} = 0.156$ billions of bills per year (or 156 million bills per year).

52. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

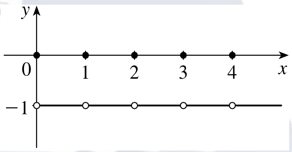
(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

53. $|f(x)| \le g(x) \quad \Leftrightarrow \quad -g(x) \le f(x) \le g(x)$ and $\lim\limits_{x\to a} g(x) = 0 = \lim\limits_{x\to a} -g(x)$.

Thus, by the Squeeze Theorem, $\lim\limits_{x\to a} f(x) = 0$.

54. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer, $[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim\limits_{x\to a} f(x)$ exists (and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.

☐ PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\displaystyle\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}\cdot\frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to 0}\frac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes $\displaystyle\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \frac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4$.

Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0}\frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0}\frac{-4x}{x} = \lim_{x\to 0}(-4) = -4$.

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$.

Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But $m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \frac{1}{2}x^2 + \frac{1}{2}$. As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. (a) For $0 < x < 1$, $[\![x]\!] = 0$, so $\dfrac{[\![x]\!]}{x} = 0$, and $\displaystyle\lim_{x\to 0^+}\frac{[\![x]\!]}{x} = 0$. For $-1 < x < 0$, $[\![x]\!] = -1$, so $\dfrac{[\![x]\!]}{x} = \dfrac{-1}{x}$, and

$\displaystyle\lim_{x\to 0^-}\frac{[\![x]\!]}{x} = \lim_{x\to 0^-}\left(\frac{-1}{x}\right) = \infty$. Since the one-sided limits are not equal, $\displaystyle\lim_{x\to 0}\frac{[\![x]\!]}{x}$ does not exist.

(b) For $x > 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \;\Rightarrow\; x(1/x-1) \le x[\![1/x]\!] \le x(1/x) \;\Rightarrow\; 1-x \le x[\![1/x]\!] \le 1$.

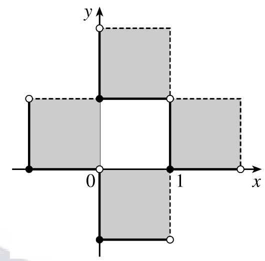
As $x \to 0^+$, $1-x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^+} x[\![1/x]\!] = 1$.

For $x < 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \;\Rightarrow\; x(1/x-1) \ge x[\![1/x]\!] \ge x(1/x) \;\Rightarrow\; 1-x \ge x[\![1/x]\!] \ge 1$.

As $x \to 0^-$, $1-x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^-} x[\![1/x]\!] = 1$.

Since the one-sided limits are equal, $\displaystyle\lim_{x\to 0} x[\![1/x]\!] = 1$.

6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:



Case (i): $[\![x]\!] = 1,\ [\![y]\!] = 0 \quad \Rightarrow 1 \le x < 2$ and $0 \le y < 1$

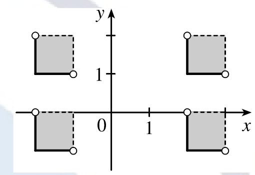
Case (ii): $[\![x]\!] = -1,\ [\![y]\!] = 0 \Rightarrow -1 \le x < 0$ and $0 \le y < 1$

Case (iii): $[\![x]\!] = 0,\ [\![y]\!] = 1 \quad \Rightarrow 0 \le x < 1$ and $1 \le y < 2$

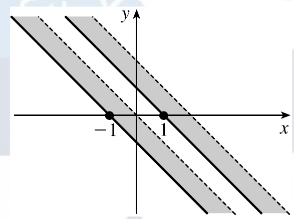
Case (iv): $[\![x]\!] = 0,\ [\![y]\!] = -1 \Rightarrow 0 \le x < 1$ and $-1 \le y < 0$

(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

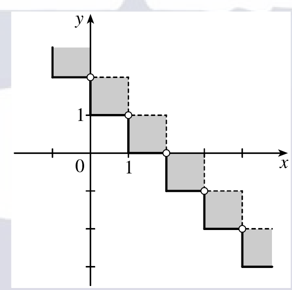
$$\{(x,y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} = \left\{ (x,y) \,\middle|\, \begin{array}{l} 2 \le x \le 3 \ \text{ or } \ -2 \le x < 1, \\ 1 \le y < 2 \ \text{ or } \ -1 \le y < 0 \end{array} \right\}.$$



(c) $[\![x+y]\!]^2 = 1 \quad \Rightarrow \quad [\![x+y]\!] = \pm 1 \quad \Rightarrow \quad 1 \le x + y < 2$ or $-1 \le x + y < 0$



(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \quad \Rightarrow \quad [\![y]\!] = 1 - n \quad \Rightarrow$ $1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.

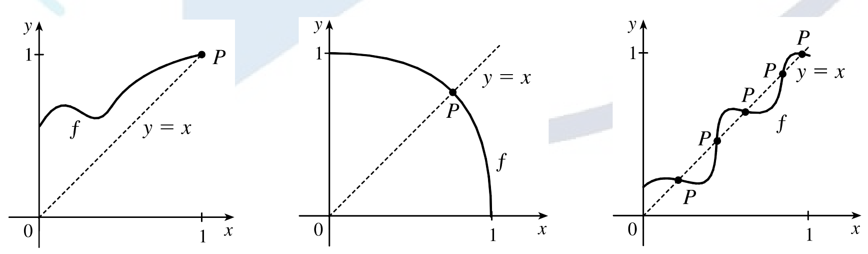


7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \quad \Rightarrow \quad \lim_{x \to a^+} x^2 = \lim_{x \to a^-} (x+1) \quad \Rightarrow \quad a^2 = a + 1 \quad \Rightarrow \quad a^2 - a - 1 = 0 \quad \Rightarrow$$

[by the quadratic formula] $\quad a = \left(1 \pm \sqrt{5}\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0,1)$, then it must either start at $(0,0)$ (in which case 0 is a fixed point) or finish at $(1,1)$ (in which case 1 is a fixed point).

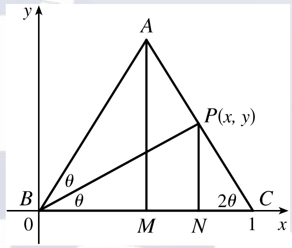
(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point. Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. $$\begin{cases} \lim\limits_{x\to a}[f(x)+g(x)]=2 \\ \lim\limits_{x\to a}[f(x)-g(x)]=1 \end{cases} \Rightarrow \begin{cases} \lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)=2 & \textbf{(1)} \\ \lim\limits_{x\to a}f(x)-\lim\limits_{x\to a}g(x)=1 & \textbf{(2)} \end{cases}$$

Adding equations **(1)** and **(2)** gives us $2\lim\limits_{x\to a}f(x)=3 \Rightarrow \lim\limits_{x\to a}f(x)=\frac{3}{2}$. From equation **(1)**, $\lim\limits_{x\to a}g(x)=\frac{1}{2}$. Thus,

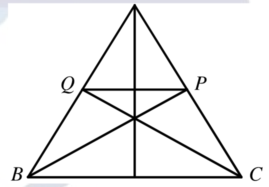
$\lim\limits_{x\to a}[f(x)\,g(x)] = \lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x) = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$. After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \Leftrightarrow y^2 = x(3x-2)$.

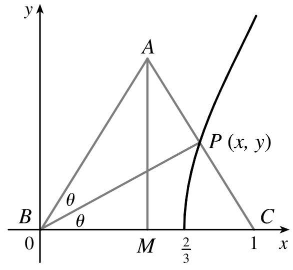


As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2) = 0$. Obviously it is not $x = 0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x - 2 = 0$, that is, $x = \frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. (a) Consider $G(x) = T(x + 180°) - T(x)$. Fix any number a. If $G(a) = 0$, we are done: Temperature at $a =$ Temperature at $a + 180°$. If $G(a) > 0$, then $G(a + 180°) = T(a + 360°) - T(a + 180°) = T(a) - T(a + 180°) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180°]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

12. $$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h)f(x+h) - xf(x)}{h} = \lim_{h\to 0} \left[\frac{xf(x+h) - xf(x)}{h} + \frac{hf(x+h)}{h}\right]$$
$$= x \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x+h) = xf'(x) + f(x)$$

because f is differentiable and therefore continuous.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad\Rightarrow\quad f(0) = 2f(0)$.
Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\left[f(0) + f(h) + 0^2h + 0h^2\right] - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{x\to 0} \frac{f(x)}{x} = 1$$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[f(x) + f(h) + x^2h + xh^2\right] - f(x)}{h} = \lim_{h\to 0} \frac{f(h) + x^2h + xh^2}{h}$$
$$= \lim_{h\to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$$

14. We are given that $|f(x)| \le x^2$ for all x. In particular, $|f(0)| \le 0$, but $|a| \ge 0$ for all a. The only conclusion is that $f(0) = 0$. Now $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{x^2}{|x|} = \dfrac{|x^2|}{|x|} = |x| \quad\Rightarrow\quad -|x| \le \dfrac{f(x) - f(0)}{x - 0} \le |x|$.

But $\lim\limits_{x\to 0} (-|x|) = 0 = \lim\limits_{x\to 0} |x|$, so by the Squeeze Theorem, $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$. So by the definition of a derivative, f is differentiable at 0 and, furthermore, $f'(0) = 0$.

3 □ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

(b)

x	$\dfrac{2.7^x - 1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

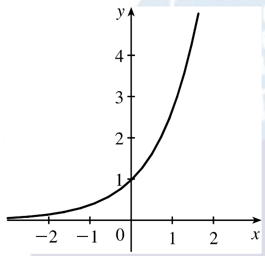
x	$\dfrac{2.8^x - 1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim\limits_{h\to 0} \dfrac{2.7^h - 1}{h} = 0.99$ and $\lim\limits_{h\to 0} \dfrac{2.8^h - 1}{h} = 1.03$.

Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

2. (a)



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

(b) $f(x) = e^x$ is an exponential function and $g(x) = x^e$ is a power function. $\dfrac{d}{dx}(e^x) = e^x$ and $\dfrac{d}{dx}(x^e) = ex^{e-1}$.

(c) $f(x) = e^x$ grows more rapidly than $g(x) = x^e$ when x is large.

3. $f(x) = 2^{40}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

4. $f(x) = e^5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

5. $f(x) = 5.2x + 2.3 \quad\Rightarrow\quad f'(x) = 5.2(1) + 0 = 5.2$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12 \quad\Rightarrow\quad g'(x) = \frac{7}{4}(2x) - 3(1) + 0 = \frac{7}{2}x - 3$

7. $f(t) = 2t^3 - 3t^2 - 4t \quad\Rightarrow\quad f'(t) = 2(3t^2) - 3(2t) - 4(1) = 6t^2 - 6t - 4$

8. $f(t) = 1.4t^5 - 2.5t^2 + 6.7 \quad\Rightarrow\quad f'(t) = 1.4(5t^4) - 2.5(2t) + 0 = 7t^4 - 5t$

9. $g(x) = x^2(1 - 2x) = x^2 - 2x^3 \quad\Rightarrow\quad g'(x) = 2x - 2(3x^2) = 2x - 6x^2$

10. $H(u) = (3u - 1)(u + 2) = 3u^2 + 5u - 2 \quad\Rightarrow\quad H'(u) = 3(2u) + 5(1) - 0 = 6u + 5$

11. $g(t) = 2t^{-3/4} \quad\Rightarrow\quad g'(t) = 2\left(-\frac{3}{4}t^{-7/4}\right) = -\frac{3}{2}t^{-7/4}$

12. $B(y) = cy^{-6} \quad\Rightarrow\quad B'(y) = c(-6y^{-7}) = -6cy^{-7}$

13. $F(r) = \dfrac{5}{r^3} = 5r^{-3} \quad\Rightarrow\quad F'(r) = 5(-3r^{-4}) = -15r^{-4} = -\dfrac{15}{r^4}$

14. $y = x^{5/3} - x^{2/3} \quad\Rightarrow\quad y' = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3}$

15. $R(a) = (3a+1)^2 = 9a^2 + 6a + 1 \quad \Rightarrow \quad R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

16. $h(t) = \sqrt[4]{t} - 4e^t = t^{1/4} - 4e^t \quad \Rightarrow \quad h'(t) = \frac{1}{4}t^{-3/4} - 4(e^t) = \frac{1}{4}t^{-3/4} - 4e^t$

17. $S(p) = \sqrt{p} - p = p^{1/2} - p \quad \Rightarrow \quad S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\dfrac{1}{2\sqrt{p}} - 1$

18. $y = \sqrt[3]{x}\,(2+x) = 2x^{1/3} + x^{4/3} \quad \Rightarrow \quad y' = 2\left(\frac{1}{3}x^{-2/3}\right) + \frac{4}{3}x^{1/3} = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$ or $\dfrac{2}{3\sqrt[3]{x^2}} + \dfrac{4}{3}\sqrt[3]{x}$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \quad \Rightarrow \quad y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$

20. $S(R) = 4\pi R^2 \quad \Rightarrow \quad S'(R) = 4\pi(2R) = 8\pi R$

21. $h(u) = Au^3 + Bu^2 + Cu \quad \Rightarrow \quad h'(u) = A(3u^2) + B(2u) + C(1) = 3Au^2 + 2Bu + C$

22. $y = \dfrac{\sqrt{x} + x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \quad \Rightarrow \quad y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \quad \Rightarrow \quad G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

25. $j(x) = x^{2.4} + e^{2.4} \quad \Rightarrow \quad j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$

26. $k(r) = e^r + r^e \quad \Rightarrow \quad k'(r) = e^r + er^{e-1}$

27. $G(q) = \left(1 + q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \quad \Rightarrow \quad G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2} = \dfrac{A}{z^2} + \dfrac{Bz}{z^2} + \dfrac{Cz^2}{z^2} = Az^{-2} + Bz^{-1} + C \quad \Rightarrow$

$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0 = -2Az^{-3} - Bz^{-2} = -\dfrac{2A}{z^3} - \dfrac{B}{z^2}$ or $-\dfrac{2A + Bz}{z^3}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v} = \dfrac{\sqrt[3]{v}}{v} - \dfrac{2ve^v}{v} = v^{-2/3} - 2e^v \quad \Rightarrow \quad f'(v) = -\frac{2}{3}v^{-5/3} - 2e^v$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3} = \dfrac{1 + 16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \quad \Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

31. $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad \Rightarrow \quad z' = -10Ay^{-11} + Be^y = -\dfrac{10A}{y^{11}} + Be^y$

32. $y = e^{x+1} + 1 = e^x e^1 + 1 = e \cdot e^x + 1 \quad \Rightarrow \quad y' = e \cdot e^x = e^{x+1}$

33. $y = 2x^3 - x^2 + 2 \quad\Rightarrow\quad y' = 6x^2 - 2x$. At $(1, 3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x - 1)$ or $y = 4x - 1$.

34. $y = 2e^x + x \quad\Rightarrow\quad y' = 2e^x + 1$. At $(0, 2)$, $y' = 2e^0 + 1 = 3$ and an equation of the tangent line is $y - 2 = 3(x - 0)$ or $y = 3x + 2$.

35. $y = x + \dfrac{2}{x} = x + 2x^{-1} \quad\Rightarrow\quad y' = 1 - 2x^{-2}$. At $(2, 3)$, $y' = 1 - 2(2)^{-2} = \frac{1}{2}$ and an equation of the tangent line is $y - 3 = \frac{1}{2}(x - 2)$ or $y = \frac{1}{2}x + 2$.

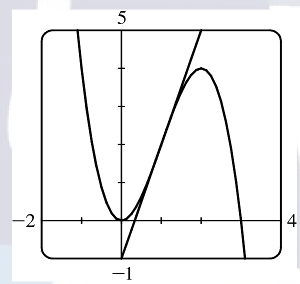
36. $y = \sqrt[4]{x} - x = x^{1/4} - x \quad\Rightarrow\quad y' = \frac{1}{4}x^{-3/4} - 1 = \dfrac{1}{4\sqrt[4]{x^3}} - 1$. At $(1, 0)$, $y' = \frac{1}{4} - 1 = -\frac{3}{4}$ and an equation of the tangent line is $y - 0 = -\frac{3}{4}(x - 1)$ or $y = -\frac{3}{4}x + \frac{3}{4}$.

37. $y = x^4 + 2e^x \quad\Rightarrow\quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is $y - 2 = -\frac{1}{2}(x - 0)$ or $y = -\frac{1}{2}x + 2$.

38. $y^2 = x^3 \quad\Rightarrow\quad y = x^{3/2}$ [since x and y are positive at $(1, 1)$] $\quad\Rightarrow\quad y' = \frac{3}{2}x^{1/2}$. At $(1, 1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x - 1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

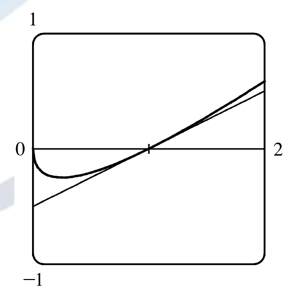
39. $y = 3x^2 - x^3 \quad\Rightarrow\quad y' = 6x - 3x^2$.

At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1$.



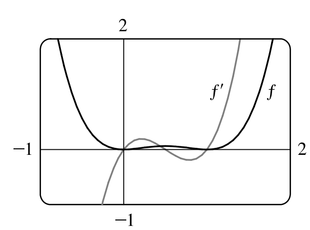
40. $y = x - \sqrt{x} \quad\Rightarrow\quad y' = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

At $(1, 0)$, $y' = \frac{1}{2}$, so an equation of the tangent line is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.



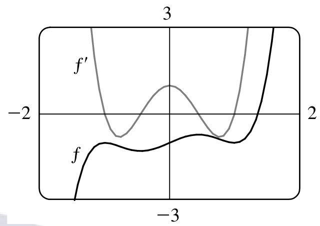
41. $f(x) = x^4 - 2x^3 + x^2 \quad\Rightarrow\quad f'(x) = 4x^3 - 6x^2 + 2x$

Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

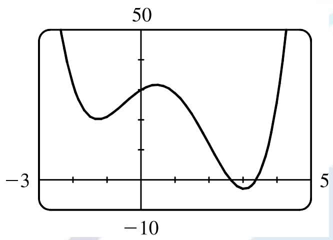


42. $f(x) = x^5 - 2x^3 + x - 1 \quad \Rightarrow \quad f'(x) = 5x^4 - 6x^2 + 1$

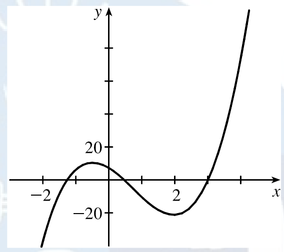
Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



43. (a)

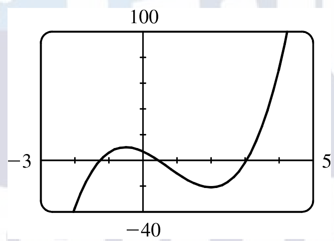


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

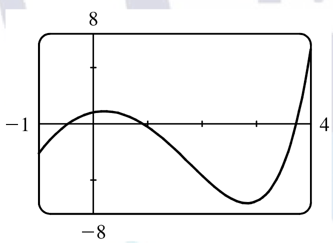


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

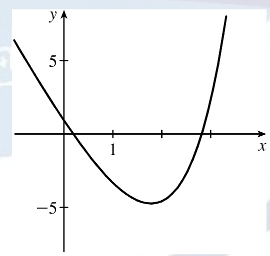
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



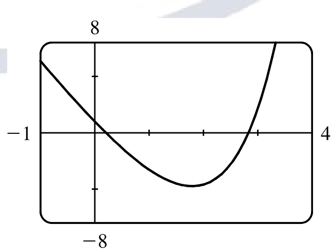
44. (a)



(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞). The slopes are negative (so f' is negative) on (x_1, x_2).



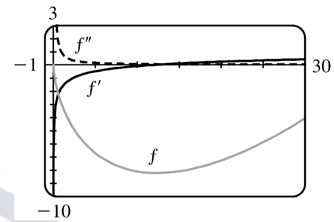
(c) $g(x) = e^x - 3x^2 \quad \Rightarrow \quad g'(x) = e^x - 6x$



45. $f(x) = 0.001x^5 - 0.02x^3 \quad \Rightarrow \quad f'(x) = 0.005x^4 - 0.06x^2 \quad \Rightarrow \quad f''(x) = 0.02x^3 - 0.12x$

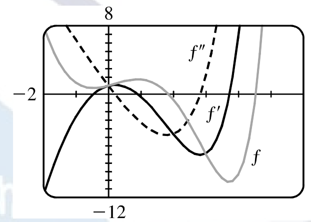
46. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad \Rightarrow \quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad \Rightarrow \quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

47. $f(x) = 2x - 5x^{3/4} \quad \Rightarrow \quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad \Rightarrow \quad f''(x) = \frac{15}{16}x^{-5/4}$



Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.

48. $f(x) = e^x - x^3 \quad \Rightarrow \quad f'(x) = e^x - 3x^2 \quad \Rightarrow \quad f''(x) = e^x - 6x$



Note that $f'(x) = 0$ when f has a horizontal tangent and that $f''(x) = 0$ when f' has a horizontal tangent.

49. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ $[t \geq 0]$ and $a(1) = 6 \text{ m/s}^2$.

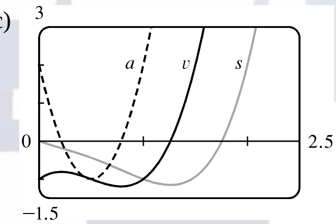
50. (a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2 \text{ m/s}^2$

(c)



51. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad \Rightarrow \quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.0465(12)^2 - 0.744(12) + 3.95 = 1.718$. The derivative is the instantaneous rate of change of the length of an Alaskan rockfish with respect to its age when its age is 12 years.

52. $S(A) = 0.882A^{0.842} \quad \Rightarrow \quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

53. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \quad \Rightarrow \quad \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at $25\,^\circ\text{C}$. Its units are m^3/kPa.

54. (a) $L = aP^2 + bP + c$, where $a \approx -0.275428$, $b \approx 19.74853$, and $c \approx -273.55234$.

(b) $\dfrac{dL}{dP} = 2aP + b$. When $P = 30$, $\dfrac{dL}{dP} \approx 3.2$, and when $P = 40$, $\dfrac{dL}{dP} \approx -2.3$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of miles)$/(\text{lb}/\text{in}^2)$. When $\dfrac{dL}{dP}$ is positive, tire life is increasing, and when $\dfrac{dL}{dP} < 0$, tire life is decreasing.

55. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \iff 6(x^2 + x - 2) = 0 \iff 6(x + 2)(x - 1) = 0 \iff x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

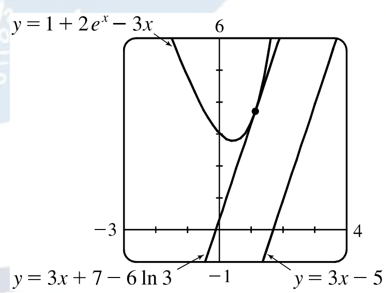
56. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$. $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$, so f has a horizontal tangent when $x = \ln 2$.

57. $y = 2e^x + 3x + 5x^3 \Rightarrow y' = 2e^x + 3 + 15x^2$. Since $2e^x > 0$ and $15x^2 \geq 0$, we must have $y' > 0 + 3 + 0 = 3$, so no tangent line can have slope 2.

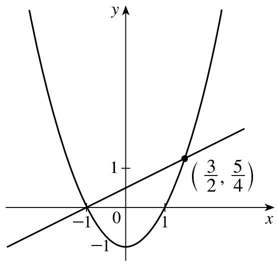
58. $y = x^4 + 1 \Rightarrow y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \iff 4x^3 = 32 \iff x^3 = 8 \iff x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

59. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3.
$y = x^3 - 3x^2 + 3x - 3 \Rightarrow y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$. Thus, $3(x - 1)^2 = 3 \Rightarrow (x - 1)^2 = 1 \Rightarrow x - 1 = \pm 1 \Rightarrow x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

60. The slope of $y = 1 + 2e^x - 3x$ is given by $m = y' = 2e^x - 3$.
The slope of $3x - y = 5 \iff y = 3x - 5$ is 3.
$m = 3 \Rightarrow 2e^x - 3 = 3 \Rightarrow e^x = 3 \Rightarrow x = \ln 3$.
This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.1, 3.7)$.

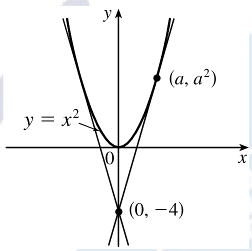


61. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \Rightarrow \sqrt{x} = 1 \Rightarrow x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.



62. $y = f(x) = x^2 - 1 \quad \Rightarrow \quad f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \quad \Leftrightarrow \quad x + 1 = 2x^2 - 2 \quad \Leftrightarrow$ $2x^2 - x - 3 = 0 \quad \Leftrightarrow \quad (2x - 3)(x + 1) = 0 \quad \Leftrightarrow \quad x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.

63.



Let $\left(a, a^2\right)$ be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad \Leftrightarrow \quad y = 2ax - 4$. Since $\left(a, a^2\right)$ also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

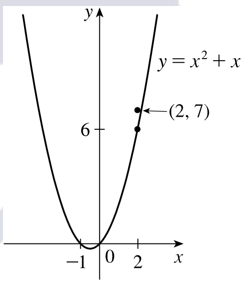
64. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $\left(a, a^2 + a\right)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



65. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h} = \lim\limits_{h \to 0} \dfrac{x - (x + h)}{hx(x + h)} = \lim\limits_{h \to 0} \dfrac{-h}{hx(x + h)} = \lim\limits_{h \to 0} \dfrac{-1}{x(x + h)} = -\dfrac{1}{x^2}$

66. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n(n - 1)x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n - 1)(n - 2) \cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3) \cdots (-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n \, n!}{x^{n+1}}$

67. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \quad \Rightarrow \quad 2a = 2 \quad \Rightarrow \quad a = 1$.

$P'(2) = 3 \quad \Rightarrow \quad 2(1)(2) + b = 3 \quad \Rightarrow \quad 4 + b = 3 \quad \Rightarrow \quad b = -1$.

$P(2) = 5 \quad \Rightarrow \quad 1(2)^2 + (-1)(2) + c = 5 \quad \Rightarrow \quad 2 + c = 5 \quad \Rightarrow \quad c = 3$. So $P(x) = x^2 - x + 3$.

68. $y = Ax^2 + Bx + C \quad \Rightarrow \quad y' = 2Ax + B \quad \Rightarrow \quad y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \quad \Rightarrow \quad A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \quad \Rightarrow$ $A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \quad \Rightarrow \quad -1 - \frac{1}{2} - 2C = 0 \quad \Rightarrow \quad C = -\frac{3}{4}$.

69. $y = f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad \Rightarrow \quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad \Rightarrow \quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad \Rightarrow \quad 3 = 16a \quad \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

70. $y = ax^2 + bx + c \quad \Rightarrow \quad y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \quad \Rightarrow$ $2a + b = 4$ **(1)** and $y'(-1) = -8 \quad \Rightarrow \quad -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \quad \Leftrightarrow \quad b = -2$. From **(1)**, $2a - 2 = 4 \quad \Leftrightarrow \quad a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \quad \Rightarrow \quad c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

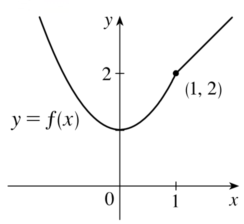
71. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

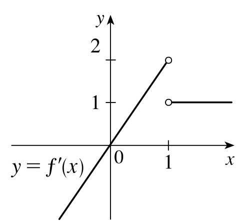
Calculate the left- and right-hand derivatives as defined in Exercise 2.8.64:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





72. $g(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \ge 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$g'_-(0) = \lim\limits_{h \to 0^-} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{2h - 2(0)}{h} = 2$ and

$g'_+(0) = \lim\limits_{h \to 0^+} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{(2h - h^2) - 2(0)}{h} = \lim\limits_{h \to 0^+} (2 - h) = 2$, so g is differentiable at $x = 0$.

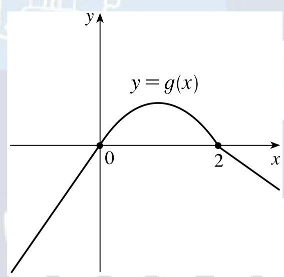
$g'_-(2) = \lim\limits_{h \to 0^-} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^-} \dfrac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim\limits_{h \to 0^-} \dfrac{-2h - h^2}{h} = \lim\limits_{h \to 0^-} (-2 - h) = -2$

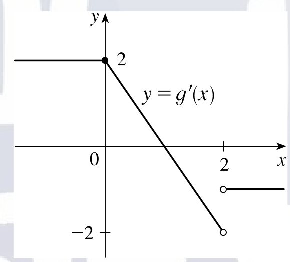
and

$g'_+(2) = \lim\limits_{h \to 0^+} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^+} \dfrac{[2 - (2+h)] - (2-2)}{h} = \lim\limits_{h \to 0^+} \dfrac{-h}{h} = \lim\limits_{h \to 0^+} (-1) = -1,$

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$g'(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$





73. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \quad \Leftrightarrow \quad |x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

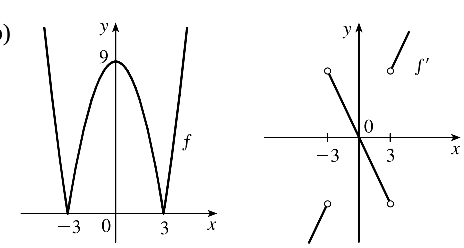
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.64.

$f'_-(3) = \lim\limits_{h \to 0^-} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^-} \dfrac{[-(3+h)^2 + 9] - 0}{h} = \lim\limits_{h \to 0^-} (-6 - h) = -6 \quad$ and

$f'_+(3) = \lim\limits_{h \to 0^+} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^+} \dfrac{\left[(3+h)^2 - 9\right] - 0}{h} = \lim\limits_{h \to 0^+} \dfrac{6h + h^2}{h} = \lim\limits_{h \to 0^+} (6 + h) = 6.$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)



74. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.

If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.

If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

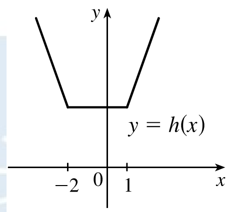
$$h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

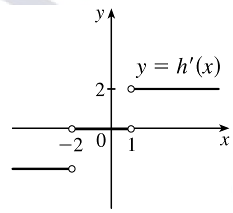
To see that $h'(1) = \lim\limits_{x \to 1} \dfrac{h(x) - h(1)}{x - 1}$ does not exist,

observe that $\lim\limits_{x \to 1^-} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{3 - 3}{3 - 1} = 0$ but

$\lim\limits_{x \to 1^+} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x - 2}{x - 1} = 2$. Similarly,

$h'(-2)$ does not exist.





75. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \Rightarrow 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

76. $y = x^4 + ax^3 + bx^2 + cx + d \Rightarrow y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \Rightarrow y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

77. $y = f(x) = ax^2 \Rightarrow f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \Leftrightarrow y = -2x + b$, is seen to be -2, so we must have $4a = -2 \Leftrightarrow a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \Leftrightarrow b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

78. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \Rightarrow c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow 3a = \frac{3}{2}a + 6 \Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

79. The line $y = 2x + 3$ has slope 2. The parabola $y = cx^2 \Rightarrow y' = 2cx$ has slope $2ca$ at $x = a$. Equating slopes gives us $2ca = 2$, or $ca = 1$. Equating y-coordinates at $x = a$ gives us $ca^2 = 2a + 3 \Leftrightarrow (ca)a = 2a + 3 \Leftrightarrow 1a = 2a + 3 \Leftrightarrow a = -3$. Thus, $c = \dfrac{1}{a} = -\dfrac{1}{3}$.

80. $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. The slope of the tangent line at $x = p$ is $2ap + b$, the slope of the tangent line at $x = q$ is $2aq + b$, and the average of those slopes is $\dfrac{(2ap + b) + (2aq + b)}{2} = ap + aq + b$. The midpoint of the interval $[p, q]$ is $\dfrac{p + q}{2}$ and the slope of the tangent line at the midpoint is $2a\left(\dfrac{p + q}{2}\right) + b = a(p + q) + b$. This is equal to $ap + aq + b$, as required.

81. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

82. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

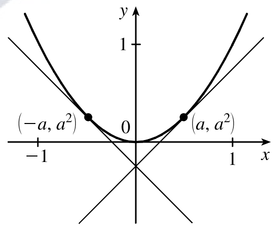
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

83. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

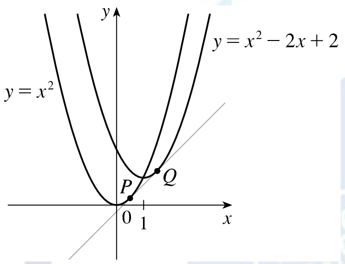
Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\begin{aligned}\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1) \\ &= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}\end{aligned}$$

84. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.



85. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \Rightarrow$ $a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

86. 

From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $[y' = 2x]$ and at Q is $2b - 2$ $[y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \Leftrightarrow a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \Rightarrow 2b - 2 = \dfrac{b^2 - 2b + 2 - (b-1)^2}{b - (b-1)} \Rightarrow 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \Rightarrow$ $2b = 3 \Rightarrow b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or $y = x - \frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

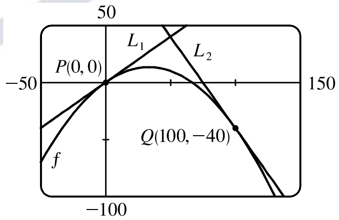
1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0,0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

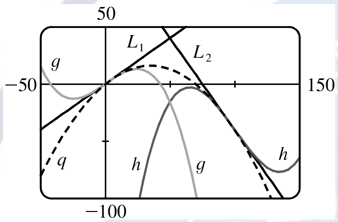
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q, g, and h.

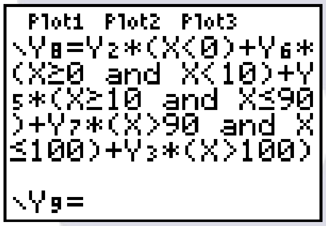
$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9} \qquad \left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

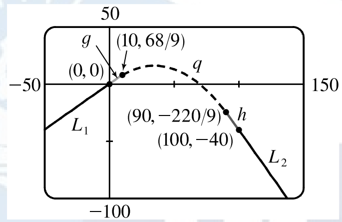
(c) Graph of L_1, q, g, h, and L_2:



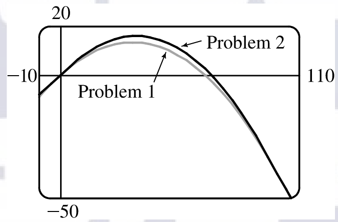
This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $\text{Y}_2 = L_1$, $\text{Y}_6 = g$, $\text{Y}_5 = q$, $\text{Y}_7 = h$, and $\text{Y}_3 = L_2$.



The graph of the five functions as a piecewise-defined function:



A comparison of the graphs in part 1(c) and part 2(c):



3.2 The Product and Quotient Rules

1. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

2. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x) = (3x^2 - 5x)e^x \quad \Rightarrow$

$$f'(x) = (3x^2 - 5x)(e^x)' + e^x(3x^2 - 5x)' = (3x^2 - 5x)e^x + e^x(6x - 5)$$
$$= e^x[(3x^2 - 5x) + (6x - 5)] = e^x(3x^2 + x - 5)$$

4. By the Product Rule, $g(x) = (x + 2\sqrt{x}\,)\,e^x \Rightarrow$

$$g'(x) = (x + 2\sqrt{x}\,)(e^x)' + e^x(x + 2\sqrt{x}\,)' = (x + 2\sqrt{x}\,)e^x + e^x\left(1 + 2 \cdot \tfrac{1}{2}x^{-1/2}\right)$$
$$= e^x\left[(x + 2\sqrt{x}\,) + \left(1 + 1/\sqrt{x}\,\right)\right] = e^x\left(x + 2\sqrt{x} + 1 + 1/\sqrt{x}\,\right)$$

5. By the Quotient Rule, $y = \dfrac{x}{e^x} \quad \Rightarrow \quad y' = \dfrac{e^x(1) - x(e^x)}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}$.

6. By the Quotient Rule, $y = \dfrac{e^x}{1 - e^x} \quad \Rightarrow \quad y' = \dfrac{(1 - e^x)e^x - e^x(-e^x)}{(1 - e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1 - e^x)^2} = \dfrac{e^x}{(1 - e^x)^2}$.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{1 + 2x}{3 - 4x} \quad \overset{\text{QR}}{\Rightarrow} \quad g'(x) = \dfrac{(3 - 4x)(2) - (1 + 2x)(-4)}{(3 - 4x)^2} = \dfrac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \dfrac{10}{(3 - 4x)^2}$

8. $G(x) = \dfrac{x^2 - 2}{2x + 1} \quad \overset{\text{QR}}{\Rightarrow} \quad G'(x) = \dfrac{(2x + 1)(2x) - (x^2 - 2)(2)}{(2x + 1)^2} = \dfrac{4x^2 + 2x - 2x^2 + 4}{(2x + 1)^2} = \dfrac{2x^2 + 2x + 4}{(2x + 1)^2}$

9. $H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) \quad \overset{\text{PR}}{\Rightarrow}$

$$H'(u) = (u - \sqrt{u}\,)\left(1 + \frac{1}{2\sqrt{u}}\right) + (u + \sqrt{u}\,)\left(1 - \frac{1}{2\sqrt{u}}\right) = u + \tfrac{1}{2}\sqrt{u} - \sqrt{u} - \tfrac{1}{2} + u - \tfrac{1}{2}\sqrt{u} + \sqrt{u} - \tfrac{1}{2} = 2u - 1.$$

An easier method is to simplify first and then differentiate as follows:

$$H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) = u^2 - (\sqrt{u}\,)^2 = u^2 - u \quad \Rightarrow \quad H'(u) = 2u - 1$$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2}) \quad \overset{\text{PR}}{\Rightarrow}$

$$J'(v) = (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2)$$
$$= -4v^{-2} - 2v^0 + 8v^{-4} + 4v^{-2} + 3v^{-2} - 2v^{-4} + 3v^0 - 2v^{-2} = 1 + v^{-2} + 6v^{-4}$$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$
$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$
$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

12. $f(z) = (1 - e^z)(z + e^z) \quad \overset{\text{PR}}{\Rightarrow}$

$$f'(z) = (1 - e^z)(1 + e^z) + (z + e^z)(-e^z) = 1^2 - (e^z)^2 - ze^z - (e^z)^2 = 1 - ze^z - 2e^{2z}$$

13. $y = \dfrac{x^2+1}{x^3-1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3-1)(2x)-(x^2+1)(3x^2)}{(x^3-1)^2} = \frac{x\left[(x^3-1)(2)-(x^2+1)(3x)\right]}{(x^3-1)^2} = \frac{x(2x^3-2-3x^3-3x)}{(x^3-1)^2} = \frac{x(-x^3-3x-2)}{(x^3-1)^2}$$

14. $y = \dfrac{\sqrt{x}}{2+x} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(2+x)\left(\dfrac{1}{2\sqrt{x}}\right)-\sqrt{x}\,(1)}{(2+x)^2} = \frac{\dfrac{1}{\sqrt{x}}+\dfrac{\sqrt{x}}{2}-\sqrt{x}}{(2+x)^2} = \frac{\dfrac{2+x-2x}{2\sqrt{x}}}{(2+x)^2} = \frac{2-x}{2\sqrt{x}(2+x)^2}$$

15. $y = \dfrac{t^3+3t}{t^2-4t+3} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^2-4t+3)(3t^2+3)-(t^3+3t)(2t-4)}{(t^2-4t+3)^2}$$

$$= \frac{3t^4+3t^2-12t^3-12t+9t^2+9-(2t^4-4t^3+6t^2-12t)}{(t^2-4t+3)^2} = \frac{t^4-8t^3+6t^2+9}{(t^2-4t+3)^2}$$

16. $y = \dfrac{1}{t^3+2t^2-1} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(t^3+2t^2-1)(0)-1(3t^2+4t)}{(t^3+2t^2-1)^2} = -\dfrac{3t^2+4t}{(t^3+2t^2-1)^2}$

17. $y = e^p(p+p\sqrt{p}\,) = e^p(p+p^{3/2}) \overset{\text{PR}}{\Rightarrow} y' = e^p\left(1+\frac{3}{2}p^{1/2}\right)+(p+p^{3/2})e^p = e^p\left(1+\frac{3}{2}\sqrt{p}+p+p\sqrt{p}\,\right)$

18. $h(r) = \dfrac{ae^r}{b+e^r} \overset{\text{QR}}{\Rightarrow} h'(r) = \dfrac{(b+e^r)(ae^r)-(ae^r)(e^r)}{(b+e^r)^2} = \dfrac{abe^r+ae^{2r}-ae^{2r}}{(b+e^r)^2} = \dfrac{abe^r}{(b+e^r)^2}$

19. $y = \dfrac{s-\sqrt{s}}{s^2} = \dfrac{s}{s^2}-\dfrac{\sqrt{s}}{s^2} = s^{-1}-s^{-3/2} \Rightarrow y' = -s^{-2}+\frac{3}{2}s^{-5/2} = \dfrac{-1}{s^2}+\dfrac{3}{2s^{5/2}} = \dfrac{3-2\sqrt{s}}{2s^{5/2}}$

20. $y = (z^2+e^z)\sqrt{z} \overset{\text{PR}}{\Rightarrow}$

$$y' = (z^2+e^z)\left(\frac{1}{2\sqrt{z}}\right)+\sqrt{z}\,(2z+e^z) = \frac{z^2}{2\sqrt{z}}+\frac{e^z}{2\sqrt{z}}+2z\sqrt{z}+\sqrt{z}\,e^z$$

$$= \frac{z^2+e^z+4z^2+2ze^z}{2\sqrt{z}} = \frac{5z^2+e^z+2ze^z}{2\sqrt{z}}$$

21. $f(t) = \dfrac{\sqrt[3]{t}}{t-3} \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right)-t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3}-t^{-2/3}-t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3}-t^{-2/3}}{(t-3)^2} = \frac{\dfrac{-2t}{3t^{2/3}}-\dfrac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t-3}{3t^{2/3}(t-3)^2}$$

22. $V(t) = \dfrac{4+t}{te^t} \overset{\text{QR}}{\Rightarrow}$

$$V'(t) = \frac{te^t(1)-(4+t)(te^t+e^t(1))}{(te^t)^2} = \frac{te^t-4te^t-4e^t-t^2e^t-te^t}{t^2e^{2t}}$$

$$= \frac{-4te^t-4e^t-t^2e^t}{t^2e^{2t}} = \frac{-e^t(t^2+4t+4)}{t^2e^{2t}} = -\frac{(t+2)^2}{t^2e^t}$$

23. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x} \overset{\text{QR}}{\Rightarrow}$

$$f'(x) = \frac{(x^2 + e^x)\left[x^2 e^x + e^x(2x)\right] - x^2 e^x(2x + e^x)}{(x^2 + e^x)^2} = \frac{x^4 e^x + 2x^3 e^x + x^2 e^{2x} + 2xe^{2x} - 2x^3 e^x - x^2 e^{2x}}{(x^2 + e^x)^2}$$

$$= \frac{x^4 e^x + 2xe^{2x}}{(x^2 + e^x)^2} = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$

24. $F(t) = \dfrac{At}{Bt^2 + Ct^3} = \dfrac{A}{Bt + Ct^2} \overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt + Ct^2)(0) - A(B + 2Ct)}{(Bt + Ct^2)^2} = \frac{-A(B + 2Ct)}{(t)^2(B + Ct)^2} = -\frac{A(B + 2Ct)}{t^2(B + Ct)^2}$$

25. $f(x) = \dfrac{x}{x + c/x} \Rightarrow f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2 + c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2 + c)^2}$

26. $f(x) = \dfrac{ax + b}{cx + d} \Rightarrow f'(x) = \dfrac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \dfrac{acx + ad - acx - bc}{(cx + d)^2} = \dfrac{ad - bc}{(cx + d)^2}$

27. $f(x) = (x^3 + 1)e^x \overset{\text{PR}}{\Rightarrow}$

$$f'(x) = (x^3 + 1)e^x + e^x(3x^2) = e^x\left[(x^3 + 1) + 3x^2\right] = e^x(x^3 + 3x^2 + 1) \overset{\text{PR}}{\Rightarrow}$$

$$f''(x) = e^x(3x^2 + 6x) + (x^3 + 3x^2 + 1)e^x = e^x\left[(3x^2 + 6x) + (x^3 + 3x^2 + 1)\right] = e^x(x^3 + 6x^2 + 6x + 1)$$

28. $f(x) = \sqrt{x}\, e^x \overset{\text{PR}}{\Rightarrow} f'(x) = \sqrt{x}\, e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right) = \left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right)e^x = \dfrac{2x + 1}{2\sqrt{x}}\, e^x.$

Using the Product Rule and $f'(x) = \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)e^x$, we get

$$f''(x) = \left(x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)e^x + e^x\left(\tfrac{1}{2}x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right) = \left(x^{1/2} + x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right)e^x = \frac{4x^2 + 4x - 1}{4x^{3/2}}\, e^x$$

29. $f(x) = \dfrac{x^2}{1 + e^x} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(1 + e^x)(2x) - x^2(e^x)}{(1 + e^x)^2} = \dfrac{x[(1 + e^x)2 - xe^x]}{(1 + e^x)^2} = \dfrac{x(2 + 2e^x - xe^x)}{(1 + e^x)^2}.$

Using the Quotient and Product Rules and $f'(x) = \dfrac{2x + 2xe^x - x^2 e^x}{(1 + e^x)^2}$, we get

$$f''(x) = \frac{(1 + e^x)^2\left[2 + 2(xe^x + e^x) - (x^2 e^x + 2xe^x)\right] - (2x + 2xe^x - x^2 e^x)\left[(1 + e^x)e^x + (1 + e^x)e^x\right]}{[(1 + e^x)^2]^2}$$

$$= \frac{(1 + e^x)\left\{\left[(1 + e^x)(2 + 2xe^x + 2e^x - x^2 e^x - 2xe^x)\right] - (2x + 2xe^x - x^2 e^x)(2e^x)\right\}}{(1 + e^x)^4}$$

$$= \frac{(1 + e^x)(2 + 2e^x - x^2 e^x) - 4xe^x - 4xe^{2x} + 2x^2 e^{2x}}{(1 + e^x)^3}$$

$$= \frac{2 + 2e^x - x^2 e^x + 2e^x + 2e^{2x} - x^2 e^{2x} - 4xe^x - 4xe^{2x} + 2x^2 e^{2x}}{(1 + e^x)^3}$$

$$= \frac{2 + 4e^x - x^2 e^x - 4xe^x + 2e^{2x} + x^2 e^{2x} - 4xe^{2x}}{(1 + e^x)^3}$$

30. $f(x) = \dfrac{x}{x^2 - 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2-1)(1) - x(2x)}{(x^2-1)^2} = \dfrac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \dfrac{-x^2-1}{(x^2-1)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2-1)^2(-2x) - (-x^2-1)(x^4 - 2x^2 + 1)'}{[(x^2-1)^2]^2} = \frac{(x^2-1)^2(-2x) + (x^2+1)(4x^3 - 4x)}{(x^2-1)^4}$$

$$= \frac{(x^2-1)^2(-2x) + (x^2+1)(4x)(x^2-1)}{(x^2-1)^4} = \frac{(x^2-1)[(x^2-1)(-2x) + (x^2+1)(4x)]}{(x^2-1)^4}$$

$$= \frac{-2x^3 + 2x + 4x^3 + 4x}{(x^2-1)^3} = \frac{2x^3 + 6x}{(x^2-1)^3}$$

31. $y = \dfrac{x^2 - 1}{x^2 + x + 1} \quad \Rightarrow$

$$y' = \frac{(x^2+x+1)(2x) - (x^2-1)(2x+1)}{(x^2+x+1)^2} = \frac{2x^3 + 2x^2 + 2x - 2x^3 - x^2 + 2x + 1}{(x^2+x+1)^2} = \frac{x^2 + 4x + 1}{(x^2+x+1)^2}.$$

At $(1, 0)$, $y' = \dfrac{6}{3^2} = \dfrac{2}{3}$, and an equation of the tangent line is $y - 0 = \frac{2}{3}(x - 1)$, or $y = \frac{2}{3}x - \frac{2}{3}$.

32. $y = \dfrac{1+x}{1+e^x} \quad \Rightarrow \quad y' = \dfrac{(1+e^x)(1) - (1+x)e^x}{(1+e^x)^2} = \dfrac{1 + e^x - e^x - xe^x}{(1+e^x)^2} = \dfrac{1 - xe^x}{(1+e^x)^2}.$

At $\left(0, \frac{1}{2}\right)$, $y' = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x - 0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

33. $y = 2xe^x \quad \Rightarrow \quad y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x+1)$.

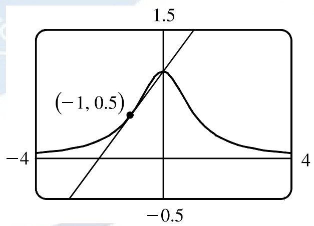
At $(0, 0)$, $y' = 2e^0(0+1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x - 0)$, or $y = -\frac{1}{2}x$.

34. $y = \dfrac{2x}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \dfrac{2 - 2x^2}{(x^2+1)^2}$. At $(1, 1)$, $y' = 0$, and an equation of the tangent line is $y - 1 = 0(x - 1)$, or $y = 1$. The slope of the normal line is undefined, so an equation of the normal line is $x = 1$.

35. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x + 1)$ or $y = \frac{1}{2}x + 1$.

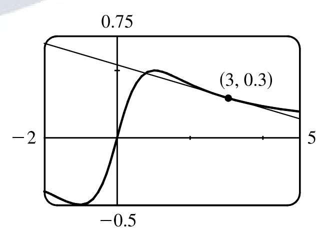
(b)



36. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad \Rightarrow$

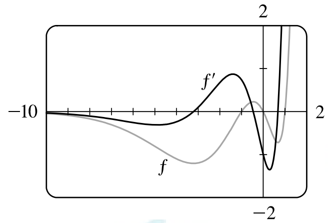
$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1 - x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x - 3)$ or $y = -0.08x + 0.54$.

(b)



37. (a) $f(x) = (x^3 - x)e^x \quad \Rightarrow \quad f'(x) = (x^3 - x)e^x + e^x(3x^2 - 1) = e^x(x^3 + 3x^2 - x - 1)$

(b)

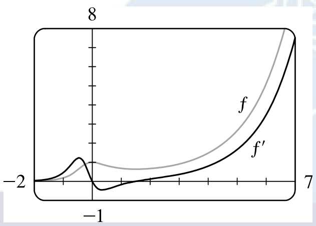


$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

38. (a) $f(x) = \dfrac{e^x}{2x^2 + x + 1} \quad \Rightarrow$

$$f'(x) = \frac{(2x^2 + x + 1)e^x - e^x(4x + 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 + x + 1 - 4x - 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 - 3x)}{(2x^2 + x + 1)^2}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

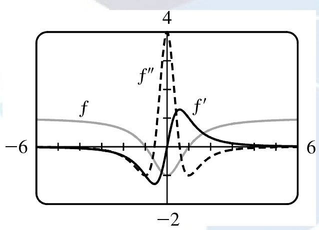
39. (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} \quad \Rightarrow$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{(2x)[(x^2 + 1) - (x^2 - 1)]}{(x^2 + 1)^2} = \frac{(2x)(2)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \quad \Rightarrow$$

$$f''(x) = \frac{(x^2 + 1)^2(4) - 4x(x^4 + 2x^2 + 1)'}{[(x^2 + 1)^2]^2} = \frac{4(x^2 + 1)^2 - 4x(4x^3 + 4x)}{(x^2 + 1)^4}$$

$$= \frac{4(x^2 + 1)^2 - 16x^2(x^2 + 1)}{(x^2 + 1)^4} = \frac{4(x^2 + 1)[(x^2 + 1) - 4x^2]}{(x^2 + 1)^4} = \frac{4(1 - 3x^2)}{(x^2 + 1)^3}$$

(b)

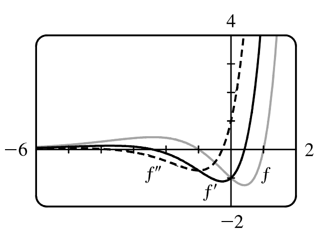


$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

40. (a) $f(x) = (x^2 - 1)e^x \quad \Rightarrow \quad f'(x) = (x^2 - 1)e^x + e^x(2x) = e^x(x^2 + 2x - 1) \quad \Rightarrow$

$f''(x) = e^x(2x + 2) + (x^2 + 2x - 1)e^x = e^x(x^2 + 4x + 1)$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

41. $f(x) = \dfrac{x^2}{1+x} \quad\Rightarrow\quad f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2+2x}{x^2+2x+1} \quad\Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$
$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

42. $g(x) = \dfrac{x}{e^x} \quad\Rightarrow\quad g'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} \quad\Rightarrow$

$$g''(x) = \frac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad\Rightarrow$$

$$g'''(x) = \frac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad\Rightarrow$$

$$g^{(4)}(x) = \frac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

43. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

44. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \quad\Rightarrow\quad h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \quad\Rightarrow\quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad\Rightarrow\quad h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$h'(4) = \dfrac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \dfrac{5(6) - 2(-3)}{5^2} = \dfrac{30+6}{25} = \dfrac{36}{25}$.

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \quad\Rightarrow$

$$h'(4) = \frac{[f(4)+g(4)]\,g'(4) - g(4)\,[f'(4)+g'(4)]}{[f(4)+g(4)]^2} = \frac{(2+5)\,(-3) - 5\,[6+(-3)]}{(2+5)^2} = \frac{-21-15}{7^2} = -\frac{36}{49}$$

45. $f(x) = e^x g(x) \quad \Rightarrow \quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5 + 2) = 7$

46. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x) \cdot 1}{x^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

47. $g(x) = xf(x) \quad \Rightarrow \quad g'(x) = xf'(x) + f(x) \cdot 1$. Now $g(3) = 3f(3) = 3 \cdot 4 = 12$ and $g'(3) = 3f'(3) + f(3) = 3(-2) + 4 = -2$. Thus, an equation of the tangent line to the graph of g at the point where $x = 3$ is $y - 12 = -2(x - 3)$, or $y = -2x + 18$.

48. $f'(x) = x^2 f(x) \quad \Rightarrow \quad f''(x) = x^2 f'(x) + f(x) \cdot 2x$. Now $f'(2) = 2^2 f(2) = 4(10) = 40$, so $f''(2) = 2^2(40) + 10(4) = 200$.

49. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4 - 0}{2 - 0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0 - 4}{2 - (-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2 \cdot (-1) + 1 \cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3 \cdot \frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

50. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1 \cdot \frac{1}{4} - 5 \cdot \left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

51. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x) \cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

52. (a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\,\right)^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

53. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

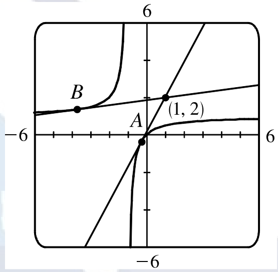
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

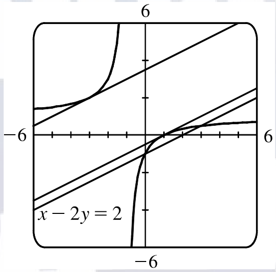
$$f\left(-2 \pm \sqrt{3}\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

54. $y = \dfrac{x-1}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects



the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad \Rightarrow$

$(a+1)^2 = 4 \quad \Rightarrow \quad a + 1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the

equation of the tangent is $y - 0 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and

the equation of the tangent is $y - 2 = \frac{1}{2}(x+3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.

55. $R = \dfrac{f}{g} \quad \Rightarrow \quad R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$,

and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1$.

56. $Q = \dfrac{f}{g} \quad \Rightarrow \quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4$.

57. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$

$$T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr})$$
$$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

58. (a) $f(20) = 10{,}000$ means that when the price of the fabric is $\$20/\text{yard}$, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past $\$20/\text{yard}$, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad \Rightarrow \quad R'(p) = pf'(p) + f(p) \cdot 1 \quad \Rightarrow \quad R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000.$

This means that as the price of the fabric increases past $\$20/\text{yard}$, the total revenue is increasing at $\$3000/(\$/\text{yard})$. Note that the Product Rule indicates that we will lose $\$7000/(\$/\text{yard})$ due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

59. $v = \dfrac{0.14[\text{S}]}{0.015 + [\text{S}]} \quad \Rightarrow \quad \dfrac{dv}{d[\text{S}]} = \dfrac{(0.015 + [\text{S}])(0.14) - (0.14[\text{S}])(1)}{(0.015 + [\text{S}])^2} = \dfrac{0.0021}{(0.015 + [\text{S}])^2}.$

$dv/d[\text{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

60. $B(t) = N(t)\,M(t) \quad \Rightarrow \quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8$ g/week.

61. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{d}{dx}(e^{3x}) = \dfrac{d}{dx}(e^x)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

62. (a) We use the Product Rule repeatedly: $F = fg \quad \Rightarrow \quad F' = f'g + fg' \quad \Rightarrow$

$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$.

(b) $F''' = f'''g + f''g' + 2\,(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad \Rightarrow$

$$\begin{aligned} F^{(4)} &= f^{(4)}g + f'''g' + 3\,(f'''g' + f''g'') + 3\,(f''g'' + f'g''') + f'g''' + fg^{(4)} \\ &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

63. For $f(x) = x^2 e^x$, $f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$f''(x) = e^x(x^2 + 4x + 2)$$
$$f'''(x) = e^x(x^2 + 6x + 6)$$
$$f^{(4)}(x) = e^x(x^2 + 8x + 12)$$
$$f^{(5)}(x) = e^x(x^2 + 10x + 20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x$$
$$= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k]$$

   This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

64. (a) $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$ [Quotient Rule] $= \dfrac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \dfrac{0 - g'(x)}{[g(x)]^2} = -\dfrac{g'(x)}{[g(x)]^2}$

(b) $\dfrac{d}{dt}\left(\dfrac{1}{t^3 + 2t^2 - 1}\right) = -\dfrac{(t^3 + 2t^2 - 1)'}{(t^3 + 2t^2 - 1)^2} = -\dfrac{3t^2 + 4t}{(t^3 + 2t^2 - 1)^2}$

(c) $\dfrac{d}{dx}(x^{-n}) = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $= -\dfrac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

3.3 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \Rightarrow f'(x) = x(-\sin x) + (\cos x)(1) + 2\sec^2 x = \cos x - x \sin x + 2\sec^2 x$

3. $f(x) = e^x \cos x \Rightarrow f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$

4. $y = 2\sec x - \csc x \Rightarrow y' = 2(\sec x \tan x) - (-\csc x \cot x) = 2\sec x \tan x + \csc x \cot x$

5. $y = \sec\theta \tan\theta \Rightarrow y' = \sec\theta(\sec^2\theta) + \tan\theta(\sec\theta\tan\theta) = \sec\theta(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta(1 + 2\tan^2\theta)$ or $\sec\theta(2\sec^2\theta - 1)$.

6. $g(\theta) = e^\theta(\tan\theta - \theta) \Rightarrow g'(\theta) = e^\theta(\sec^2\theta - 1) + (\tan\theta - \theta)e^\theta = e^\theta(\sec^2\theta - 1 + \tan\theta - \theta)$

7. $y = c\cos t + t^2 \sin t \quad \Rightarrow \quad y' = c(-\sin t) + t^2(\cos t) + \sin t\,(2t) = -c\sin t + t(t\cos t + 2\sin t)$

8. $f(t) = \dfrac{\cot t}{e^t} \quad \Rightarrow \quad f'(t) = \dfrac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \dfrac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\dfrac{\csc^2 t + \cot t}{e^t}$

9. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

10. $y = \sin\theta\,\cos\theta \quad \Rightarrow \quad y' = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta) = \cos^2\theta - \sin^2\theta \quad$ [or $\cos 2\theta$]

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos\theta)\cos\theta - (\sin\theta)(-\sin\theta)}{(1 + \cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1 + \cos\theta)^2} = \frac{\cos\theta + 1}{(1 + \cos\theta)^2} = \frac{1}{1 + \cos\theta}$$

12. $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

13. $y = \dfrac{t\sin t}{1 + t} \quad \Rightarrow$

$$y' = \frac{(1 + t)(t\cos t + \sin t) - t\sin t(1)}{(1 + t)^2} = \frac{t\cos t + \sin t + t^2\cos t + t\sin t - t\sin t}{(1 + t)^2} = \frac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$$

14. $y = \dfrac{\sin t}{1 + \tan t} \quad \Rightarrow$

$$y' = \frac{(1 + \tan t)\cos t - (\sin t)\sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t\sec t}{(1 + \tan t)^2}$$

15. Using Exercise 3.2.61(a) , $f(\theta) = \theta\cos\theta\,\sin\theta \quad \Rightarrow$

$$f'(\theta) = 1\cos\theta\,\sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta\,\sin\theta - \theta\sin^2\theta + \theta\cos^2\theta$$

$$= \sin\theta\,\cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta\cos 2\theta$$ [using double-angle formulas]

16. Using Exercise 3.2.61(a), $f(t) = te^t\cot t \quad \Rightarrow$

$$f'(t) = 1e^t\cot t + te^t\cot t + te^t(-\csc^2 t) = e^t(\cot t + t\cot t - t\csc^2 t)$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\,\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\,\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \, \cos h - \sin x \, \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left(\cos x \, \frac{\cos h - 1}{h} - \sin x \, \frac{\sin h}{h} \right) = \cos x \, \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \, \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

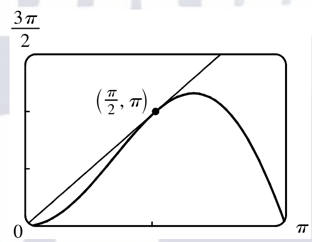
22. $y = e^x \cos x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x) \quad \Rightarrow$ the slope of the tangent line at $(0, 1)$ is $e^0(\cos 0 - \sin 0) = 1(1 - 0) = 1$ and an equation is $y - 1 = 1(x - 0)$ or $y = x + 1$.

23. $y = \cos x - \sin x \quad \Rightarrow \quad y' = -\sin x - \cos x$, so $y'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1$. An equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

24. $y = x + \tan x \quad \Rightarrow \quad y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

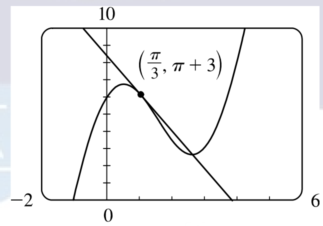
25. (a) $y = 2x \sin x \quad \Rightarrow \quad y' = 2(x \cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



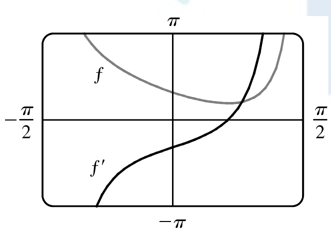
26. (a) $y = 3x + 6\cos x \quad \Rightarrow \quad y' = 3 - 6\sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6\sin\frac{\pi}{3} = 3 - 6\,\frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right)x + 3 + \pi\sqrt{3}$.

(b)



27. (a) $f(x) = \sec x - x \quad \Rightarrow \quad f'(x) = \sec x \, \tan x - 1$

(b)

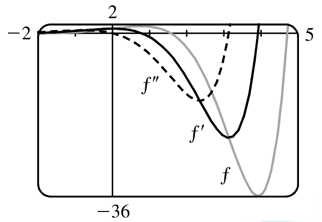


Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

28. (a) $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)\,e^x = e^x(\cos x - \sin x) \quad \Rightarrow$

$$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x)\,e^x = e^x(-\sin x - \cos x + \cos x - \sin x) = -2e^x \sin x$$

(b)



Note that $f' = 0$ where f has a minimum and $f'' = 0$ where f' has a minimum. Also note that f' is negative when f is decreasing and f'' is negative when f' is decreasing.

29. $H(\theta) = \theta\sin\theta \quad\Rightarrow\quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad\Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

30. $f(t) = \sec t \quad\Rightarrow\quad f'(t) = \sec t\,\tan t \quad\Rightarrow\quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\,\tan t = \sec^3 t + \sec t\,\tan^2 t$, so

$f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}.$

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1+\tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1+\tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

32. (a) $g(x) = f(x)\sin x \quad\Rightarrow\quad g'(x) = f(x)\cos x + \sin x\cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x\cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3}+1}{16} = \frac{1-2\sqrt{3}}{16}$$

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad\Leftrightarrow\quad 1 + 2\cos x = 0 \quad\Leftrightarrow\quad \cos x = -\frac{1}{2} \quad\Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

34. $f(x) = e^x\cos x$ has a horizontal tangent when $f'(x) = 0$. $f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$.

$f'(x) = 0 \quad\Leftrightarrow\quad \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x \quad\Leftrightarrow\quad \tan x = 1 \quad\Leftrightarrow\quad x = \frac{\pi}{4} + n\pi$, n an integer.

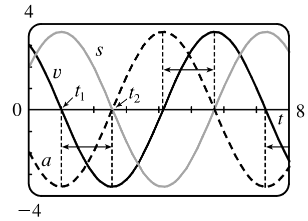
35. (a) $x(t) = 8\sin t \quad\Rightarrow\quad v(t) = x'(t) = 8\cos t \quad\Rightarrow\quad a(t) = x''(t) = -8\sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

36. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t \quad \Rightarrow$

$a(t) = -2\cos t - 3\sin t$

(b)

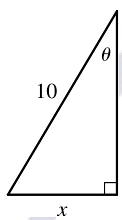


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55.$ So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98, s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

37.

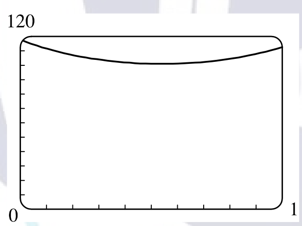


From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

38. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad \Leftrightarrow \quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad \Leftrightarrow \quad \sin\theta = \mu\cos\theta \quad \Leftrightarrow \quad \tan\theta = \mu \quad \Leftrightarrow \quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

39. $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad [\theta = 5x] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

40. $\displaystyle\lim_{x\to 0}\frac{\sin x}{\sin\pi x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\pi x}{\sin\pi x}\cdot\frac{1}{\pi} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1}{\pi} \quad [\theta = \pi x]$

$\displaystyle = 1\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin\theta}{\theta}}\cdot\frac{1}{\pi} = 1\cdot 1\cdot\frac{1}{\pi} = \frac{1}{\pi}$

41. $\displaystyle\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$

$\displaystyle = 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3$

42. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}}{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$

43. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x^3-4x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{3x}\cdot\dfrac{3}{5x^2-4}\right)=\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\cdot\lim\limits_{x\to 0}\dfrac{3}{5x^2-4}=1\cdot\left(\dfrac{3}{-4}\right)=-\dfrac{3}{4}$

44. $\lim\limits_{x\to 0}\dfrac{\sin 3x\sin 5x}{x^2}=\lim\limits_{x\to 0}\left(\dfrac{3\sin 3x}{3x}\cdot\dfrac{5\sin 5x}{5x}\right)=\lim\limits_{x\to 0}\dfrac{3\sin 3x}{3x}\cdot\lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x}$

$=3\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\cdot 5\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}=3(1)\cdot 5(1)=15$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}=\frac{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1+\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

46. $\lim\limits_{x\to 0}\csc x\,\sin(\sin x)=\lim\limits_{x\to 0}\dfrac{\sin(\sin x)}{\sin x}=\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ [As $x\to 0$, $\theta=\sin x\to 0$.] $=1$

47. $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{2\theta^2}=\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{2\theta^2}\cdot\dfrac{\cos\theta+1}{\cos\theta+1}=\lim\limits_{\theta\to 0}\dfrac{\cos^2\theta-1}{2\theta^2(\cos\theta+1)}=\lim\limits_{\theta\to 0}\dfrac{-\sin^2\theta}{2\theta^2(\cos\theta+1)}$

$=-\dfrac{1}{2}\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta+1}=-\dfrac{1}{2}\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta+1}$

$=-\dfrac{1}{2}\cdot 1\cdot 1\cdot\dfrac{1}{1+1}=-\dfrac{1}{4}$

48. $\lim\limits_{x\to 0}\dfrac{\sin(x^2)}{x}=\lim\limits_{x\to 0}\left[x\cdot\dfrac{\sin(x^2)}{x\cdot x}\right]=\lim\limits_{x\to 0}x\cdot\lim\limits_{x\to 0}\dfrac{\sin(x^2)}{x^2}=0\cdot\lim\limits_{y\to 0^+}\dfrac{\sin y}{y}$ [where $y=x^2$] $=0\cdot 1=0$

49. $\lim\limits_{x\to\pi/4}\dfrac{1-\tan x}{\sin x-\cos x}=\lim\limits_{x\to\pi/4}\dfrac{\left(1-\dfrac{\sin x}{\cos x}\right)\cdot\cos x}{(\sin x-\cos x)\cdot\cos x}=\lim\limits_{x\to\pi/4}\dfrac{\cos x-\sin x}{(\sin x-\cos x)\cos x}=\lim\limits_{x\to\pi/4}\dfrac{-1}{\cos x}=\dfrac{-1}{1/\sqrt{2}}=-\sqrt{2}$

50. $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2+x-2}=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{(x+2)(x-1)}=\lim\limits_{x\to 1}\dfrac{1}{x+2}\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1}=\frac{1}{3}\cdot 1=\frac{1}{3}$

51. $\dfrac{d}{dx}(\sin x)=\cos x\ \Rightarrow\ \dfrac{d^2}{dx^2}(\sin x)=-\sin x\ \Rightarrow\ \dfrac{d^3}{dx^3}(\sin x)=-\cos x\ \Rightarrow\ \dfrac{d^4}{dx^4}(\sin x)=\sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99=4(24)+3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x)=\dfrac{d^3}{dx^3}(\sin x)=-\cos x.$

52. Let $f(x)=x\sin x$ and $h(x)=\sin x$, so $f(x)=xh(x)$. Then $f'(x)=h(x)+xh'(x)$,

$f''(x)=h'(x)+h'(x)+xh''(x)=2h'(x)+xh''(x)$,

$f'''(x)=2h''(x)+h''(x)+xh'''(x)=3h''(x)+xh'''(x),\cdots,f^{(n)}(x)=nh^{(n-1)}(x)+xh^{(n)}(x).$

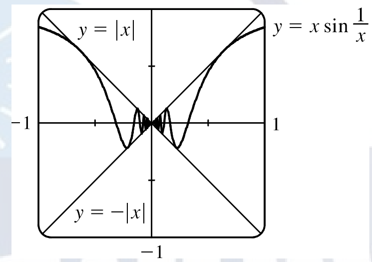
Since $34=4(8)+2$, we have $h^{(34)}(x)=h^{(2)}(x)=\dfrac{d^2}{dx^2}(\sin x)=-\sin x$ and $h^{(35)}(x)=-\cos x$.

Thus, $\dfrac{d^{35}}{dx^{35}}(x\sin x)=35h^{(34)}(x)+xh^{(35)}(x)=-35\sin x-x\cos x.$

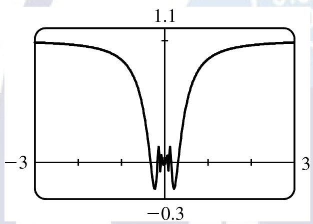
53. $y = A\sin x + B\cos x \quad \Rightarrow \quad y' = A\cos x - B\sin x \quad \Rightarrow \quad y'' = -A\sin x - B\cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us
$(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \quad \Leftrightarrow$
$-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \quad \Leftrightarrow \quad (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

54. (a) Let $\theta = \dfrac{1}{x}$. Then as $x \to \infty$, $\theta \to 0^+$, and $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0^+}\frac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(b) Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\displaystyle\lim_{x\to 0}(|x|) = 0$ and $\displaystyle\lim_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0} x\sin(1/x) = 0$.



(c)



55. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \quad \Rightarrow \quad \sec^2 x = \dfrac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad \Rightarrow \quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1 + \cot x}{\csc x} \quad \Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1 + \cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1 + \cot x)\,\cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

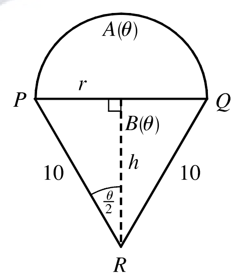
So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

56. We get the following formulas for r and h in terms of θ:

$$\sin\frac{\theta}{2} = \frac{r}{10} \quad \Rightarrow \quad r = 10\sin\frac{\theta}{2} \quad \text{and} \quad \cos\frac{\theta}{2} = \frac{h}{10} \quad \Rightarrow \quad h = 10\cos\frac{\theta}{2}$$

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

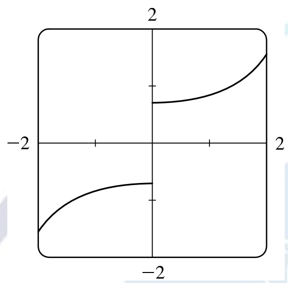
$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$



57. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$. So $\lim\limits_{\theta\to 0^+}\frac{s}{d} = \lim\limits_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim\limits_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim\limits_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

58. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,
$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} = \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)}$$
$$= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

3.4 The Chain Rule

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \dfrac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\pi) = \pi\sec^2\pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\dfrac{e^x}{2\sqrt{2-e^x}}$.

7. $F(x) = (5x^6 + 2x^3)^4 \Rightarrow F'(x) = 4(5x^6+2x^3)^3\cdot\dfrac{d}{dx}(5x^6+2x^3) = 4(5x^6+2x^3)^3(30x^5+6x^2)$.

We can factor as follows: $4(x^3)^3(5x^3+2)^3 6x^2(5x^3+1) = 24x^{11}(5x^3+2)^3(5x^3+1)$

8. $F(x) = (1 + x + x^2)^{99} \quad \Rightarrow \quad F'(x) = 99(1 + x + x^2)^{98} \cdot \dfrac{d}{dx}\left(1 + x + x^2\right) = 99(1 + x + x^2)^{98}(1 + 2x)$

9. $f(x) = \sqrt{5x + 1} = (5x + 1)^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}(5x + 1)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x + 1}}$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-1/3} \quad \Rightarrow \quad f'(x) = -\dfrac{1}{3}(x^2 - 1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2 - 1)^{4/3}}$

11. $f(\theta) = \cos(\theta^2) \quad \Rightarrow \quad f'(\theta) = -\sin(\theta^2) \cdot \dfrac{d}{d\theta}(\theta^2) = -\sin(\theta^2) \cdot (2\theta) = -2\theta\sin(\theta^2)$

12. $g(\theta) = \cos^2\theta = (\cos\theta)^2 \quad \Rightarrow \quad g'(\theta) = 2(\cos\theta)^1(-\sin\theta) = -2\sin\theta\,\cos\theta = -\sin 2\theta$

13. $y = x^2e^{-3x} \quad \Rightarrow \quad y' = x^2e^{-3x}(-3) + e^{-3x}(2x) = e^{-3x}(-3x^2 + 2x) = xe^{-3x}(2 - 3x)$

14. $f(t) = t\sin\pi t \quad \Rightarrow \quad f'(t) = t(\cos\pi t) \cdot \pi + (\sin\pi t) \cdot 1 = \pi t\cos\pi t + \sin\pi t$

15. $f(t) = e^{at}\sin bt \quad \Rightarrow \quad f'(t) = e^{at}(\cos bt) \cdot b + (\sin bt)e^{at} \cdot a = e^{at}(b\cos bt + a\sin bt)$

16. $g(x) = e^{x^2 - x} \quad \Rightarrow \quad g'(x) = e^{x^2 - x}(2x - 1)$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5 \quad \Rightarrow$

$$\begin{aligned} f'(x) &= (2x - 3)^4 \cdot 5(x^2 + x + 1)^4(2x + 1) + (x^2 + x + 1)^5 \cdot 4(2x - 3)^3 \cdot 2 \\ &= (2x - 3)^3(x^2 + x + 1)^4[(2x - 3) \cdot 5(2x + 1) + (x^2 + x + 1) \cdot 8] \\ &= (2x - 3)^3(x^2 + x + 1)^4(20x^2 - 20x - 15 + 8x^2 + 8x + 8) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7) \end{aligned}$$

18. $g(x) = (x^2 + 1)^3(x^2 + 2)^6 \quad \Rightarrow$

$$\begin{aligned} g'(x) &= (x^2 + 1)^3 \cdot 6(x^2 + 2)^5 \cdot 2x + (x^2 + 2)^6 \cdot 3(x^2 + 1)^2 \cdot 2x \\ &= 6x(x^2 + 1)^2(x^2 + 2)^5[2(x^2 + 1) + (x^2 + 2)] = 6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4) \end{aligned}$$

19. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3 \quad \Rightarrow$

$$\begin{aligned} h'(t) &= (t + 1)^{2/3} \cdot 3(2t^2 - 1)^2 \cdot 4t + (2t^2 - 1)^3 \cdot \tfrac{2}{3}(t + 1)^{-1/3} = \tfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2[18t(t + 1) + (2t^2 - 1)] \\ &= \tfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1) \end{aligned}$$

20. $F(t) = (3t - 1)^4(2t + 1)^{-3} \quad \Rightarrow$

$$\begin{aligned} F'(t) &= (3t - 1)^4(-3)(2t + 1)^{-4}(2) + (2t + 1)^{-3} \cdot 4(3t - 1)^3(3) \\ &= 6(3t - 1)^3(2t + 1)^{-4}[-(3t - 1) + 2(2t + 1)] = 6(3t - 1)^3(2t + 1)^{-4}(t + 3) \end{aligned}$$

21. $y = \sqrt{\dfrac{x}{x + 1}} = \left(\dfrac{x}{x + 1}\right)^{1/2} \quad \Rightarrow$

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{x}{x + 1}\right)^{-1/2}\frac{d}{dx}\left(\frac{x}{x + 1}\right) = \frac{1}{2}\,\frac{x^{-1/2}}{(x + 1)^{-1/2}}\,\frac{(x + 1)(1) - x(1)}{(x + 1)^2} \\ &= \frac{1}{2}\,\frac{(x + 1)^{1/2}}{x^{1/2}}\,\frac{1}{(x + 1)^2} = \frac{1}{2\sqrt{x}(x + 1)^{3/2}} \end{aligned}$$

22. $y = \left(x + \dfrac{1}{x}\right)^5 \quad \Rightarrow \quad y' = 5\left(x + \dfrac{1}{x}\right)^4 \dfrac{d}{dx}\left(x + \dfrac{1}{x}\right) = 5\left(x + \dfrac{1}{x}\right)^4\left(1 - \dfrac{1}{x^2}\right)$.

Another form of the answer is $\dfrac{5(x^2+1)^4(x^2-1)}{x^6}$.

23. $y = e^{\tan\theta} \quad \Rightarrow \quad y' = e^{\tan\theta}\dfrac{d}{d\theta}(\tan\theta) = (\sec^2\theta)e^{\tan\theta}$

24. Using Formula 5 and the Chain Rule, $f(t) = 2^{t^3} \quad \Rightarrow \quad f'(t) = 2^{t^3}\ln 2\,\dfrac{d}{dt}(t^3) = 3(\ln 2)t^2 2^{t^3}$.

25. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \quad \Rightarrow$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$
$$= 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

26. $s(t) = \sqrt{\dfrac{1+\sin t}{1+\cos t}} = \left(\dfrac{1+\sin t}{1+\cos t}\right)^{1/2} \quad \Rightarrow$

$$s'(t) = \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-1/2} \frac{(1+\cos t)\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}$$
$$= \frac{1}{2}\,\frac{(1+\sin t)^{-1/2}}{(1+\cos t)^{-1/2}}\,\frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^2} = \frac{\cos t + \sin t + 1}{2\sqrt{1+\sin t}\,(1+\cos t)^{3/2}}$$

27. Using Formula 5 and the Chain Rule, $r(t) = 10^{2\sqrt{t}} \quad \Rightarrow$

$$r'(t) = 10^{2\sqrt{t}}\ln 10\,\frac{d}{dt}\left(2\sqrt{t}\right) = 10^{2\sqrt{t}}\ln 10\left(2\cdot\tfrac{1}{2}t^{-1/2}\right) = \frac{(\ln 10)\,10^{2\sqrt{t}}}{\sqrt{t}}$$

28. $f(z) = e^{z/(z-1)} \quad \Rightarrow \quad f'(z) = e^{z/(z-1)}\dfrac{d}{dz}\dfrac{z}{z-1} = e^{z/(z-1)}\dfrac{(z-1)(1)-z(1)}{(z-1)^2} = -\dfrac{e^{z/(z-1)}}{(z-1)^2}$

29. $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5} \quad \Rightarrow$

$$H'(r) = \frac{(2r+1)^5\cdot 3(r^2-1)^2(2r) - (r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2} = \frac{2(2r+1)^4(r^2-1)^2[3r(2r+1)-5(r^2-1)]}{(2r+1)^{10}}$$
$$= \frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6} = \frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

30. $J(\theta) = \tan^2(n\theta) = [\tan(n\theta)]^2 \quad \Rightarrow$

$$J'(\theta) = 2\,[\tan(n\theta)]^1\,\frac{d}{d\theta}\tan(n\theta) = 2\tan(n\theta)\,\sec^2(n\theta)\cdot n = 2n\tan(n\theta)\,\sec^2(n\theta)$$

31. By (9), $F(t) = e^{t\sin 2t} \quad \Rightarrow$

$$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t\cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$$

32. $F(t) = \dfrac{t^2}{\sqrt{t^3+1}} \quad \Rightarrow$

$$F'(t) = \frac{(t^3+1)^{1/2}(2t) - t^2 \cdot \frac{1}{2}(t^3+1)^{-1/2}(3t^2)}{\left(\sqrt{t^3+1}\right)^2} = \frac{t(t^3+1)^{-1/2}\left[2(t^3+1) - \frac{3}{2}t^3\right]}{(t^3+1)^1}$$

$$= \frac{t\left(\frac{1}{2}t^3+2\right)}{(t^3+1)^{3/2}} = \frac{t(t^3+4)}{2(t^3+1)^{3/2}}$$

33. Using Formula 5 and the Chain Rule, $G(x) = 4^{C/x} \quad \Rightarrow$

$$G'(x) = 4^{C/x}(\ln 4)\frac{d}{dx}\frac{C}{x} \quad \left[\frac{C}{x} = Cx^{-1}\right] \quad = 4^{C/x}(\ln 4)\left(-Cx^{-2}\right) = -C(\ln 4)\frac{4^{C/x}}{x^2}$$

34. $U(y) = \left(\dfrac{y^4+1}{y^2+1}\right)^5 \quad \Rightarrow$

$$U'(y) = 5\left(\frac{y^4+1}{y^2+1}\right)^4 \frac{(y^2+1)(4y^3) - (y^4+1)(2y)}{(y^2+1)^2} = \frac{5(y^4+1)^4 2y[2y^2(y^2+1) - (y^4+1)]}{(y^2+1)^4(y^2+1)^2}$$

$$= \frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$$

35. $y = \cos\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right) \quad \Rightarrow$

$$y' = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{d}{dx}\left(\frac{1-e^{2x}}{1+e^{2x}}\right) = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{(1+e^{2x})(-2e^{2x}) - (1-e^{2x})(2e^{2x})}{(1+e^{2x})^2}$$

$$= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{-2e^{2x}\left[(1+e^{2x}) + (1-e^{2x})\right]}{(1+e^{2x})^2} = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{-2e^{2x}(2)}{(1+e^{2x})^2} = \frac{4e^{2x}}{(1+e^{2x})^2} \cdot \sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)$$

36. $y = x^2 e^{-1/x} \quad \Rightarrow \quad y' = x^2 e^{-1/x}\left(\dfrac{1}{x^2}\right) + e^{-1/x}(2x) = e^{-1/x} + 2xe^{-1/x} = e^{-1/x}(1+2x)$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad \Rightarrow$

$$y' = 2[\cot(\sin\theta)] \cdot \frac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta) \cdot [-\csc^2(\sin\theta) \cdot \cos\theta] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

38. $y = \sqrt{1 + xe^{-2x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(1 + xe^{-2x}\right)^{-1/2}\left[x\left(-2e^{-2x}\right) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

39. $f(t) = \tan(\sec(\cos t)) \quad \Rightarrow$

$$f'(t) = \sec^2(\sec(\cos t))\frac{d}{dt}\sec(\cos t) = \sec^2(\sec(\cos t))[\sec(\cos t)\tan(\cos t)]\frac{d}{dt}\cos t$$

$$= -\sec^2(\sec(\cos t))\sec(\cos t)\tan(\cos t)\sin t$$

40. $y = e^{\sin 2x} + \sin(e^{2x}) \quad \Rightarrow$

$$y' = e^{\sin 2x}\frac{d}{dx}\sin 2x + \cos(e^{2x})\frac{d}{dx}e^{2x} = e^{\sin 2x}(\cos 2x) \cdot 2 + \cos(e^{2x})\,e^{2x} \cdot 2$$

$$= 2\cos 2x\,e^{\sin 2x} + 2e^{2x}\cos(e^{2x})$$

41. $f(t) = \sin^2\left(e^{\sin^2 t}\right) = \left[\sin\left(e^{\sin^2 t}\right)\right]^2 \quad \Rightarrow$

$$f'(t) = 2\left[\sin\left(e^{\sin^2 t}\right)\right] \cdot \frac{d}{dt} \sin\left(e^{\sin^2 t}\right) = 2\sin\left(e^{\sin^2 t}\right) \cdot \cos\left(e^{\sin^2 t}\right) \cdot \frac{d}{dt}\, e^{\sin^2 t}$$

$$= 2\sin\left(e^{\sin^2 t}\right) \cos\left(e^{\sin^2 t}\right) \cdot e^{\sin^2 t} \cdot \frac{d}{dt}\, \sin^2 t = 2\sin\left(e^{\sin^2 t}\right) \cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \cdot 2\sin t\, \cos t$$

$$= 4\sin\left(e^{\sin^2 t}\right) \cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \sin t\, \cos t$$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \left[1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right)\right]$

43. $g(x) = (2ra^{rx} + n)^p \quad \Rightarrow$

$$g'(x) = p(2ra^{rx} + n)^{p-1} \cdot \frac{d}{dx}(2ra^{rx} + n) = p(2ra^{rx} + n)^{p-1} \cdot 2ra^{rx}(\ln a) \cdot r = 2r^2 p(\ln a)(2ra^{rx} + n)^{p-1} a^{rx}$$

44. $y = 2^{3^{4^x}} \quad \Rightarrow$

$$y' = 2^{3^{4^x}} (\ln 2)\, \frac{d}{dx} 3^{4^x} = 2^{3^{4^x}} (\ln 2)\, 3^{4^x} (\ln 3)\, \frac{d}{dx} 4^x = 2^{3^{4^x}} (\ln 2)\, 3^{4^x} (\ln 3)\, 4^x (\ln 4) = (\ln 2)(\ln 3)(\ln 4) 4^x 3^{4^x} 2^{3^{4^x}}$$

45. $y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$

$$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx}\, (\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx}\, (\sin(\tan \pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx} \tan \pi x = \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi \cos(\tan \pi x) \sec^2(\pi x) \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$$

46. $y = \left[x + (x + \sin^2 x)^3\right]^4 \quad \Rightarrow \quad y' = 4\left[x + (x + \sin^2 x)^3\right]^3 \cdot \left[1 + 3(x + \sin^2 x)^2 \cdot (1 + 2\sin x\, \cos x)\right]$

47. $y = \cos(\sin 3\theta) \quad \Rightarrow \quad y' = -\sin(\sin 3\theta) \cdot (\cos 3\theta) \cdot 3 = -3\cos 3\theta\, \sin(\sin 3\theta) \quad \Rightarrow$

$$y'' = -3\left[(\cos 3\theta)\cos(\sin 3\theta)(\cos 3\theta) \cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta) \cdot 3\right] = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$$

48. $y = \dfrac{1}{(1 + \tan x)^2} = (1 + \tan x)^{-2} \quad \Rightarrow \quad y' = -2(1 + \tan x)^{-3} \sec^2 x = \dfrac{-2\sec^2 x}{(1 + \tan x)^3}.$

Using the Product Rule with $y' = \left[-2(1 + \tan x)^{-3}\right](\sec x)^2$, we get

$$y'' = -2(1 + \tan x)^{-3} \cdot 2(\sec x)(\sec x\, \tan x) + (\sec x)^2 \cdot 6(1 + \tan x)^{-4} \sec^2 x$$

$$= 2\sec^2 x\, (1 + \tan x)^{-4} \left[-2(1 + \tan x)\tan x + 3\sec^2 x\right] \qquad \left[\begin{array}{c}\text{2 is the lesser exponent for } \sec x \\ \text{and } -4 \text{ for } (1 + \tan x)\end{array}\right]$$

$$= 2\sec^2 x\, (1 + \tan x)^{-4} \left[-2\tan x - 2\tan^2 x + 3(\tan^2 x + 1)\right]$$

$$= \frac{2\sec^2 x\, (\tan^2 x - 2\tan x + 3)}{(1 + \tan x)^4}$$

49. $y = \sqrt{1 - \sec t} \quad \Rightarrow \quad y' = \frac{1}{2}(1 - \sec t)^{-1/2}(-\sec t\, \tan t) = \dfrac{-\sec t\, \tan t}{2\sqrt{1 - \sec t}}.$

Using the Product Rule with $y' = \left(-\frac{1}{2}\sec t\, \tan t\right)(1 - \sec t)^{-1/2}$, we get

$$y'' = \left(-\tfrac{1}{2}\sec t\,\tan t\right)\left[-\tfrac{1}{2}(1-\sec t)^{-3/2}(-\sec t\,\tan t)\right] + (1-\sec t)^{-1/2}\left(-\tfrac{1}{2}\right)[\sec t\,\sec^2 t + \tan t\,\sec t\,\tan t].$$

Now factor out $-\frac{1}{2}\sec t(1-\sec t)^{-3/2}$. Note that $-\frac{3}{2}$ is the lesser exponent on $(1-\sec t)$. Continuing,

$$\begin{aligned} y'' &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[\tfrac{1}{2}\sec t\,\tan^2 t + (1-\sec t)(\sec^2 t + \tan^2 t)\right] \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(\tfrac{1}{2}\sec t\,\tan^2 t + \sec^2 t + \tan^2 t - \sec^3 t - \sec t\,\tan^2 t\right) \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[-\tfrac{1}{2}\sec t\,(\sec^2 t - 1) + \sec^2 t + (\sec^2 t - 1) - \sec^3 t\right] \\ &= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(-\tfrac{3}{2}\sec^3 t + 2\sec^2 t + \tfrac{1}{2}\sec t - 1\right) \\ &= \sec t\,(1-\sec t)^{-3/2}\left(\tfrac{3}{4}\sec^3 t - \sec^2 t - \tfrac{1}{4}\sec t + \tfrac{1}{2}\right) \\ &= \frac{\sec t\,(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1-\sec t)^{3/2}} \end{aligned}$$

There are many other correct forms of y'', such as $y'' = \dfrac{\sec t\,(3\sec t + 2)\sqrt{1-\sec t}}{4}$. We chose to find a factored form with only secants in the final form.

50. $y = e^{e^x} \quad\Rightarrow\quad y' = e^{e^x}\cdot(e^x)' = e^{e^x}\cdot e^x \quad\Rightarrow$

$y'' = e^{e^x}\cdot(e^x)' + e^x\cdot\left(e^{e^x}\right)' = e^{e^x}\cdot e^x + e^x\cdot e^{e^x}\cdot e^x = e^{e^x}\cdot e^x(1+e^x)$ or $e^{e^x+x}(1+e^x)$

51. $y = 2^x \quad\Rightarrow\quad y' = 2^x\ln 2$. At $(0,1)$, $y' = 2^0\ln 2 = \ln 2$, and an equation of the tangent line is $y - 1 = (\ln 2)(x-0)$ or $y = (\ln 2)x + 1$.

52. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \quad\Rightarrow\quad y' = \frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2 = \dfrac{3x^2}{2\sqrt{1+x^3}}$. At $(2,3)$, $y' = \dfrac{3\cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x-2)$, or $y = 2x - 1$.

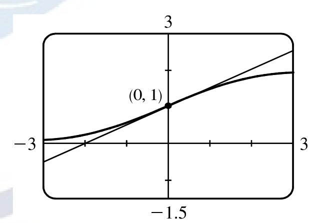
53. $y = \sin(\sin x) \quad\Rightarrow\quad y' = \cos(\sin x)\cdot\cos x$. At $(\pi,0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x + \pi$.

54. $y = xe^{-x^2} \quad\Rightarrow\quad y' = xe^{-x^2}(-2x) + e^{-x^2}(1) = e^{-x^2}(-2x^2+1)$. At $(0,0)$, $y' = e^0(1) = 1$, and an equation of the tangent line is $y - 0 = 1(x-0)$ or $y = x$.

55. (a) $y = \dfrac{2}{1+e^{-x}} \quad\Rightarrow\quad y' = \dfrac{(1+e^{-x})(0) - 2(-e^{-x})}{(1+e^{-x})^2} = \dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0,1)$, $y' = \dfrac{2e^0}{(1+e^0)^2} = \dfrac{2(1)}{(1+1)^2} = \dfrac{2}{2^2} = \dfrac{1}{2}$. So an equation of the tangent line is $y - 1 = \frac{1}{2}(x-0)$ or $y = \frac{1}{2}x + 1$.

(b)

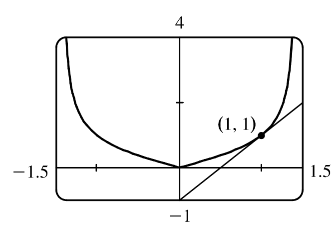


56. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \quad\Rightarrow$

$$\begin{aligned} f'(x) &= \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\,\right)^2}\cdot\frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}} \\ &= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}} \end{aligned}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

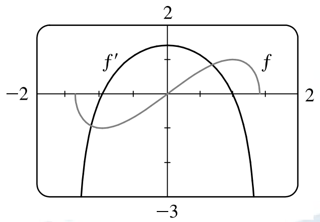
(b)



57. (a) $f(x) = x\sqrt{2 - x^2} = x(2 - x^2)^{1/2} \quad \Rightarrow$

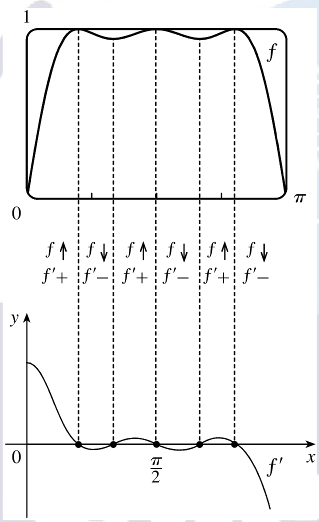
$$f'(x) = x \cdot \tfrac{1}{2}(2 - x^2)^{-1/2}(-2x) + (2 - x^2)^{1/2} \cdot 1 = (2 - x^2)^{-1/2}\left[-x^2 + (2 - x^2)\right] = \frac{2 - 2x^2}{\sqrt{2 - x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

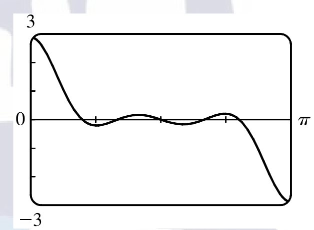
58. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \quad \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2\cos 2x)$$



59. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\,\cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

60. $y = \sqrt{1 + 2x} \quad \Rightarrow \quad y' = \frac{1}{2}(1 + 2x)^{-1/2} \cdot 2 = \dfrac{1}{\sqrt{1 + 2x}}$. The line $6x + 2y = 1$ $\left(\text{or } y = -3x + \frac{1}{2}\right)$ has slope -3, so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\dfrac{1}{3} = \dfrac{1}{\sqrt{1 + 2x}} \quad \Leftrightarrow \quad \sqrt{1 + 2x} = 3 \quad \Rightarrow \quad 1 + 2x = 9 \quad \Leftrightarrow$ $2x = 8 \quad \Leftrightarrow \quad x = 4$. When $x = 4$, $y = \sqrt{1 + 2(4)} = 3$, so the point is $(4, 3)$.

61. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

62. $h(x) = \sqrt{4 + 3f(x)} \quad \Rightarrow \quad h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$.

63. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

64. (a) $F(x) = f(f(x)) \quad \Rightarrow \quad F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \quad \Rightarrow \quad G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

65. (a) $u(x) = f(g(x)) \quad \Rightarrow \quad u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$ to $(2, 0)$, so its slope is $\dfrac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = (-\frac{1}{4})(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \quad \Rightarrow \quad v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \quad \Rightarrow \quad w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

66. (a) $h(x) = f(f(x)) \quad \Rightarrow \quad h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = f'(x^2) \cdot \dfrac{d}{dx}(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

67. The point $(3, 2)$ is on the graph of f, so $f(3) = 2$. The tangent line at $(3, 2)$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{-4}{6} = -\dfrac{2}{3}$.

$g(x) = \sqrt{f(x)} \quad \Rightarrow \quad g'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) \quad \Rightarrow$

$g'(3) = \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\dfrac{1}{3\sqrt{2}}$ or $-\frac{1}{6}\sqrt{2}$.

68. (a) $F(x) = f(x^\alpha) \quad \Rightarrow \quad F'(x) = f'(x^\alpha)\dfrac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \quad \Rightarrow \quad G'(x) = \alpha\,[f(x)]^{\alpha-1}\,f'(x)$

69. (a) $F(x) = f(e^x) \quad \Rightarrow \quad F'(x) = f'(e^x)\dfrac{d}{dx}(e^x) = f'(e^x)e^x$

(b) $G(x) = e^{f(x)} \quad \Rightarrow \quad G'(x) = e^{f(x)}\dfrac{d}{dx}f(x) = e^{f(x)}f'(x)$

70. (a) $g(x) = e^{cx} + f(x) \quad \Rightarrow \quad g'(x) = e^{cx} \cdot c + f'(x) \quad \Rightarrow \quad g'(0) = e^0 \cdot c + f'(0) = c + 5$.

$g'(x) = ce^{cx} + f'(x) \quad \Rightarrow \quad g''(x) = ce^{cx} \cdot c + f''(x) \quad \Rightarrow \quad g''(0) = c^2e^0 + f''(0) = c^2 - 2$.

(b) $h(x) = e^{kx}f(x) \quad \Rightarrow \quad h'(x) = e^{kx}f'(x) + f(x) \cdot ke^{kx} \quad \Rightarrow \quad h'(0) = e^0f'(0) + f(0) \cdot ke^0 = 5 + 3k$.

An equation of the tangent line to the graph of h at the point $(0, h(0)) = (0, f(0)) = (0, 3)$ is

$y - 3 = (5 + 3k)(x - 0)$ or $y = (5 + 3k)x + 3$.

71. $r(x) = f(g(h(x))) \quad \Rightarrow \quad r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so

$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$

72. $f(x) = xg(x^2) \quad \Rightarrow \quad f'(x) = xg'(x^2)\,2x + g(x^2) \cdot 1 = 2x^2 g'(x^2) + g(x^2) \quad \Rightarrow$

$f''(x) = 2x^2 g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3 g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3 g''(x^2)$

73. $F(x) = f(3f(4f(x))) \quad \Rightarrow$

$$F'(x) = f'(3f(4f(x))) \cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot \frac{d}{dx}(4f(x))$$
$$= f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot 4f'(x), \quad \text{so}$$

$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0) = f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$

74. $F(x) = f(xf(xf(x))) \quad \Rightarrow$

$$F'(x) = f'(xf(xf(x))) \cdot \frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x))) \cdot \left[x \cdot f'(xf(x)) \cdot \frac{d}{dx}(xf(x)) + f(xf(x)) \cdot 1\right]$$
$$= f'(xf(xf(x))) \cdot [xf'(xf(x)) \cdot (xf'(x) + f(x) \cdot 1) + f(xf(x))], \quad \text{so}$$

$$F'(1) = f'(f(f(1))) \cdot [f'(f(1)) \cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2)) \cdot [f'(2) \cdot (4+2) + f(2)]$$
$$= f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot 33 = 198.$$

75. $y = e^{2x}(A\cos 3x + B\sin 3x) \quad \Rightarrow$

$$y' = e^{2x}(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x) \cdot 2e^{2x}$$
$$= e^{2x}(-3A\sin 3x + 3B\cos 3x + 2A\cos 3x + 2B\sin 3x)$$
$$= e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x] \quad \Rightarrow$$

$$y'' = e^{2x}[-3(2A+3B)\sin 3x + 3(2B-3A)\cos 3x] + [(2A+3B)\cos 3x + (2B-3A)\sin 3x] \cdot 2e^{2x}$$
$$= e^{2x}\{[-3(2A+3B) + 2(2B-3A)]\sin 3x + [3(2B-3A) + 2(2A+3B)]\cos 3x\}$$
$$= e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x]$$

Substitute the expressions for y, y', and y'' in $y'' - 4y' + 13y$ to get

$$y'' - 4y' + 13y = e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x]$$
$$- 4e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x] + 13e^{2x}(A\cos 3x + B\sin 3x)$$
$$= e^{2x}[(-12A-5B-8B+12A+13B)\sin 3x + (-5A+12B-8A-12B+13A)\cos 3x]$$
$$= e^{2x}[(0)\sin 3x + (0)\cos 3x] = 0$$

Thus, the function y satisfies the differential equation $y'' - 4y' + 13y = 0$.

76. $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}$. Substituting y, y', and y'' into $y'' - 4y' + y = 0$ gives us $r^2 e^{rx} - 4re^{rx} + e^{rx} = 0 \quad \Rightarrow \quad e^{rx}(r^2 - 4r + 1) = 0$. Since $e^{rx} \neq 0$, we must have

$$r^2 - 4r + 1 = 0 \quad \Rightarrow \quad r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

77. The use of D, D^2, ..., D^n is just a derivative notation (see text page 159). In general, $Df(2x) = 2f'(2x)$, $D^2f(2x) = 4f''(2x)$, ..., $D^nf(2x) = 2^nf^{(n)}(2x)$. Since $f(x) = \cos x$ and $50 = 4(12) + 2$, we have $f^{(50)}(x) = f^{(2)}(x) = -\cos x$, so $D^{50}\cos 2x = -2^{50}\cos 2x$.

78. $f(x) = xe^{-x}$, $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, $f''(x) = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$. Similarly, $f'''(x) = (3-x)e^{-x}$, $f^{(4)}(x) = (x-4)e^{-x}$, ..., $f^{(1000)}(x) = (x-1000)e^{-x}$.

79. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \quad\Rightarrow\quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

80. (a) $s = A\cos(\omega t + \delta) \quad\Rightarrow\quad$ velocity $= s' = -\omega A\sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad\Leftrightarrow\quad \sin(\omega t + \delta) = 0 \quad\Leftrightarrow\quad \omega t + \delta = n\pi \quad\Leftrightarrow\quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

81. (a) $B(t) = 4.0 + 0.35\sin\dfrac{2\pi t}{5.4} \quad\Rightarrow\quad \dfrac{dB}{dt} = \left(0.35\cos\dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4}\cos\dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

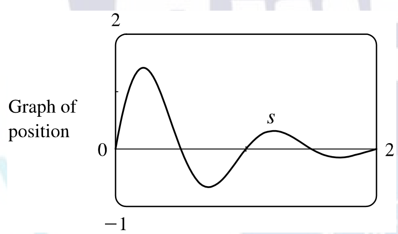
(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi}{5.4} \approx 0.16$.

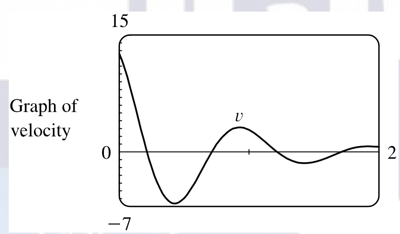
82. $L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t-80)\right) \quad\Rightarrow\quad L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

83. $s(t) = 2e^{-1.5t}\sin 2\pi t \quad\Rightarrow$

$v(t) = s'(t) = 2[e^{-1.5t}(\cos 2\pi t)(2\pi) + (\sin 2\pi t)e^{-1.5t}(-1.5)] = 2e^{-1.5t}(2\pi\cos 2\pi t - 1.5\sin 2\pi t)$

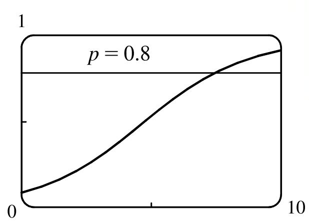




84. (a) $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\frac{1}{1+ae^{-kt}} = \frac{1}{1+a\cdot 0} = 1$, since $k > 0 \quad\Rightarrow\quad -kt \to -\infty \quad\Rightarrow\quad e^{-kt} \to 0$.

(b) $p(t) = (1+ae^{-kt})^{-1} \quad\Rightarrow\quad \dfrac{dp}{dt} = -(1+ae^{-kt})^{-2}(-kae^{-kt}) = \dfrac{kae^{-kt}}{(1+ae^{-kt})^2}$

(c)



From the graph of $p(t) = (1+10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

85. (a) Use $C(t) = ate^{bt}$ with $a = 0.0225$ and $b = -0.0467$ to get $C'(t) = a(te^{bt}\cdot b + e^{bt}\cdot 1) = a(bt+1)e^{bt}$.

$C'(10) = 0.0225(0.533)e^{-0.467} \approx 0.0075$, so the BAC was increasing at approximately 0.0075 (mg/mL)/min after 10 minutes.

(b) A half an hour later gives us $t = 10 + 30 = 40$. $C'(40) = 0.0225(-0.868)e^{-18.68} \approx -0.0030$, so the BAC was decreasing at approximately 0.0030 $(\text{mg/mL})/\text{min}$ after 40 minutes.

86. $P(t) = (1436.53) \cdot (1.01395)^t \quad \Rightarrow \quad P'(t) = (1436.53) \cdot (1.01395)^t(\ln 1.01395)$. The units for $P'(t)$ are millions of people per year. The rates of increase for 1920, 1950, and 2000 are $P'(20) \approx 26.25$, $P'(50) \approx 39.78$, and $P'(100) \approx 79.53$, respectively.

87. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}\,v(t) = v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

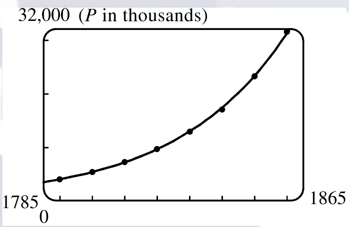
88. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2\,\dfrac{dr}{dt}$.

89. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a \approx 100.0124369$ and $b \approx 0.000045145933$.

(b) Use $Q'(t) = ab^t \ln b$ (from Formula 5) with the values of a and b from part (a) to get $Q'(0.04) \approx -670.63\ \mu\text{A}$. The result of Example 2.1.2 was $-670\ \mu\text{A}$.

90. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 7) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

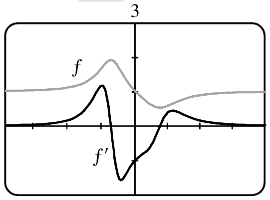
91. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the expression given by Derive.

(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x+1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x+1)^4(x^3 - x + 1)^4 + 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use

Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

92. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{(3x^4 - 1)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\,(x^4 + x + 1)^2}$ after simplification.



(b) $f'(x) = 0 \iff 3x^4 - 1 = 0 \iff x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

(c) Yes. $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.

93. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

94.
$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)\,[g(x)]^{-1}\right\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

This is an alternative derivation of the *formula* in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g. The proof in Section 3.2 does that; this one doesn't.

95. (a)
$$\begin{aligned} \frac{d}{dx}(\sin^n x\,\cos nx) &= n\sin^{n-1} x\,\cos x\,\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]} \\ &= n\sin^{n-1} x\,(\cos nx\,\cos x - \sin nx\,\sin x) && [\text{factor out } n\sin^{n-1} x] \\ &= n\sin^{n-1} x\cos(nx + x) && \text{[Addition Formula for cosine]} \\ &= n\sin^{n-1} x\cos[(n+1)x] && [\text{factor out } x] \end{aligned}$$

(b)
$$\begin{aligned} \frac{d}{dx}(\cos^n x\,\cos nx) &= n\cos^{n-1} x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]} \\ &= -n\cos^{n-1} x\,(\cos nx\,\sin x + \sin nx\,\cos x) && [\text{factor out } -n\cos^{n-1} x] \\ &= -n\cos^{n-1} x\sin(nx + x) && \text{[Addition Formula for sine]} \\ &= -n\cos^{n-1} x\sin[(n+1)x] && [\text{factor out } x] \end{aligned}$$

96. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x" $\iff$ $\dfrac{d}{dx}y^5 = 80\dfrac{dy}{dx} \iff 5y^4\dfrac{dy}{dx} = 80\dfrac{dy}{dx} \iff 5y^4 = 80$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\iff y^4 = 16 \iff y = 2$ (since $y > 0$ for all x)

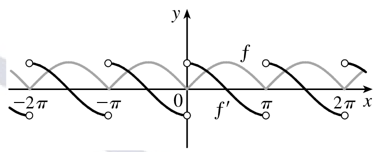
97. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

98. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

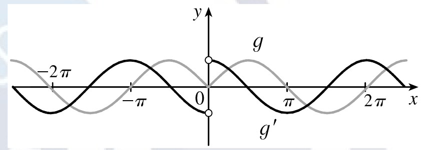
$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2} 2\sin x\,\cos x = \frac{\sin x}{|\sin x|}\cos x = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow$

$$g'(x) = \cos|x| \cdot \frac{x}{|x|} = \frac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$



g is not differentiable at 0.

99. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

100. From Exercise 99, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \quad\Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$

$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \quad\Rightarrow\quad P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell\,(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \quad\Rightarrow\quad a = -\dfrac{2b}{3\ell}$.

Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \quad\Rightarrow$

$-\dfrac{2}{3}b\ell^2 + b\ell^2 = h \quad\Rightarrow\quad \dfrac{1}{3}b\ell^2 = h \quad\Rightarrow\quad b = \dfrac{3h}{\ell^2} \quad\Rightarrow\quad a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x\,(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule,

we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}\left(3x^2\right)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}\left(2x\right)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2} \quad (\text{for } x \le \ell) \quad\Rightarrow$

$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}\left(2x\right)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

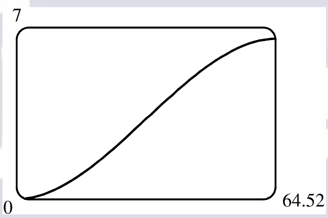
3. We substitute $k = 860$ mi/h^2, $h = 35{,}000 \text{ ft} \times \dfrac{1 \text{ mi}}{5280 \text{ ft}}$, and $v = 300$ mi/h into the result of part (b):

$$\frac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \quad\Rightarrow\quad \ell \ge 300\sqrt{6 \cdot \frac{35{,}000}{5280 \cdot 860}} \approx 64.5 \text{ miles.}$$

4. Substituting the values of h and ℓ in Problem 3 into

$P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$,

where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



3.5 Implicit Differentiation

1. (a) $\dfrac{d}{dx}(9x^2 - y^2) = \dfrac{d}{dx}(1) \quad\Rightarrow\quad 18x - 2y\,y' = 0 \quad\Rightarrow\quad 2y\,y' = 18x \quad\Rightarrow\quad y' = \dfrac{9x}{y}$

(b) $9x^2 - y^2 = 1 \quad\Rightarrow\quad y^2 = 9x^2 - 1 \quad\Rightarrow\quad y = \pm\sqrt{9x^2 - 1}$, so $y' = \pm\frac{1}{2}(9x^2 - 1)^{-1/2}(18x) = \pm\dfrac{9x}{\sqrt{9x^2 - 1}}$.

(c) From part (a), $y' = \dfrac{9x}{y} = \dfrac{9x}{\pm\sqrt{9x^2 - 1}}$, which agrees with part (b).

2. (a) $\dfrac{d}{dx}(2x^2 + x + xy) = \dfrac{d}{dx}(1) \quad\Rightarrow\quad 4x + 1 + xy' + y \cdot 1 = 0 \quad\Rightarrow\quad xy' = -4x - y - 1 \quad\Rightarrow\quad y' = -\dfrac{4x + y + 1}{x}$

(b) $2x^2 + x + xy = 1 \quad\Rightarrow\quad xy = 1 - 2x^2 - x \quad\Rightarrow\quad y = \dfrac{1}{x} - 2x - 1$, so $y' = -\dfrac{1}{x^2} - 2$

(c) From part (a),

$y' = -\dfrac{4x + y + 1}{x} = -4 - \dfrac{1}{x}y - \dfrac{1}{x} = -4 - \dfrac{1}{x}\left(\dfrac{1}{x} - 2x - 1 - \dfrac{1}{x}\right) = -4 - \dfrac{1}{x^2} + 2 + \dfrac{1}{x} - \dfrac{1}{x} = -\dfrac{1}{x^2} - 2$, which

agrees with part (b).

3. (a) $\dfrac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\dfrac{d}{dx}(1) \;\Rightarrow\; \dfrac{1}{2}x^{-1/2}+\dfrac{1}{2}y^{-1/2}y'=0 \;\Rightarrow\; \dfrac{1}{2\sqrt{y}}\,y'=-\dfrac{1}{2\sqrt{x}} \;\Rightarrow\; y'=-\dfrac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x}+\sqrt{y}=1 \;\Rightarrow\; \sqrt{y}=1-\sqrt{x} \;\Rightarrow\; y=\left(1-\sqrt{x}\right)^2 \;\Rightarrow\; y=1-2\sqrt{x}+x$, so

$y'=-2\cdot\dfrac{1}{2}x^{-1/2}+1=1-\dfrac{1}{\sqrt{x}}$.

(c) From part (a), $y'=-\dfrac{\sqrt{y}}{\sqrt{x}}=-\dfrac{1-\sqrt{x}}{\sqrt{x}}$ [from part (b)] $=-\dfrac{1}{\sqrt{x}}+1$, which agrees with part (b).

4. (a) $\dfrac{d}{dx}\left(\dfrac{2}{x}-\dfrac{1}{y}\right)=\dfrac{d}{dx}(4) \;\Rightarrow\; -2x^{-2}+y^{-2}y'=0 \;\Rightarrow\; \dfrac{1}{y^2}\,y'=\dfrac{2}{x^2} \;\Rightarrow\; y'=\dfrac{2y^2}{x^2}$

(b) $\dfrac{2}{x}-\dfrac{1}{y}=4 \;\Rightarrow\; \dfrac{1}{y}=\dfrac{2}{x}-4 \;\Rightarrow\; \dfrac{1}{y}=\dfrac{2-4x}{x} \;\Rightarrow\; y=\dfrac{x}{2-4x}$, so

$y'=\dfrac{(2-4x)(1)-x(-4)}{(2-4x)^2}=\dfrac{2}{(2-4x)^2}\;\left[\text{or }\dfrac{1}{2(1-2x)^2}\right]$.

(c) From part (a), $y'=\dfrac{2y^2}{x^2}=\dfrac{2\left(\dfrac{x}{2-4x}\right)^2}{x^2}$ [from part (b)] $=\dfrac{2x^2}{x^2(2-4x)^2}=\dfrac{2}{(2-4x)^2}$, which agrees with part (b).

5. $\dfrac{d}{dx}(x^2-4xy+y^2)=\dfrac{d}{dx}(4) \;\Rightarrow\; 2x-4[xy'+y(1)]+2y\,y'=0 \;\Rightarrow\; 2y\,y'-4xy'=4y-2x \;\Rightarrow$

$y'(y-2x)=2y-x \;\Rightarrow\; y'=\dfrac{2y-x}{y-2x}$

6. $\dfrac{d}{dx}(2x^2+xy-y^2)=\dfrac{d}{dx}(2) \;\Rightarrow\; 4x+xy'+y(1)-2y\,y'=0 \;\Rightarrow\; xy'-2y\,y'=-4x-y \;\Rightarrow$

$(x-2y)y'=-4x-y \;\Rightarrow\; y'=\dfrac{-4x-y}{x-2y}$

7. $\dfrac{d}{dx}(x^4+x^2y^2+y^3)=\dfrac{d}{dx}(5) \;\Rightarrow\; 4x^3+x^2\cdot 2y\,y'+y^2\cdot 2x+3y^2y'=0 \;\Rightarrow\; 2x^2y\,y'+3y^2y'=-4x^3-2xy^2 \;\Rightarrow$

$(2x^2y+3y^2)y'=-4x^3-2xy^2 \;\Rightarrow\; y'=\dfrac{-4x^3-2xy^2}{2x^2y+3y^2}=-\dfrac{2x(2x^2+y^2)}{y(2x^2+3y)}$

8. $\dfrac{d}{dx}(x^3-xy^2+y^3)=\dfrac{d}{dx}(1) \;\Rightarrow\; 3x^2-x\cdot 2y\,y'-y^2\cdot 1+3y^2y'=0 \;\Rightarrow\; 3y^2y'-2x\,y\,y'=y^2-3x^2 \;\Rightarrow$

$(3y^2-2xy)\,y'=y^2-3x^2 \;\Rightarrow\; y'=\dfrac{y^2-3x^2}{3y^2-2xy}=\dfrac{y^2-3x^2}{y(3y-2x)}$

9. $\dfrac{d}{dx}\left(\dfrac{x^2}{x+y}\right)=\dfrac{d}{dx}(y^2+1) \;\Rightarrow\; \dfrac{(x+y)(2x)-x^2(1+y')}{(x+y)^2}=2y\,y' \;\Rightarrow$

$2x^2+2xy-x^2-x^2\,y'=2y(x+y)^2\,y' \;\Rightarrow\; x^2+2xy=2y(x+y)^2\,y'+x^2\,y' \;\Rightarrow$

$x(x+2y)=[2y(x^2+2xy+y^2)+x^2]\,y' \;\Rightarrow\; y'=\dfrac{x(x+2y)}{2x^2y+4xy^2+2y^3+x^2}$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\dfrac{d}{dx}(xe^y)=\dfrac{d}{dx}(x-y) \;\Rightarrow\; xe^y\,y'+e^y\cdot 1=1-y' \;\Rightarrow\; xe^y\,y'+y'=1-e^y \;\Rightarrow\; y'(xe^y+1)=1-e^y \;\Rightarrow$

$y'=\dfrac{1-e^y}{xe^y+1}$

11. $\dfrac{d}{dx}(y\cos x) = \dfrac{d}{dx}(x^2+y^2) \;\Rightarrow\; y(-\sin x)+\cos x\cdot y' = 2x+2y\,y' \;\Rightarrow\; \cos x\cdot y' - 2y\,y' = 2x + y\sin x \;\Rightarrow$

$y'(\cos x - 2y) = 2x + y\sin x \;\Rightarrow\; y' = \dfrac{2x+y\sin x}{\cos x - 2y}$

12. $\dfrac{d}{dx}\cos(xy) = \dfrac{d}{dx}(1+\sin y) \;\Rightarrow\; -\sin(xy)(xy'+y\cdot 1) = \cos y\cdot y' \;\Rightarrow\; -xy'\sin(xy) - \cos y\cdot y' = y\sin(xy) \;\Rightarrow$

$y'[-x\sin(xy)-\cos y] = y\sin(xy) \;\Rightarrow\; y' = \dfrac{y\sin(xy)}{-x\sin(xy)-\cos y} = -\dfrac{y\sin(xy)}{x\sin(xy)+\cos y}$

13. $\dfrac{d}{dx}\sqrt{x+y} = \dfrac{d}{dx}\left(x^4+y^4\right) \;\Rightarrow\; \frac{1}{2}(x+y)^{-1/2}(1+y') = 4x^3+4y^3y' \;\Rightarrow$

$\dfrac{1}{2\sqrt{x+y}} + \dfrac{1}{2\sqrt{x+y}}\,y' = 4x^3+4y^3y' \;\Rightarrow\; \dfrac{1}{2\sqrt{x+y}} - 4x^3 = 4y^3y' - \dfrac{1}{2\sqrt{x+y}}y' \;\Rightarrow$

$\dfrac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}} = \dfrac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y' \;\Rightarrow\; y' = \dfrac{1-8x^3\sqrt{x+y}}{8y^3\sqrt{x+y}-1}$

14. $\dfrac{d}{dx}(e^y\sin x) = \dfrac{d}{dx}(x+xy) \;\Rightarrow\; e^y\cos x + \sin x\cdot e^y y' = 1 + xy' + y\cdot 1 \;\Rightarrow$

$e^y\sin x\cdot y' - xy' = 1 + y - e^y\cos x \;\Rightarrow\; y'(e^y\sin x - x) = 1 + y - e^y\cos x \;\Rightarrow\; y' = \dfrac{1+y-e^y\cos x}{e^y\sin x - x}$

15. $\dfrac{d}{dx}(e^{x/y}) = \dfrac{d}{dx}(x-y) \;\Rightarrow\; e^{x/y}\cdot\dfrac{d}{dx}\left(\dfrac{x}{y}\right) = 1 - y' \;\Rightarrow$

$e^{x/y}\cdot\dfrac{y\cdot 1 - x\cdot y'}{y^2} = 1 - y' \;\Rightarrow\; e^{x/y}\cdot\dfrac{1}{y} - \dfrac{xe^{x/y}}{y^2}\cdot y' = 1 - y' \;\Rightarrow\; y' - \dfrac{xe^{x/y}}{y^2}\cdot y' = 1 - \dfrac{e^{x/y}}{y} \;\Rightarrow$

$y'\left(1 - \dfrac{xe^{x/y}}{y^2}\right) = \dfrac{y - e^{x/y}}{y} \;\Rightarrow\; y' = \dfrac{\dfrac{y-e^{x/y}}{y}}{\dfrac{y^2 - xe^{x/y}}{y^2}} = \dfrac{y(y-e^{x/y})}{y^2 - xe^{x/y}}$

16. $\dfrac{d}{dx}(xy) = \dfrac{d}{dx}\sqrt{x^2+y^2} \;\Rightarrow\; xy' + y(1) = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\,y') \;\Rightarrow$

$xy' + y = \dfrac{x}{\sqrt{x^2+y^2}} + \dfrac{y}{\sqrt{x^2+y^2}}\,y' \;\Rightarrow\; xy' - \dfrac{y}{\sqrt{x^2+y^2}}\,y' = \dfrac{x}{\sqrt{x^2+y^2}} - y \;\Rightarrow$

$\dfrac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}\,y' = \dfrac{x - y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \;\Rightarrow\; y' = \dfrac{x - y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$

17. $\dfrac{d}{dx}\tan^{-1}(x^2y) = \dfrac{d}{dx}(x+xy^2) \;\Rightarrow\; \dfrac{1}{1+(x^2y)^2}(x^2y' + y\cdot 2x) = 1 + x\cdot 2y\,y' + y^2\cdot 1 \;\Rightarrow$

$\dfrac{x^2}{1+x^4y^2}\,y' - 2xy\,y' = 1 + y^2 - \dfrac{2xy}{1+x^4y^2} \;\Rightarrow\; y'\left(\dfrac{x^2}{1+x^4y^2} - 2xy\right) = 1 + y^2 - \dfrac{2xy}{1+x^4y^2} \;\Rightarrow$

$y' = \dfrac{1+y^2-\dfrac{2xy}{1+x^4y^2}}{\dfrac{x^2}{1+x^4y^2}-2xy}$ or $y' = \dfrac{1+x^4y^2+y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$

18. $\frac{d}{dx}(x\sin y + y\sin x) = \frac{d}{dx}(1) \Rightarrow x\cos y\cdot y' + \sin y\cdot 1 + y\cos x + \sin x\cdot y' = 0 \Rightarrow$

$x\cos y\cdot y' + \sin x\cdot y' = -\sin y - y\cos x \Rightarrow y'(x\cos y + \sin x) = -\sin y - y\cos x \Rightarrow y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}$

19. $\frac{d}{dx}\sin(xy) = \frac{d}{dx}\cos(x+y) \Rightarrow \cos(xy)\cdot(xy' + y\cdot 1) = -\sin(x+y)\cdot(1+y') \Rightarrow$

$x\cos(xy)\,y' + y\cos(xy) = -\sin(x+y) - y'\sin(x+y) \Rightarrow$

$x\cos(xy)\,y' + y'\sin(x+y) = -y\cos(xy) - \sin(x+y) \Rightarrow$

$[x\cos(xy) + \sin(x+y)]\,y' = -1\,[y\cos(xy) + \sin(x+y)] \Rightarrow y' = -\frac{y\cos(xy)+\sin(x+y)}{x\cos(xy)+\sin(x+y)}$

20. $\tan(x-y) = \frac{y}{1+x^2} \Rightarrow (1+x^2)\tan(x-y) = y \Rightarrow (1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \Rightarrow$

$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y' \Rightarrow$

$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = \left[1 + (1+x^2)\sec^2(x-y)\right]\cdot y' \Rightarrow$

$y' = \frac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

21. $\frac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \frac{d}{dx}(10) \Rightarrow f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \Rightarrow f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \Rightarrow$

$f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}.$

22. $\frac{d}{dx}\,[g(x) + x\sin g(x)] = \frac{d}{dx}\,(x^2) \Rightarrow g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x.$ If $x = 0$, we have

$g'(0) + 0 + \sin g(0) = 2(0) \Rightarrow g'(0) + \sin 0 = 0 \Rightarrow g'(0) + 0 = 0 \Rightarrow g'(0) = 0.$

23. $\frac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \frac{d}{dy}(0) \Rightarrow x^4\cdot 2y + y^2\cdot 4x^3\,x' - (x^3\cdot 1 + y\cdot 3x^2\,x') + 2(x\cdot 3y^2 + y^3\cdot x') = 0 \Rightarrow$

$4x^3y^2\,x' - 3x^2y\,x' + 2y^3\,x' = -2x^4y + x^3 - 6xy^2 \Rightarrow (4x^3y^2 - 3x^2y + 2y^3)\,x' = -2x^4y + x^3 - 6xy^2 \Rightarrow$

$x' = \frac{dx}{dy} = \frac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

24. $\frac{d}{dy}(y\sec x) = \frac{d}{dy}(x\tan y) \Rightarrow y\cdot\sec x\,\tan x\cdot x' + \sec x\cdot 1 = x\cdot\sec^2 y + \tan y\cdot x' \Rightarrow$

$y\sec x\,\tan x\cdot x' - \tan y\cdot x' = x\sec^2 y - \sec x \Rightarrow (y\sec x\,\tan x - \tan y)\,x' = x\sec^2 y - \sec x \Rightarrow$

$x' = \frac{dx}{dy} = \frac{x\sec^2 y - \sec x}{y\sec x\,\tan x - \tan y}$

25. $y\sin 2x = x\cos 2y \Rightarrow y\cdot\cos 2x\cdot 2 + \sin 2x\cdot y' = x(-\sin 2y\cdot 2y') + \cos(2y)\cdot 1 \Rightarrow$

$\sin 2x\cdot y' + 2x\sin 2y\cdot y' = -2y\cos 2x + \cos 2y \Rightarrow y'(\sin 2x + 2x\sin 2y) = -2y\cos 2x + \cos 2y \Rightarrow$

$y' = \dfrac{-2y\cos 2x + \cos 2y}{\sin 2x + 2x\sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \dfrac{(-\pi/2)(-1) + 0}{0 + \pi \cdot 1} = \dfrac{\pi/2}{\pi} = \dfrac{1}{2}$, so an equation of the tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

26. $\sin(x+y) = 2x - 2y \quad \Rightarrow \quad \cos(x+y)\cdot(1+y') = 2 - 2y' \quad \Rightarrow \quad \cos(x+y)\cdot y' + 2y' = 2 - \cos(x+y) \quad \Rightarrow$

$y'[\cos(x+y) + 2] = 2 - \cos(x+y) \quad \Rightarrow \quad y' = \dfrac{2 - \cos(x+y)}{\cos(x+y) + 2}$. When $x = \pi$ and $y = \pi$, we have $y' = \dfrac{2-1}{1+2} = \dfrac{1}{3}$, so an equation of the tangent line is $y - \pi = \frac{1}{3}(x - \pi)$, or $y = \frac{1}{3}x + \frac{2\pi}{3}$.

27. $x^2 - xy - y^2 = 1 \quad \Rightarrow \quad 2x - (xy' + y\cdot 1) - 2y\,y' = 0 \quad \Rightarrow \quad 2x - xy' - y - 2y\,y' = 0 \quad \Rightarrow \quad 2x - y = xy' + 2y\,y' \quad \Rightarrow$

$2x - y = (x + 2y)\,y' \quad \Rightarrow \quad y' = \dfrac{2x - y}{x + 2y}$. When $x = 2$ and $y = 1$, we have $y' = \dfrac{4-1}{2+2} = \dfrac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x - 2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

28. $x^2 + 2xy + 4y^2 = 12 \quad \Rightarrow \quad 2x + 2x\,y' + 2y + 8y\,y' = 0 \quad \Rightarrow \quad 2x\,y' + 8y\,y' = -2x - 2y \quad \Rightarrow$

$(x + 4y)\,y' = -x - y \quad \Rightarrow \quad y' = -\dfrac{x+y}{x+4y}$. When $x = 2$ and $y = 1$, we have $y' = -\dfrac{2+1}{2+4} = -\dfrac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad \Rightarrow \quad 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \quad \Rightarrow \quad y' = 2y' - 1 \quad \Rightarrow \quad y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

30. $x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad \Rightarrow \quad \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \quad \Rightarrow \quad y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}} = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is

$y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

31. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \quad \Rightarrow \quad 4(x^2 + y^2)(2x + 2y\,y') = 25(2x - 2y\,y') \quad \Rightarrow$

$4(x + y\,y')(x^2 + y^2) = 25(x - y\,y') \quad \Rightarrow \quad 4y\,y'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \quad \Rightarrow$

$y' = \dfrac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$,

so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

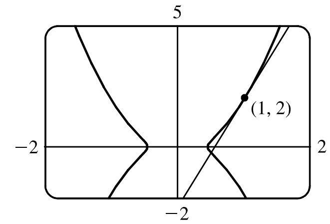
32. $y^2(y^2 - 4) = x^2(x^2 - 5) \quad \Rightarrow \quad y^4 - 4y^2 = x^4 - 5x^2 \quad \Rightarrow \quad 4y^3\,y' - 8y\,y' = 4x^3 - 10x$.

When $x = 0$ and $y = -2$, we have $-32y' + 16y' = 0 \quad \Rightarrow \quad -16y' = 0 \quad \Rightarrow \quad y' = 0$, so an equation of the tangent line is $y + 2 = 0(x - 0)$ or $y = -2$.

33. (a) $y^2 = 5x^4 - x^2 \quad \Rightarrow \quad 2y\,y' = 5(4x^3) - 2x \quad \Rightarrow \quad y' = \dfrac{10x^3 - x}{y}$.

So at the point $(1, 2)$ we have $y' = \dfrac{10(1)^3 - 1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}$.

(b)



34. (a) $y^2 = x^3 + 3x^2 \quad \Rightarrow \quad 2y\,y' = 3x^2 + 3(2x) \quad \Rightarrow \quad y' = \dfrac{3x^2 + 6x}{2y}$. So at the point $(1, -2)$ we have

$y' = \dfrac{3(1)^2 + 6(1)}{2(-2)} = -\dfrac{9}{4}$, and an equation of the tangent line is $y + 2 = -\frac{9}{4}(x - 1)$ or $y = -\frac{9}{4}x + \frac{1}{4}$.

(b) The curve has a horizontal tangent where $y' = 0 \quad \Leftrightarrow$

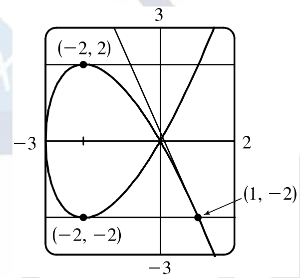
$3x^2 + 6x = 0 \quad \Leftrightarrow \quad 3x(x + 2) = 0 \quad \Leftrightarrow \quad x = 0$ or $x = -2$.

But note that at $x = 0$, $y = 0$ also, so the derivative does not exist.

At $x = -2$, $y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$, so $y = \pm 2$.

So the two points at which the curve has a horizontal tangent are $(-2, -2)$ and $(-2, 2)$.

(c)



35. $x^2 + 4y^2 = 4 \quad \Rightarrow \quad 2x + 8y\,y' = 0 \quad \Rightarrow \quad y' = -x/(4y) \quad \Rightarrow$

$$y'' = -\frac{1}{4}\frac{y \cdot 1 - x \cdot y'}{y^2} = -\frac{1}{4}\frac{y - x[-x/(4y)]}{y^2} = -\frac{1}{4}\frac{4y^2 + x^2}{4y^3} = -\frac{1}{4}\frac{4}{4y^3} \qquad \left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the} \\ \text{original equation } x^2 + 4y^2 = 4\end{array}\right]$$

Thus, $y'' = -\dfrac{1}{4y^3}$.

36. $x^2 + xy + y^2 = 3 \quad \Rightarrow \quad 2x + xy' + y + 2y\,y' = 0 \quad \Rightarrow \quad (x + 2y)y' = -2x - y \quad \Rightarrow \quad y' = \dfrac{-2x - y}{x + 2y}$.

Differentiating $2x + xy' + y + 2y\,y' = 0$ to find y'' gives $2 + xy'' + y' + y' + 2y\,y'' + 2y'y' = 0 \quad \Rightarrow$

$$(x + 2y)\,y'' = -2 - 2y' - 2(y')^2 = -2\left[1 - \frac{2x + y}{x + 2y} + \left(\frac{2x + y}{x + 2y}\right)^2\right] \quad \Rightarrow$$

$$\begin{aligned} y'' &= -\frac{2}{x + 2y}\left[\frac{(x + 2y)^2 - (2x + y)(x + 2y) + (2x + y)^2}{(x + 2y)^2}\right] \\ &= -\frac{2}{(x + 2y)^3}\,(x^2 + 4xy + 4y^2 - 2x^2 - 4xy - xy - 2y^2 + 4x^2 + 4xy + y^2) \\ &= -\frac{2}{(x + 2y)^3}\,(3x^2 + 3xy + 3y^2) = -\frac{2}{(x + 2y)^3}\,(9) \qquad \left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the} \\ \text{original equation } x^2 + xy + y^2 = 3\end{array}\right] \end{aligned}$$

Thus, $y'' = -\dfrac{18}{(x + 2y)^3}$.

37. $\sin y + \cos x = 1 \quad \Rightarrow \quad \cos y \cdot y' - \sin x = 0 \quad \Rightarrow \quad y' = \dfrac{\sin x}{\cos y} \quad \Rightarrow$

$$y'' = \frac{\cos y\ \cos x - \sin x(-\sin y)\, y'}{(\cos y)^2} = \frac{\cos y\ \cos x + \sin x\ \sin y(\sin x/\cos y)}{\cos^2 y}$$

$$= \frac{\cos^2 y\ \cos x + \sin^2 x\ \sin y}{\cos^2 y\ \cos y} = \frac{\cos^2 y\ \cos x + \sin^2 x\ \sin y}{\cos^3 y}$$

Using $\sin y + \cos x = 1$, the expression for y'' can be simplified to $y'' = (\cos^2 x + \sin y)/\cos^3 y$.

38. $x^3 - y^3 = 7 \quad \Rightarrow \quad 3x^2 - 3y^2 y' = 0 \quad \Rightarrow \quad y' = \dfrac{x^2}{y^2} \quad \Rightarrow$

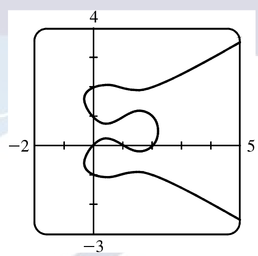
$$y'' = \frac{y^2(2x) - x^2(2y\,y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

39. If $x = 0$ in $xy + e^y = e$, then we get $0 + e^y = e$, so $y = 1$ and the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $xy' + y \cdot 1 + e^y y' = 0$. Substituting 0 for x and 1 for y gives us $0 + 1 + ey' = 0 \quad \Rightarrow \quad ey' = -1 \quad \Rightarrow \quad y' = -1/e$. Differentiating $xy' + y + e^y y' = 0$ implicitly with respect to x gives us $xy'' + y' \cdot 1 + y' + e^y y'' + y' \cdot e^y y' = 0$. Now substitute 0 for x, 1 for y, and $-1/e$ for y'.

$$0 + \left(-\frac{1}{e}\right) + \left(-\frac{1}{e}\right) + ey'' + \left(-\frac{1}{e}\right)(e)\left(-\frac{1}{e}\right) = 0 \quad \Rightarrow \quad -\frac{2}{e} + ey'' + \frac{1}{e} = 0 \quad \Rightarrow \quad ey'' = \frac{1}{e} \quad \Rightarrow \quad y'' = \frac{1}{e^2}.$$

40. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2 + 1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2 y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'.
$2 + y'' + (-2) + (-2) + 3(0 + 0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2 y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2 y''' + y'' \cdot 2yy') + 6[y \cdot 2y'y'' + (y')^2 y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

41. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

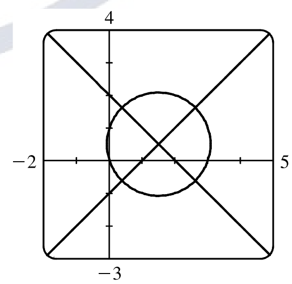


(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

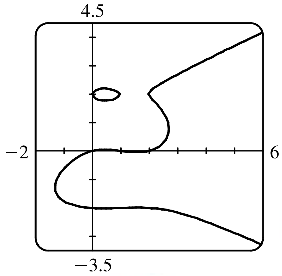
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

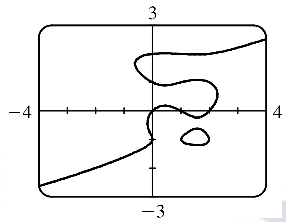
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



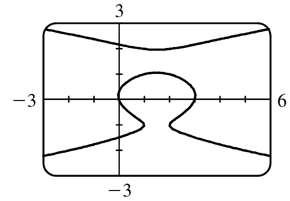
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



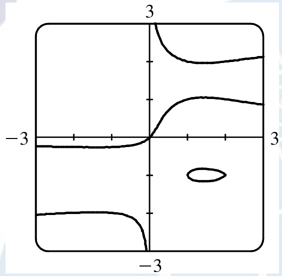
$y(y^2-4)(y-2)$
$=x(x-1)(x-2)$



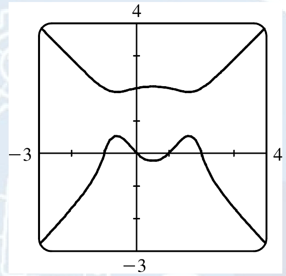
$y(y+1)(y^2-1)(y-2)$
$=x(x-1)(x-2)$



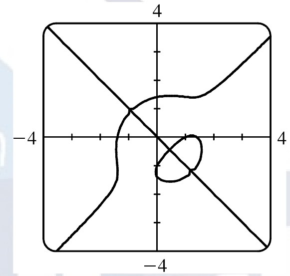
$(y+1)(y^2-1)(y-2)$
$=(x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)$
$=y(x-1)(x-2)$

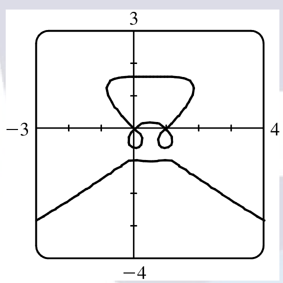


$y(y^2+1)(y-2)$
$=x(x^2-1)(x-2)$



$y(y+1)(y^2-2)$
$=x(x-1)(x^2-2)$

42. (a)



(b) $\dfrac{d}{dx}(2y^3+y^2-y^5)=\dfrac{d}{dx}(x^4-2x^3+x^2) \quad\Rightarrow$

$6y^2y'+2y\,y'-5y^4y'=4x^3-6x^2+2x \quad\Rightarrow$

$y'=\dfrac{2x(2x^2-3x+1)}{6y^2+2y-5y^4}=\dfrac{2x(2x-1)(x-1)}{y(6y+2-5y^3)}$. From the graph and the values for which $y'=0$, we speculate that there are 9 points with horizontal tangents: 3 at $x=0$, 3 at $x=\frac{1}{2}$, and 3 at $x=1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

43. From Exercise 31, a tangent to the lemniscate will be horizontal if $y'=0 \quad\Rightarrow\quad 25x-4x(x^2+y^2)=0 \quad\Rightarrow$ $x[25-4(x^2+y^2)]=0 \quad\Rightarrow\quad x^2+y^2=\frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for x^2+y^2 in the equation of the lemniscate, $2(x^2+y^2)^2=25(x^2-y^2)$, we get $x^2-y^2=\frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2=\frac{75}{16}$ and $y^2=\frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4},\pm\frac{5}{4}\right)$.

44. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0 \quad\Rightarrow\quad y'=-\dfrac{b^2x}{a^2y} \quad\Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=\dfrac{-b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2}-\dfrac{y_0^2}{b^2}=-\dfrac{x_0x}{a^2}+\dfrac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the ellipse, we have $\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1$.

45. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{2x}{a^2}-\dfrac{2yy'}{b^2}=0 \;\Rightarrow\; y'=\dfrac{b^2x}{a^2y} \;\Rightarrow$ an equation of the tangent line at (x_0,y_0) is

$y-y_0=\dfrac{b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2}-\dfrac{y_0^2}{b^2}=\dfrac{x_0x}{a^2}-\dfrac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the hyperbola,

we have $\dfrac{x_0x}{a^2}-\dfrac{y_0y}{b^2}=\dfrac{x_0^2}{a^2}-\dfrac{y_0^2}{b^2}=1$.

46. $\sqrt{x}+\sqrt{y}=\sqrt{c} \;\Rightarrow\; \dfrac{1}{2\sqrt{x}}+\dfrac{y'}{2\sqrt{y}}=0 \;\Rightarrow\; y'=-\dfrac{\sqrt{y}}{\sqrt{x}} \;\Rightarrow$ an equation of the tangent line at (x_0,y_0)

is $y-y_0=-\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$. Now $x=0 \;\Rightarrow\; y=y_0-\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)=y_0+\sqrt{x_0}\sqrt{y_0}$, so the y-intercept is

$y_0+\sqrt{x_0}\sqrt{y_0}$. And $y=0 \;\Rightarrow\; -y_0=-\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0) \;\Rightarrow\; x-x_0=\dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \;\Rightarrow$

$x=x_0+\sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0+\sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0+\sqrt{x_0}\sqrt{y_0}\right)+\left(x_0+\sqrt{x_0}\sqrt{y_0}\right)=x_0+2\sqrt{x_0}\sqrt{y_0}+y_0=\left(\sqrt{x_0}+\sqrt{y_0}\right)^2=\left(\sqrt{c}\right)^2=c.$$

47. If the circle has radius r, its equation is $x^2+y^2=r^2 \;\Rightarrow\; 2x+2yy'=0 \;\Rightarrow\; y'=-\dfrac{x}{y}$, so the slope of the tangent line

at $P(x_0,y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0}=\dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at

P is perpendicular to the radius OP.

48. $y^q=x^p \;\Rightarrow\; qy^{q-1}y'=px^{p-1} \;\Rightarrow\; y'=\dfrac{px^{p-1}}{qy^{q-1}}=\dfrac{px^{p-1}y}{qy^q}=\dfrac{px^{p-1}x^{p/q}}{qx^p}=\dfrac{p}{q}x^{(p/q)-1}$

49. $y=(\tan^{-1}x)^2 \;\Rightarrow\; y'=2(\tan^{-1}x)^1\cdot\dfrac{d}{dx}(\tan^{-1}x)=2\tan^{-1}x\cdot\dfrac{1}{1+x^2}=\dfrac{2\tan^{-1}x}{1+x^2}$

50. $y=\tan^{-1}(x^2) \;\Rightarrow\; y'=\dfrac{1}{1+(x^2)^2}\cdot\dfrac{d}{dx}(x^2)=\dfrac{1}{1+x^4}\cdot 2x=\dfrac{2x}{1+x^4}$

51. $y=\sin^{-1}(2x+1) \;\Rightarrow$

$$y'=\frac{1}{\sqrt{1-(2x+1)^2}}\cdot\frac{d}{dx}(2x+1)=\frac{1}{\sqrt{1-(4x^2+4x+1)}}\cdot 2=\frac{2}{\sqrt{-4x^2-4x}}=\frac{1}{\sqrt{-x^2-x}}$$

52. $g(x)=\arccos\sqrt{x} \;\Rightarrow\; g'(x)=-\dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\dfrac{d}{dx}\sqrt{x}=-\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1}{2}x^{-1/2}\right)=-\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

53. $F(x)=x\sec^{-1}(x^3) \;\overset{\text{PR}}{\Rightarrow}$

$$F'(x)=x\cdot\frac{1}{x^3\sqrt{(x^3)^2-1}}\frac{d}{dx}(x^3)+\sec^{-1}(x^3)\cdot 1=\frac{x(3x^2)}{x^3\sqrt{x^6-1}}+\sec^{-1}(x^3)=\frac{3}{\sqrt{x^6-1}}+\sec^{-1}(x^3)$$

54. $y = \tan^{-1}\left(x - \sqrt{x^2+1}\right) \quad \Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]} = \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]}$$

$$= \frac{1}{2(1+x^2)}$$

55. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad \Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = \dfrac{3\pi}{2}$ for $t < 0$.

56. $R(t) = \arcsin(1/t) \quad \Rightarrow$

$$R'(t) = \frac{1}{\sqrt{1-(1/t)^2}}\,\frac{d}{dt}\frac{1}{t} = \frac{1}{\sqrt{1-1/t^2}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-1/t^2}}\,\frac{1}{\sqrt{t^4}}$$

$$= -\frac{1}{\sqrt{t^4-t^2}} = -\frac{1}{\sqrt{t^2(t^2-1)}} = -\frac{1}{|\,t\,|\sqrt{t^2-1}}$$

57. $y = x\sin^{-1}x + \sqrt{1-x^2} \quad \Rightarrow$

$$y' = x\cdot\frac{1}{\sqrt{1-x^2}} + (\sin^{-1}x)(1) + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

58. $y = \cos^{-1}(\sin^{-1}t) \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{d}{dt}\sin^{-1}t = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{1}{\sqrt{1-t^2}}$

59. $y = \arccos\left(\dfrac{b+a\cos x}{a+b\cos x}\right) \quad \Rightarrow$

$$y' = -\frac{1}{\sqrt{1-\left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\,\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2+b^2\cos^2 x-b^2-a^2\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|}$$

$$= \frac{1}{\sqrt{a^2-b^2}\sqrt{1-\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}}{|a+b\cos x|}\,\frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \quad \Rightarrow \quad b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

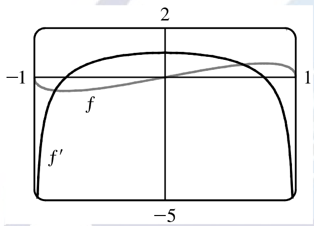
60. $y = \arctan\sqrt{\dfrac{1-x}{1+x}} = \arctan\left(\dfrac{1-x}{1+x}\right)^{1/2} \quad \Rightarrow$

$$y' = \frac{1}{1+\left(\sqrt{\dfrac{1-x}{1+x}}\right)^2}\cdot\frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{1+\dfrac{1-x}{1+x}}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\cdot\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$= \frac{1}{\dfrac{1+x}{1+x}+\dfrac{1-x}{1+x}}\cdot\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\cdot\frac{-2}{(1+x)^2} = \frac{1+x}{2}\cdot\frac{1}{2}\cdot\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\cdot\frac{-2}{(1+x)^2}$$

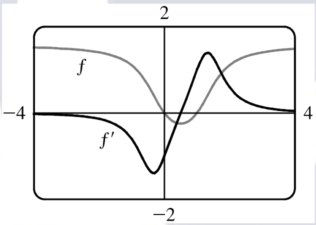
$$= \frac{-1}{2(1-x)^{1/2}(1+x)^{1/2}} = \frac{-1}{2\sqrt{1-x^2}}$$

61. $f(x) = \sqrt{1-x^2}\arcsin x \quad \Rightarrow \quad f'(x) = \sqrt{1-x^2}\cdot\dfrac{1}{\sqrt{1-x^2}} + \arcsin x\cdot\dfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = 1 - \dfrac{x\arcsin x}{\sqrt{1-x^2}}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

62. $f(x) = \arctan(x^2 - x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{1+(x^2-x)^2}\cdot\dfrac{d}{dx}(x^2-x) = \dfrac{2x-1}{1+(x^2-x)^2}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

63. Let $y = \cos^{-1}x$. Then $\cos y = x$ and $0 \le y \le \pi \quad \Rightarrow \quad -\sin y\,\dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$. [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

64. (a) Let $y = \sec^{-1}x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y\,\tan y\left(\dfrac{dy}{dx}\right) = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{\sec y\sqrt{\sec^2 y - 1}} = \dfrac{1}{x\sqrt{x^2-1}}$. Note that $\tan^2 y = \sec^2 y - 1 \quad \Rightarrow \quad \tan y = \sqrt{\sec^2 y - 1}$

since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

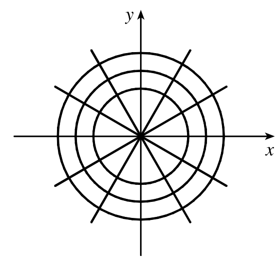
(b) $y = \sec^{-1}x \quad \Rightarrow \quad \sec y = x \quad \Rightarrow \quad \sec y\,\tan y\,\dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$,

so $\tan y = \pm\sqrt{x^2-1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \ge 1$, so $\sec y = x = |x|$ and $\tan y \ge 0 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \le -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1} \quad \Rightarrow$

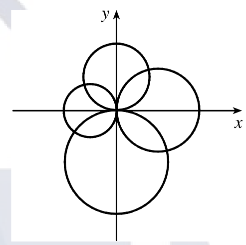
$\dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{x\left(-\sqrt{x^2-1}\right)} = \dfrac{1}{(-x)\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$.

65. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

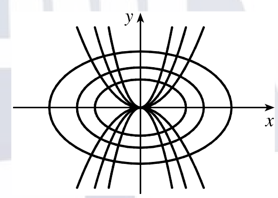


66. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ **(1)** and $x_0^2 + y_0^2 = by_0$ **(2)**.

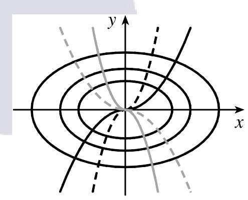
Now $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \Rightarrow$ $2x + 2yy' = by' \Rightarrow y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \Leftrightarrow$ $\dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \Leftrightarrow 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \Leftrightarrow ax_0 + by_0 = 2(x_0^2 + y_0^2)$, which is true by **(1)** and **(2)**.



67. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\Rightarrow 2x + 4yy' = 0 \Rightarrow$ $2yy' = -x \Rightarrow y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if $c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.



68. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\Rightarrow 2x + 6yy' = 0 \Rightarrow$ $3yy' = -x \Rightarrow y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal if $a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intesects $x^2 + 3y^2 = b$ orthogonally at $\left(\pm\sqrt{b}, 0\right)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.



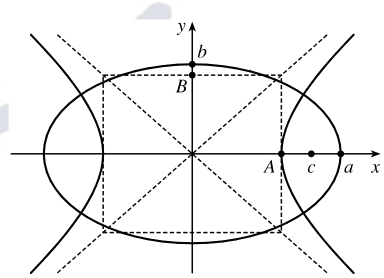
69. Since $A^2 < a^2$, we are assured that there are four points of intersection.

(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \Rightarrow \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \Rightarrow$

$y' = m_1 = -\dfrac{xb^2}{ya^2}$.

(2) $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \Rightarrow \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \Rightarrow \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \Rightarrow$

$y' = m_2 = \dfrac{xB^2}{yA^2}$.



Now $m_1 m_2 = -\dfrac{xb^2}{ya^2} \cdot \dfrac{xB^2}{yA^2} = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{x^2}{y^2}$ **(3)**. Subtracting equations, **(1)** − **(2)**, gives us $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 0 \Rightarrow$

$\dfrac{y^2}{b^2}+\dfrac{y^2}{B^2}=\dfrac{x^2}{A^2}-\dfrac{x^2}{a^2} \quad\Rightarrow\quad \dfrac{y^2B^2+y^2b^2}{b^2B^2}=\dfrac{x^2a^2-x^2A^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{y^2(b^2+B^2)}{b^2B^2}=\dfrac{x^2(a^2-A^2)}{a^2A^2}$ **(4)**. Since $a^2-b^2=A^2+B^2$, we have $a^2-A^2=b^2+B^2$. Thus, equation **(4)** becomes $\dfrac{y^2}{b^2B^2}=\dfrac{x^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{x^2}{y^2}=\dfrac{A^2a^2}{b^2B^2}$, and substituting for $\dfrac{x^2}{y^2}$ in equation **(3)** gives us $m_1m_2=-\dfrac{b^2B^2}{a^2A^2}\cdot\dfrac{a^2A^2}{b^2B^2}=-1$. Hence, the ellipse and hyperbola are orthogonal trajectories.

70. $y=(x+c)^{-1} \quad\Rightarrow\quad y'=-(x+c)^{-2}$ and $y=a(x+k)^{1/3} \quad\Rightarrow\quad y'=\frac{1}{3}a(x+k)^{-2/3}$, so the curves are othogonal if the product of the slopes is -1, that is, $\dfrac{-1}{(x+c)^2}\cdot\dfrac{a}{3(x+k)^{2/3}}=-1 \quad\Rightarrow\quad a=3(x+c)^2(x+k)^{2/3} \quad\Rightarrow$

$a=3\left(\dfrac{1}{y}\right)^2\left(\dfrac{y}{a}\right)^2$ [since $y^2=(x+c)^{-2}$ and $y^2=a^2(x+k)^{2/3}$] $\quad\Rightarrow\quad a=3\left(\dfrac{1}{a^2}\right) \quad\Rightarrow\quad a^3=3 \quad\Rightarrow\quad a=\sqrt[3]{3}$.

71. (a) $\left(P+\dfrac{n^2a}{V^2}\right)(V-nb)=nRT \quad\Rightarrow\quad PV-Pnb+\dfrac{n^2a}{V}-\dfrac{n^3ab}{V^2}=nRT \quad\Rightarrow$

$$\frac{d}{dP}(PV-Pnb+n^2aV^{-1}-n^3abV^{-2})=\frac{d}{dP}(nRT) \quad\Rightarrow$$

$$PV'+V\cdot 1-nb-n^2aV^{-2}\cdot V'+2n^3abV^{-3}\cdot V'=0 \quad\Rightarrow\quad V'(P-n^2aV^{-2}+2n^3abV^{-3})=nb-V \quad\Rightarrow$$

$$V'=\frac{nb-V}{P-n^2aV^{-2}+2n^3abV^{-3}} \text{ or } \frac{dV}{dP}=\frac{V^3(nb-V)}{PV^3-n^2aV+2n^3ab}$$

(b) Using the last expression for dV/dP from part (a), we get

$$\frac{dV}{dP}=\frac{(10\text{ L})^3[(1\text{ mole})(0.04267\text{ L/mole})-10\text{ L}]}{\left[\begin{array}{l}(2.5\text{ atm})(10\text{ L})^3-(1\text{ mole})^2(3.592\text{ L}^2\text{-atm/ mole}^2)(10\text{ L})\\ \qquad +2(1\text{ mole})^3(3.592\text{ L}^2\text{-atm/ mole}^2)(0.04267\text{ L/ mole})\end{array}\right]}$$

$$=\frac{-9957.33\text{ L}^4}{2464.386541\text{ L}^3\text{-atm}}\approx -4.04\text{ L/ atm}.$$

72. (a) $x^2+xy+y^2+1=0 \quad\Rightarrow\quad 2x+xy'+y\cdot 1+2yy'+0=0 \quad\Rightarrow\quad y'(x+2y)=-2x-y \quad\Rightarrow\quad y'=\dfrac{-2x-y}{x+2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $x^2+xy+y^2+1=0 \quad\Rightarrow\quad x^2+2xy+y^2-xy+1=0 \quad\Rightarrow\quad (x+y)^2=xy-1$. The left side of the last equation is nonnegative, but the right side is at most -1, so that proves there are no points that satisfy the equation.

Another solution: $x^2+xy+y^2+1=\frac{1}{2}x^2+xy+\frac{1}{2}y^2+\frac{1}{2}x^2+\frac{1}{2}y^2+1=\frac{1}{2}(x^2+2xy+y^2)+\frac{1}{2}(x^2+y^2)+1$

$$=\tfrac{1}{2}(x+y)^2+\tfrac{1}{2}(x^2+y^2)+1\geq 1$$

Another solution: Regarding $x^2+xy+y^2+1=0$ as a quadratic in x, the discriminant is $y^2-4(y^2+1)=-3y^2-4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x, and therefore it is meaningless to consider y'.

73. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

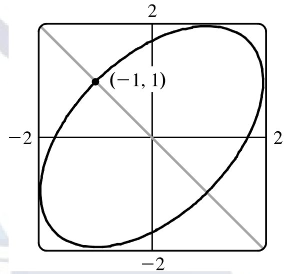
$y = 0 \quad \Rightarrow \quad x^2 - x(0) + 0^2 = 3 \quad \Leftrightarrow \quad x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $(\pm\sqrt{3}, 0)$.

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \quad \Rightarrow \quad y'(2y - x) = y - 2x \quad \Leftrightarrow \quad y' = \dfrac{y - 2x}{2y - x}$.

So y' at $(\sqrt{3}, 0)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

74. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 73. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \quad \Leftrightarrow$ $y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \quad \Rightarrow \quad 3x^2 = 3 \quad \Leftrightarrow \quad x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



75. $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \quad \Leftrightarrow \quad y'(2x^2y + x) = -2xy^2 - y \quad \Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \quad \Leftrightarrow \quad 2xy^2 + y = 2x^2y + x \quad \Leftrightarrow \quad y(2xy + 1) = x(2xy + 1) \quad \Leftrightarrow$

$y(2xy + 1) - x(2xy + 1) = 0 \quad \Leftrightarrow \quad (2xy + 1)(y - x) = 0 \quad \Leftrightarrow \quad xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \quad \Rightarrow$

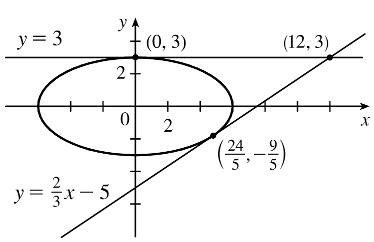
$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^4 + x^2 = 2 \quad \Leftrightarrow \quad x^4 + x^2 - 2 = 0 \quad \Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

76. $x^2 + 4y^2 = 36 \quad \Rightarrow \quad 2x + 8yy' = 0 \quad \Rightarrow \quad y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \quad \Rightarrow \quad a + b = 3 \quad \Rightarrow$ $b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \quad \Leftrightarrow \quad a^2 + 36 - 24a + 4a^2 = 36 \quad \Leftrightarrow$ $5a^2 - 24a = 0 \quad \Leftrightarrow \quad a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \quad \Leftrightarrow \quad y - 3 = \frac{2}{3}(x - 12) \quad \Leftrightarrow \quad y = \frac{2}{3}x - 5$.

A graph of the ellipse and the tangent lines confirms our results.



77. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\,\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad\Rightarrow\quad \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \quad\Rightarrow\quad f^{-1}(5) = 4$. By part (a), $\left(f^{-1}\right)'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = 1\big/\left(\frac{2}{3}\right) = \frac{3}{2}$.

78. (a) Assume $a < b$. Since e^x is an increasing function, $e^a < e^b$, and hence, $a + e^a < b + e^b$; that is, $f(a) < f(b)$. So $f(x) = x + e^x$ is an increasing function and therefore one-to-one.

(b) $f^{-1}(1) = a \quad\Leftrightarrow\quad f(a) = 1$, so we need to find a such that $f(a) = 1$. By inspection, we see that $f(0) = 0 + e^0 = 1$, so $a = 0$, and hence, $f^{-1}(1) = 0$.

(c) $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)}$ [by part (b)]. Now $f(x) = x + e^x \quad\Rightarrow\quad f'(x) = 1 + e^x$, so $f'(0) = 1 + e^0 = 2$. Thus, $(f^{-1})'(1) = \frac{1}{2}$.

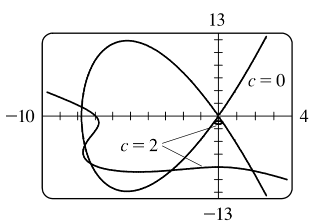
79. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \quad\Rightarrow\quad xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

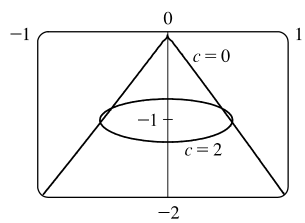
(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \quad\Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \quad\Rightarrow\quad 2J''(0) = -1 \quad\Rightarrow\quad J''(0) = -\frac{1}{2}$.

80. $x^2 + 4y^2 = 5 \quad\Rightarrow\quad 2x + 4(2yy') = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \quad\Leftrightarrow\quad 4b^2 = -a^2 - 5a \quad\Leftrightarrow\quad a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \quad\Leftrightarrow\quad a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \quad\Rightarrow\quad b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \quad\Rightarrow\quad h = 2$. So the lamp is located 2 units above the x-axis.

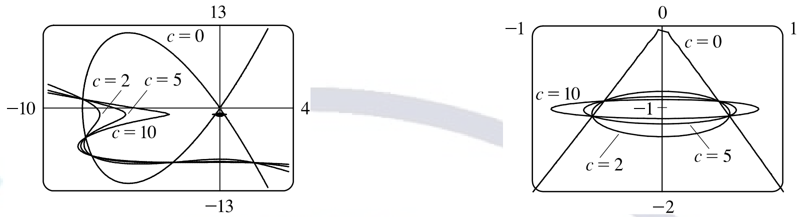
LABORATORY PROJECT Families of Implicit Curves

1. (a) There appear to be nine points of intersection. The "inner four" near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The "outer five" are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.





(b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c, that the curves change shape but the nine points of intersection are the same.

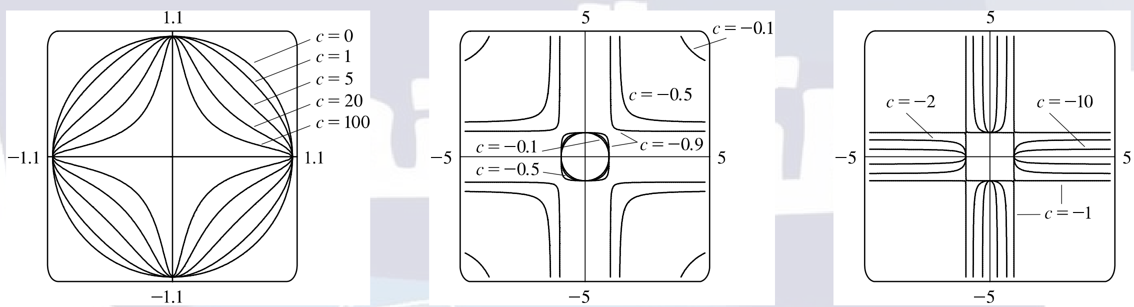


2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

For $-1 < c < 0$ (see the graph), there are four hyperboliclike branches as well as an ellipticlike curve bounded by $|x| \leq 1$ and $|y| \leq 1$ for values of c close to 0. As c gets closer to -1, the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.

$\boldsymbol{c \geq 0}$ $\boldsymbol{-1 < c < 0}$ $\boldsymbol{c \leq -1}$



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get $x^2 + y^2 - x^2y^2 = 1 \quad \Leftrightarrow$ $0 = x^2y^2 - x^2 - y^2 + 1 \quad \Leftrightarrow \quad 0 = (x^2 - 1)(y^2 - 1) \quad \Leftrightarrow \quad x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

(c) $\dfrac{d}{dx}(x^2 + y^2 + cx^2y^2) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 2x + 2y\,y' + c(x^2 \cdot 2y\,y' + y^2 \cdot 2x) = 0 \quad \Rightarrow$

$2y\,y' + 2cx^2y\,y' = -2x - 2cxy^2 \quad \Rightarrow \quad 2y(1 + cx^2)y' = -2x(1 + cy^2) \quad \Rightarrow \quad y' = -\dfrac{x(1 + cy^2)}{y(1 + cx^2)}$.

For $c = -1$, $y' = -\dfrac{x(1 - y^2)}{y(1 - x^2)} = -\dfrac{x(1 + y)(1 - y)}{y(1 + x)(1 - x)}$, so $y' = 0$ when $y = \pm 1$ or $x = 0$ (which leads to $y = \pm 1$) and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y'.

3.6 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = x\ln x - x \quad\Rightarrow\quad f'(x) = x\cdot\frac{1}{x} + (\ln x)\cdot 1 - 1 = 1 + \ln x - 1 = \ln x$

3. $f(x) = \sin(\ln x) \quad\Rightarrow\quad f'(x) = \cos(\ln x)\cdot\frac{d}{dx}\ln x = \cos(\ln x)\cdot\frac{1}{x} = \frac{\cos(\ln x)}{x}$

4. $f(x) = \ln(\sin^2 x) = \ln(\sin x)^2 = 2\ln|\sin x| \quad\Rightarrow\quad f'(x) = 2\cdot\frac{1}{\sin x}\cdot\cos x = 2\cot x$

5. $f(x) = \ln\frac{1}{x} \quad\Rightarrow\quad f'(x) = \frac{1}{1/x}\,\frac{d}{dx}\left(\frac{1}{x}\right) = x\left(-\frac{1}{x^2}\right) = -\frac{1}{x}.$

Another solution: $f(x) = \ln\frac{1}{x} = \ln 1 - \ln x = -\ln x \quad\Rightarrow\quad f'(x) = -\frac{1}{x}.$

6. $y = \frac{1}{\ln x} = (\ln x)^{-1} \quad\Rightarrow\quad y' = -1(\ln x)^{-2}\cdot\frac{1}{x} = \frac{-1}{x(\ln x)^2}$

7. $f(x) = \log_{10}(1+\cos x) \quad\Rightarrow\quad f'(x) = \frac{1}{(1+\cos x)\ln 10}\,\frac{d}{dx}(1+\cos x) = \frac{-\sin x}{(1+\cos x)\ln 10}$

8. $f(x) = \log_{10}\sqrt{x} \quad\Rightarrow\quad f'(x) = \frac{1}{\sqrt{x}\,\ln 10}\,\frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{x}\,\ln 10}\,\frac{1}{2\sqrt{x}} = \frac{1}{2(\ln 10)x}$

Or: $f(x) = \log_{10}\sqrt{x} = \log_{10} x^{1/2} = \frac{1}{2}\log_{10} x \quad\Rightarrow\quad f'(x) = \frac{1}{2}\,\frac{1}{x\ln 10} = \frac{1}{2\,(\ln 10)\,x}$

9. $g(x) = \ln(xe^{-2x}) = \ln x + \ln e^{-2x} = \ln x - 2x \quad\Rightarrow\quad g'(x) = \frac{1}{x} - 2$

10. $g(t) = \sqrt{1+\ln t} \quad\Rightarrow\quad g'(t) = \frac{1}{2}(1+\ln t)^{-1/2}\,\frac{d}{dt}(1+\ln t) = \frac{1}{2\sqrt{1+\ln t}}\cdot\frac{1}{t} = \frac{1}{2t\sqrt{1+\ln t}}$

11. $F(t) = (\ln t)^2\sin t \quad\Rightarrow\quad F'(t) = (\ln t)^2\cos t + \sin t\cdot 2\ln t\cdot\frac{1}{t} = \ln t\left(\ln t\,\cos t + \frac{2\sin t}{t}\right)$

12. $h(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad\Rightarrow\quad h'(x) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$

13. $G(y) = \ln\frac{(2y+1)^5}{\sqrt{y^2+1}} = \ln(2y+1)^5 - \ln(y^2+1)^{1/2} = 5\ln(2y+1) - \frac{1}{2}\ln(y^2+1) \quad\Rightarrow$

$G'(y) = 5\cdot\frac{1}{2y+1}\cdot 2 - \frac{1}{2}\cdot\frac{1}{y^2+1}\cdot 2y = \frac{10}{2y+1} - \frac{y}{y^2+1} \quad \left[\text{or } \frac{8y^2-y+10}{(2y+1)(y^2+1)}\right]$

14. $P(v) = \frac{\ln v}{1-v} \quad\Rightarrow\quad P'(v) = \frac{(1-v)(1/v) - (\ln v)(-1)}{(1-v)^2}\cdot\frac{v}{v} = \frac{1 - v + v\ln v}{v(1-v)^2}$

15. $F(s) = \ln\ln s \quad\Rightarrow\quad F'(s) = \frac{1}{\ln s}\,\frac{d}{ds}\ln s = \frac{1}{\ln s}\cdot\frac{1}{s} = \frac{1}{s\ln s}$

16. $y = \ln\left|1+t-t^3\right| \quad\Rightarrow\quad y' = \dfrac{1}{1+t-t^3}\,\dfrac{d}{dt}\,(1+t-t^3) = \dfrac{1-3t^2}{1+t-t^3}$

17. $T(z) = 2^z \log_2 z \quad\Rightarrow\quad T'(z) = 2^z\dfrac{1}{z\ln 2} + \log_2 z\cdot 2^z\ln 2 = 2^z\left(\dfrac{1}{z\ln 2} + \log_2 z\,(\ln 2)\right).$

Note that $\log_2 z\,(\ln 2) = \dfrac{\ln z}{\ln 2}\,(\ln 2) = \ln z$ by the change of base theorem. Thus, $T'(z) = 2^z\left(\dfrac{1}{z\ln 2} + \ln z\right).$

18. $y = \ln(\csc x - \cot x) \quad\Rightarrow$

$$y' = \frac{1}{\csc x-\cot x}\,\frac{d}{dx}\,(\csc x-\cot x) = \frac{1}{\csc x-\cot x}(-\csc x\,\cot x+\csc^2 x) = \frac{\csc x(\csc x-\cot x)}{\csc x-\cot x} = \csc x$$

19. $y = \ln(e^{-x}+xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x+\ln(1+x) \quad\Rightarrow$

$$y' = -1+\frac{1}{1+x} = \frac{-1-x+1}{1+x} = -\frac{x}{1+x}$$

20. $H(z) = \ln\sqrt{\dfrac{a^2-z^2}{a^2+z^2}} = \ln\left(\dfrac{a^2-z^2}{a^2+z^2}\right)^{1/2} = \dfrac{1}{2}\ln\left(\dfrac{a^2-z^2}{a^2+z^2}\right) = \frac{1}{2}\ln(a^2-z^2) - \frac{1}{2}\ln(a^2+z^2) \quad\Rightarrow$

$$H'(z) = \frac{1}{2}\cdot\frac{1}{a^2-z^2}\cdot(-2z) - \frac{1}{2}\cdot\frac{1}{a^2+z^2}\cdot(2z) = \frac{z}{z^2-a^2} - \frac{z}{z^2+a^2} = \frac{z(z^2+a^2)-z(z^2-a^2)}{(z^2-a^2)(z^2+a^2)}$$

$$= \frac{z^3+za^2-z^3+za^2}{(z^2-a^2)(z^2+a^2)} = \frac{2a^2z}{z^4-a^4}$$

21. $y = \tan\left[\ln(ax+b)\right] \quad\Rightarrow\quad y' = \sec^2[\ln(ax+b)]\cdot\dfrac{1}{ax+b}\cdot a = \sec^2[\ln(ax+b)]\,\dfrac{a}{ax+b}$

22. $y = \log_2(x\log_5 x) \quad\Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)}\,\frac{d}{dx}\,(x\log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x\cdot\frac{1}{x\ln 5} + \log_5 x\right) = \frac{1}{(x\log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \dfrac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base theorem. Thus, $y' = \dfrac{1}{x\ln x\ln 2} + \dfrac{1}{x\ln 2} = \dfrac{1+\ln x}{x\ln x\ln 2}.$

23. $y = \sqrt{x}\,\ln x \quad\Rightarrow\quad y' = \sqrt{x}\cdot\dfrac{1}{x} + (\ln x)\,\dfrac{1}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}} \quad\Rightarrow$

$$y'' = \frac{2\sqrt{x}\,(1/x) - (2+\ln x)(1/\sqrt{x}\,)}{(2\sqrt{x}\,)^2} = \frac{2/\sqrt{x} - (2+\ln x)(1/\sqrt{x}\,)}{4x} = \frac{2-(2+\ln x)}{\sqrt{x}(4x)} = -\frac{\ln x}{4x\sqrt{x}}$$

24. $y = \dfrac{\ln x}{1+\ln x} \quad\Rightarrow\quad y' = \dfrac{(1+\ln x)(1/x) - (\ln x)(1/x)}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2} \quad\Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)^2]}{[x(1+\ln x)^2]^2}\quad\text{[Reciprocal Rule]}\quad = -\frac{x\cdot 2(1+\ln x)\cdot(1/x) + (1+\ln x)^2}{x^2(1+\ln x)^4}$$

$$= -\frac{(1+\ln x)[2+(1+\ln x)]}{x^2(1+\ln x)^4} = -\frac{3+\ln x}{x^2(1+\ln x)^3}$$

25. $y = \ln|\sec x| \quad\Rightarrow\quad y' = \dfrac{1}{\sec x}\,\dfrac{d}{dx}\sec x = \dfrac{1}{\sec x}\sec x\,\tan x = \tan x \quad\Rightarrow\quad y'' = \sec^2 x$

26. $y = \ln(1 + \ln x) \quad \Rightarrow \quad y' = \dfrac{1}{1 + \ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x(1 + \ln x)} \quad \Rightarrow$

$$y'' = -\frac{\frac{d}{dx}[x(1+\ln x)]}{[x(1+\ln x)]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x(1/x) + (1+\ln x)(1)}{x^2(1+\ln x)^2} = -\frac{1+1+\ln x}{x^2(1+\ln x)^2} = -\frac{2+\ln x}{x^2(1+\ln x)^2}$$

27. $f(x) = \dfrac{x}{1 - \ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)] \cdot 1 - x \cdot \dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)] + x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x - 1 - (x-1)\ln(x-1) + x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x - 1 > 0 \ \text{ and } \ 1 - \ln(x-1) \neq 0\} = \{x \mid x > 1 \ \text{ and } \ \ln(x-1) \neq 1\}$

$= \{x \mid x > 1 \ \text{ and } \ x - 1 \neq e^1\} = \{x \mid x > 1 \ \text{ and } \ x \neq 1 + e\} = (1, 1+e) \cup (1+e, \infty)$

28. $f(x) = \sqrt{2 + \ln x} = (2 + \ln x)^{1/2} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2}(2 + \ln x)^{-1/2} \cdot \dfrac{1}{x} = \dfrac{1}{2x\sqrt{2 + \ln x}}$

$\text{Dom}(f) = \{x \mid 2 + \ln x \geq 0\} = \{x \mid \ln x \geq -2\} = \{x \mid x \geq e^{-2}\} = [e^{-2}, \infty)$.

29. $f(x) = \ln(x^2 - 2x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 - 2x}(2x - 2) = \dfrac{2(x-1)}{x(x-2)}$.

$\text{Dom}(f) = \{x \mid x(x-2) > 0\} = (-\infty, 0) \cup (2, \infty)$.

30. $f(x) = \ln \ln \ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln \ln x} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

31. $f(x) = \ln(x + \ln x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x + \ln x} \dfrac{d}{dx}(x + \ln x) = \dfrac{1}{x + \ln x}\left(1 + \dfrac{1}{x}\right)$.

Substitute 1 for x to get $f'(1) = \dfrac{1}{1 + \ln 1}\left(1 + \dfrac{1}{1}\right) = \dfrac{1}{1+0}(1+1) = 1 \cdot 2 = 2$.

32. $f(x) = \cos(\ln x^2) \quad \Rightarrow \quad f'(x) = -\sin(\ln x^2) \dfrac{d}{dx} \ln x^2 = -\sin(\ln x^2) \dfrac{1}{x^2}(2x) = -\dfrac{2\sin(\ln x^2)}{x}$.

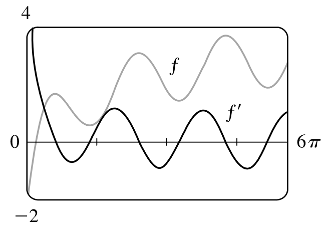
Substitute 1 for x to get $f'(1) = -\dfrac{2\sin(\ln 1^2)}{1} = -2\sin 0 = 0$.

33. $y = \ln(x^2 - 3x + 1) \quad \Rightarrow \quad y' = \dfrac{1}{x^2 - 3x + 1} \cdot (2x - 3) \quad \Rightarrow \quad y'(3) = \frac{1}{1} \cdot 3 = 3$, so an equation of a tangent line at $(3, 0)$ is $y - 0 = 3(x - 3)$, or $y = 3x - 9$.

34. $y = x^2 \ln x \quad \Rightarrow \quad y' = x^2 \cdot \dfrac{1}{x} + (\ln x)(2x) \quad \Rightarrow \quad y'(1) = 1 + 0 = 1$, so an equation of a tangent line at $(1, 0)$ is $y - 0 = 1(x - 1)$, or $y = x - 1$.

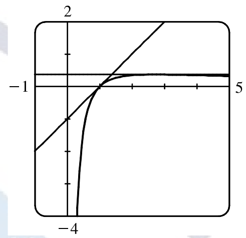
35. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



36. $y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

$y'(1) = \dfrac{1-0}{1^2} = 1$ and $y'(e) = \dfrac{1-1}{e^2} = 0 \quad \Rightarrow \quad$ equations of tangent lines are $y - 0 = 1(x-1)$ or $y = x - 1$ and $y - 1/e = 0(x - e)$ or $y = 1/e$.



37. $f(x) = cx + \ln(\cos x) \quad \Rightarrow \quad f'(x) = c + \dfrac{1}{\cos x} \cdot (-\sin x) = c - \tan x$.

$f'(\frac{\pi}{4}) = 6 \quad \Rightarrow \quad c - \tan\frac{\pi}{4} = 6 \quad \Rightarrow \quad c - 1 = 6 \quad \Rightarrow \quad c = 7$.

38. $f(x) = \log_a(3x^2 - 2) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(3x^2-2)\ln a} \cdot 6x$.

$f'(1) = 3 \quad \Rightarrow \quad \dfrac{1}{\ln a} \cdot 6 = 3 \quad \Rightarrow \quad 2 = \ln a \quad \Rightarrow \quad a = e^2$.

39. $y = (x^2+2)^2(x^4+4)^4 \quad \Rightarrow \quad \ln y = \ln[(x^2+2)^2(x^4+4)^4] \quad \Rightarrow \quad \ln y = 2\ln(x^2+2) + 4\ln(x^4+4) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = 2 \cdot \dfrac{1}{x^2+2} \cdot 2x + 4 \cdot \dfrac{1}{x^4+4} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right) \quad \Rightarrow$

$y' = (x^2+2)^2(x^4+4)^4\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right)$

40. $y = \dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow \quad \ln y = \ln \dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow$

$\ln y = \ln e^{-x} + \ln|\cos x|^2 - \ln(x^2+x+1) = -x + 2\ln|\cos x| - \ln(x^2+x+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = -1 + 2 \cdot \dfrac{1}{\cos x}(-\sin x) - \dfrac{1}{x^2+x+1}(2x+1) \quad \Rightarrow \quad y' = y\left(-1 - 2\tan x - \dfrac{2x+1}{x^2+x+1}\right) \quad \Rightarrow$

$y' = -\dfrac{e^{-x}\cos^2 x}{x^2+x+1}\left(1 + 2\tan x + \dfrac{2x+1}{x^2+x+1}\right)$

41. $y = \sqrt{\dfrac{x-1}{x^4+1}} \quad \Rightarrow \quad \ln y = \ln\left(\dfrac{x-1}{x^4+1}\right)^{1/2} \quad \Rightarrow \quad \ln y = \dfrac{1}{2}\ln(x-1) - \dfrac{1}{2}\ln(x^4+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{2}\dfrac{1}{x-1} - \dfrac{1}{2}\dfrac{1}{x^4+1} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{1}{2(x-1)} - \dfrac{2x^3}{x^4+1}\right) \quad \Rightarrow \quad y' = \sqrt{\dfrac{x-1}{x^4+1}}\left(\dfrac{1}{2x-2} - \dfrac{2x^3}{x^4+1}\right)$

42. $y = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3} \quad \Rightarrow \quad \ln y = \ln\left[x^{1/2}e^{x^2-x}(x+1)^{2/3}\right] \quad \Rightarrow$

$\ln y = \frac{1}{2}\ln x + (x^2 - x) + \frac{2}{3}\ln(x+1) \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \dfrac{1}{2} \cdot \dfrac{1}{x} + 2x - 1 + \dfrac{2}{3} \cdot \dfrac{1}{x+1} \quad \Rightarrow$

$y' = y\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right) \quad \Rightarrow \quad y' = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right)$

43. $y = x^x \quad\Rightarrow\quad \ln y = \ln x^x \quad\Rightarrow\quad \ln y = x\ln x \quad\Rightarrow\quad y'/y = x(1/x) + (\ln x)\cdot 1 \quad\Rightarrow\quad y' = y(1+\ln x) \quad\Rightarrow$
$y' = x^x(1+\ln x)$

44. $y = x^{\cos x} \quad\Rightarrow\quad \ln y = \ln x^{\cos x} \quad\Rightarrow\quad \ln y = \cos x \ln x \quad\Rightarrow\quad \dfrac{1}{y}\,y' = \cos x \cdot \dfrac{1}{x} + \ln x \cdot (-\sin x) \quad\Rightarrow$

$y' = y\left(\dfrac{\cos x}{x} - \ln x \sin x\right) \quad\Rightarrow\quad y' = x^{\cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

45. $y = x^{\sin x} \quad\Rightarrow\quad \ln y = \ln x^{\sin x} \quad\Rightarrow\quad \ln y = \sin x \ln x \quad\Rightarrow\quad \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \quad\Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x \cos x\right) \quad\Rightarrow\quad y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\,\cos x\right)$

46. $y = \sqrt{x}^{\,x} \quad\Rightarrow\quad \ln y = \ln \sqrt{x}^{\,x} \quad\Rightarrow\quad \ln y = x\ln x^{1/2} \quad\Rightarrow\quad \ln y = \tfrac{1}{2}x\ln x \quad\Rightarrow\quad \dfrac{1}{y}\,y' = \dfrac{1}{2}x\cdot\dfrac{1}{x} + \ln x \cdot \dfrac{1}{2} \quad\Rightarrow$

$y' = y\left(\tfrac{1}{2} + \tfrac{1}{2}\ln x\right) \quad\Rightarrow\quad y' = \tfrac{1}{2}\sqrt{x}^{\,x}(1+\ln x)$

47. $y = (\cos x)^x \quad\Rightarrow\quad \ln y = \ln(\cos x)^x \quad\Rightarrow\quad \ln y = x\ln\cos x \quad\Rightarrow\quad \dfrac{1}{y}\,y' = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x \cdot 1 \quad\Rightarrow$

$y' = y\left(\ln\cos x - \dfrac{x\sin x}{\cos x}\right) \quad\Rightarrow\quad y' = (\cos x)^x(\ln\cos x - x\tan x)$

48. $y = (\sin x)^{\ln x} \quad\Rightarrow\quad \ln y = \ln(\sin x)^{\ln x} \quad\Rightarrow\quad \ln y = \ln x \cdot \ln\sin x \quad\Rightarrow\quad \dfrac{1}{y}\,y' = \ln x \cdot \dfrac{1}{\sin x}\cdot\cos x + \ln\sin x \cdot \dfrac{1}{x} \quad\Rightarrow$

$y' = y\left(\ln x \cdot \dfrac{\cos x}{\sin x} + \dfrac{\ln\sin x}{x}\right) \quad\Rightarrow\quad y' = (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln\sin x}{x}\right)$

49. $y = (\tan x)^{1/x} \quad\Rightarrow\quad \ln y = \ln(\tan x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln\tan x \quad\Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{x}\cdot\dfrac{1}{\tan x}\cdot\sec^2 x + \ln\tan x\cdot\left(-\dfrac{1}{x^2}\right) \quad\Rightarrow\quad y' = y\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right) \quad\Rightarrow$

$y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right)$ or $y' = (\tan x)^{1/x}\cdot\dfrac{1}{x}\left(\csc x\sec x - \dfrac{\ln\tan x}{x}\right)$

50. $y = (\ln x)^{\cos x} \quad\Rightarrow\quad \ln y = \cos x\ln(\ln x) \quad\Rightarrow\quad \dfrac{y'}{y} = \cos x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)(-\sin x) \quad\Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x\ln x} - \sin x\ln\ln x\right)$

51. $y = \ln(x^2+y^2) \quad\Rightarrow\quad y' = \dfrac{1}{x^2+y^2}\,\dfrac{d}{dx}(x^2+y^2) \quad\Rightarrow\quad y' = \dfrac{2x+2yy'}{x^2+y^2} \quad\Rightarrow\quad x^2y' + y^2y' = 2x + 2yy' \quad\Rightarrow$

$x^2y' + y^2y' - 2yy' = 2x \quad\Rightarrow\quad (x^2+y^2-2y)y' = 2x \quad\Rightarrow\quad y' = \dfrac{2x}{x^2+y^2-2y}$

52. $x^y = y^x \quad\Rightarrow\quad y\ln x = x\ln y \quad\Rightarrow\quad y\cdot\dfrac{1}{x} + (\ln x)\cdot y' = x\cdot\dfrac{1}{y}\cdot y' + \ln y \quad\Rightarrow\quad y'\ln x - \dfrac{x}{y}\,y' = \ln y - \dfrac{y}{x} \quad\Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

53. $f(x) = \ln(x-1) \quad\Rightarrow\quad f'(x) = \dfrac{1}{(x-1)} = (x-1)^{-1} \quad\Rightarrow\quad f''(x) = -(x-1)^{-2} \quad\Rightarrow\quad f'''(x) = 2(x-1)^{-3} \quad\Rightarrow$

$$f^{(4)}(x) = -2\cdot 3(x-1)^{-4} \quad\Rightarrow\quad \cdots \quad\Rightarrow\quad f^{(n)}(x) = (-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot(n-1)(x-1)^{-n} = (-1)^{n-1}\frac{(n-1)!}{(x-1)^n}$$

54. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

$$D^8(8x^7 \ln x) = D^7(8\cdot 7x^6 \ln x + 8x^6) = D^7(8\cdot 7x^6 \ln x) = D^6(8\cdot 7\cdot 6x^5 \ln x) = \cdots = D(8!\,x^0 \ln x) = 8!/x.$$

55. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1.$

56. Let $m = n/x$. Then $n = xm$, and as $n\to\infty$, $m\to\infty$.

Therefore, $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^x = e^x$ by Equation 6.

3.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 8t^2 + 24t$ (in feet) $\quad\Rightarrow\quad v(t) = f'(t) = 3t^2 - 16t + 24$ (in ft/s)

(b) $v(1) = 3(1)^2 - 16(1) + 24 = 11$ ft/s

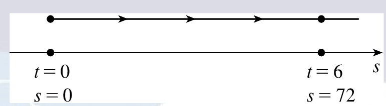
(c) The particle is at rest when $v(t) = 0$. $3t^2 - 16t + 24 = 0 \quad\Rightarrow\quad \dfrac{-(-16)\pm\sqrt{(-16)^2 - 4(3)(24)}}{2(3)} = \dfrac{16\pm\sqrt{-32}}{6}$.

The negative discriminant indicates that v is never 0 and that the particle never rests.

(d) From parts (b) and (c), we see that $v(t) > 0$ for all t, so the particle is always moving in the positive direction.

(e) The total distance traveled during the first 6 seconds (since the particle doesn't change direction) is $f(6) - f(0) = 72 - 0 = 72$ ft.

(f)

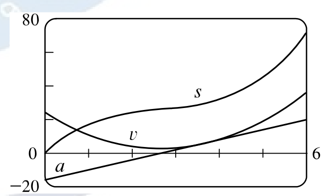


(g) $v(t) = 3t^2 - 16t + 24 \quad\Rightarrow$

$a(t) = v'(t) = 6t - 16$ (in (ft/s)/s or ft/s^2).

$a(1) = 6(1) - 16 = -10$ ft/s^2

(h)



(i) The particle is speeding up when v and a have the same sign. v is always positive and a is positive when $6t - 16 > 0 \quad\Rightarrow$ $t > \frac{8}{3}$, so the particle is speeding up when $t > \frac{8}{3}$. It is slowing down when v and a have opposite signs; that is, when $0 \le t < \frac{8}{3}$.

2. (a) $s = f(t) = \dfrac{9t}{t^2+9}$ (in feet) $\quad\Rightarrow\quad v(t) = f'(t) = \dfrac{(t^2+9)(9) - 9t(2t)}{(t^2+9)^2} = \dfrac{-9t^2+81}{(t^2+9)^2} = \dfrac{-9(t^2-9)}{(t^2+9)^2}$ (in ft/s)

(b) $v(1) = \dfrac{-9(1-9)}{(1+9)^2} = \dfrac{72}{100} = 0.72$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\dfrac{-9(t^2-9)}{(t^2+9)^2} = 0 \quad \Leftrightarrow \quad t^2 - 9 = 0 \quad \Rightarrow \quad t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$$\frac{-9(t^2-9)}{(t^2+9)^2} > 0 \quad \Rightarrow \quad -9(t^2-9) > 0 \quad \Rightarrow \quad t^2 - 9 < 0 \quad \Rightarrow \quad t^2 < 9 \quad \Rightarrow \quad 0 \leq t < 3.$$

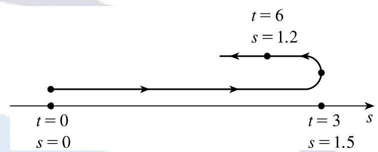
(e) Since the particle is moving in the positve direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0,3]$ and $[3,6]$, respectively.

$|f(3) - f(0)| = \left|\frac{27}{18} - 0\right| = \frac{3}{2}$

$|f(6) - f(3)| = \left|\frac{54}{45} - \frac{27}{18}\right| = \frac{3}{10}$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 ft.

(f)

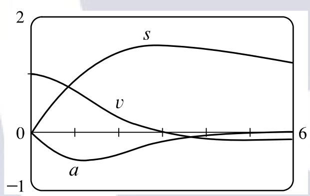


(g) $v(t) = -9\,\dfrac{t^2-9}{(t^2+9)^2} \quad \Rightarrow$

$$a(t) = v'(t) = -9\,\frac{(t^2+9)^2(2t) - (t^2-9)2(t^2+9)(2t)}{[(t^2+9)^2]^2} = -9\,\frac{2t(t^2+9)[(t^2+9) - 2(t^2-9)]}{(t^2+9)^4} = \frac{18t(t^2-27)}{(t^2+9)^3}.$$

$a(1) = \dfrac{18(-26)}{10^3} = -0.468 \text{ ft/s}^2$

(h)



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27}$ [≈ 5.2], so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2}\cos(\pi t/2)$ (in ft/s)

(b) $v(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ ft/s

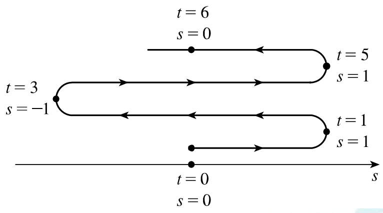
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2}\cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \quad \Leftrightarrow \quad t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3, and 5 in the interval $[0,6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned} |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| &= |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1| \\ &= 1 + 2 + 2 + 1 = 6 \text{ ft} \end{aligned}$$

(f)

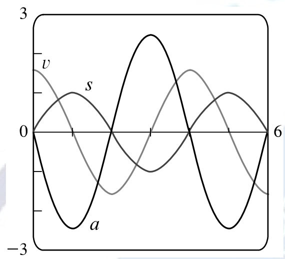


(g) $v(t) = \frac{\pi}{2}\cos(\pi t/2) \quad \Rightarrow$

$$a(t) = v'(t) = \tfrac{\pi}{2}\left[-\sin(\pi t/2)\cdot(\pi/2)\right]$$
$$= (-\pi^2/4)\sin(\pi t/2)\ \text{ft/s}^2$$

$$a(1) = (-\pi^2/4)\sin(\pi/2) = -\pi^2/4\ \text{ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = t^2e^{-t}$ (in feet) $\quad\Rightarrow\quad v(t) = f'(t) = t^2(-e^{-t}) + e^{-t}(2t) = te^{-t}(-t+2)$ (in ft/s)

(b) $v(1) = (1)e^{-1}(-1+2) = 1/e$ ft/s

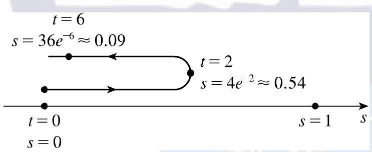
(c) The particle is at rest when $v(t) = 0$. $v(t) = 0 \quad\Leftrightarrow\quad t = 0$ or 2 s.

(d) The particle is moving in the positive direction when $v(t) > 0 \quad\Leftrightarrow\quad te^{-t}(-t+2) > 0 \quad\Leftrightarrow\quad t(-t+2) > 0 \quad\Leftrightarrow$ $0 < t < 2$.

(e) v changes sign at $t = 2$ in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$|f(2) - f(0)| + |f(6) - f(2)| = \left|4e^{-2} - 0\right| + \left|36e^{-6} - 4e^{-2}\right| = 4e^{-2} + 4e^{-2} - 36e^{-6}$$
$$= 8e^{-2} - 36e^{-6} \approx 0.99\ \text{ft}$$

(f)

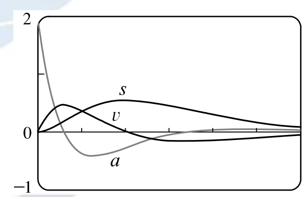


(g) $v(t) = (2t - t^2)e^{-t} \quad\Rightarrow$

$$a(t) = v'(t) = (2t - t^2)(-e^{-t}) + e^{-t}(2 - 2t)$$
$$= e^{-t}\left[-(2t - t^2) + (2 - 2t)\right]$$
$$= e^{-t}(t^2 - 4t + 2)\ \text{ft/s}^2$$

$$a(1) = e^{-1}(1 - 4 + 2) = -1/e\ \text{ft/s}^2$$

(h)



(i) $a(t) = 0 \quad\Leftrightarrow\quad t^2 - 4t + 2 = 0 \quad [e^{-t} \neq 0] \quad\Leftrightarrow\quad t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \quad [\approx 0.6$ and $3.4]$. The particle is speeding up when v and a have the same sign. Using the previous information and the figure in part (h), we see that v and a are both positive when $0 < t < 2 - \sqrt{2}$ and both negative when $2 < t < 2 + \sqrt{2}$. The particle is slowing down when v and a have opposite signs. This occurs when $2 - \sqrt{2} < t < 2$ and $t > 2 + \sqrt{2}$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ $[v > 0, a > 0]$ and on $(3, 4)$ $[v < 0, a < 0]$. The particle is slowing down on $(0, 1)$ and $(2, 3)$ $[v > 0, a < 0]$.

6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0, 1)$ and $(3, 4)$; and it is negative when s is decreasing, that is, on the interval $(1, 3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2, 4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v > 0, a < 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

(b) The velocity v is positive on $(3, 4)$ and negative on $(0, 3)$. The acceleration a is positive on $(0, 1)$ and $(2, 4)$ and negative on $(1, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v < 0, a > 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

7. (a) $h(t) = 2 + 24.5t - 4.9t^2 \quad \Rightarrow \quad v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $v(t) = 0 \quad \Leftrightarrow \quad 24.5 - 9.8t = 0 \quad \Leftrightarrow$ $t = \dfrac{24.5}{9.8} = 2.5$ s.

(c) The maximum height occurs when $t = 2.5$. $h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625$ m $\left[\text{or } 32\frac{5}{8} \text{ m}\right]$.

(d) The projectile hits the ground when $h = 0 \quad \Leftrightarrow \quad 2 + 24.5t - 4.9t^2 = 0 \quad \Leftrightarrow$

$$t = \frac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \quad \Rightarrow \quad t = t_f \approx 5.08 \text{ s} \;\; [\text{since } t \geq 0].$$

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3$ m/s [downward].

8. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \quad \Leftrightarrow \quad 32t = 80 \quad \Leftrightarrow \quad t = \frac{5}{2}$.
So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \quad \Leftrightarrow \quad 16t^2 - 80t + 96 = 0 \quad \Leftrightarrow \quad 16(t^2 - 5t + 6) = 0 \quad \Leftrightarrow \quad 16(t - 3)(t - 2) = 0$.
So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

9. (a) $h(t) = 15t - 1.86t^2 \quad \Rightarrow \quad v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56$ m/s.

(b) $25 = h \quad \Leftrightarrow \quad 1.86t^2 - 15t + 25 = 0 \quad \Leftrightarrow \quad t = \dfrac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \quad \Leftrightarrow \quad t = t_1 \approx 2.35$ or $t = t_2 \approx 5.71$.

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24$ m/s [upward] and $v(t_2) = 15 - 3.72t_2 \approx -6.24$ m/s [downward].

10. (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \quad\Rightarrow\quad v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20. \quad v = 20 \quad\Leftrightarrow$

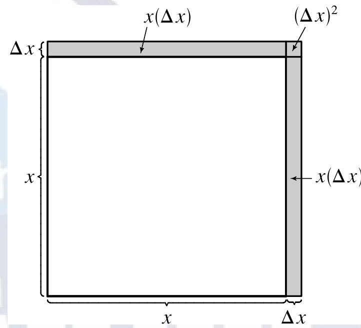
$4t^3 - 12t^2 - 40t + 20 = 20 \quad\Leftrightarrow\quad 4t^3 - 12t^2 - 40t = 0 \quad\Leftrightarrow\quad 4t(t^2 - 3t - 10) = 0 \quad\Leftrightarrow$

$4t(t-5)(t+2) = 0 \quad\Leftrightarrow\quad t = 0$ s or 5 s [for $t \geq 0$].

(b) $a(t) = v'(t) = 12t^2 - 24t - 40. \quad a = 0 \quad\Leftrightarrow\quad 12t^2 - 24t - 40 = 0 \quad\Leftrightarrow\quad 4(3t^2 - 6t - 10) = 0 \quad\Leftrightarrow$

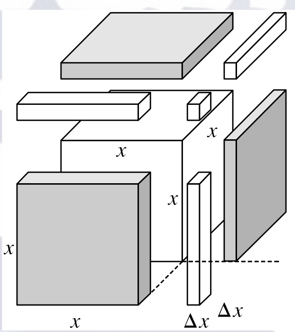
$t = \dfrac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \dfrac{1}{3}\sqrt{39} \approx 3.08$ s [for $t \geq 0$]. At this time, the acceleration changes from negative to positive and the velocity attains its minimum value.

11. (a) $A(x) = x^2 \quad\Rightarrow\quad A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \quad\Rightarrow\quad \dfrac{dV}{dx} = 3x^2. \quad \left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

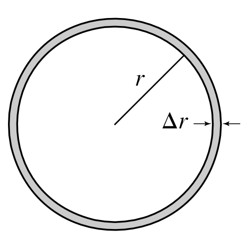
13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad\Rightarrow\quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \quad \Rightarrow \quad A'(t) = 7200\pi t \quad \Rightarrow$

(a) $A'(1) = 7200\pi \text{ cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi \text{ cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi \text{ cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \quad \Rightarrow \quad S'(r) = 8\pi r \quad \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \quad \Rightarrow \quad V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75 \text{ gal/min}$ (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5 \text{ gal/min}$

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125 \text{ gal/min}$ (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0 \text{ gal/min}$

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250 \text{ gal/min}$. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad \Rightarrow \quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \quad \Rightarrow \quad GmM = 20{,}000^3$.

$$F'(10,000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

21. With $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$,

$$F = \frac{d}{dt}(mv) = m\,\frac{d}{dt}(v) + v\,\frac{d}{dt}(m) = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot a + v \cdot m_0\left[-\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\right]\left(-\frac{2v}{c^2}\right)\frac{d}{dt}(v)$$

$$= m_0\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot a\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\dfrac{d}{dt}(v) = a$.

22. (a) $D(t) = 7 + 5\cos[0.503(t - 6.75)] \quad \Rightarrow \quad D'(t) = -5\sin[0.503(t - 6.75)](0.503) = -2.515\sin[0.503(t - 6.75)]$.

At 3:00 AM, $t = 3$, and $D'(3) = -2.515\sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515\sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515\sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515\sin[0.503(5.25)] \approx -1.21$ m/h (falling).

23. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad \Rightarrow \quad V = \frac{C}{P} \quad \Rightarrow \quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

24. (a) $[\text{C}] = \dfrac{a^2kt}{akt + 1} \quad \Rightarrow \quad \text{rate of reaction} = \dfrac{d[\text{C}]}{dt} = \dfrac{(akt + 1)(a^2k) - (a^2kt)(ak)}{(akt + 1)^2} = \dfrac{a^2k(akt + 1 - akt)}{(akt + 1)^2} = \dfrac{a^2k}{(akt + 1)^2}$

(b) If $x = [\text{C}]$, then $a - x = a - \dfrac{a^2kt}{akt + 1} = \dfrac{a^2kt + a - a^2kt}{akt + 1} = \dfrac{a}{akt + 1}$.

So $k(a - x)^2 = k\left(\dfrac{a}{akt + 1}\right)^2 = \dfrac{a^2k}{(akt + 1)^2} = \dfrac{d[\text{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

(c) As $t \to \infty$, $[\text{C}] = \dfrac{a^2kt}{akt + 1} = \dfrac{(a^2kt)/t}{(akt + 1)/t} = \dfrac{a^2k}{ak + (1/t)} \to \dfrac{a^2k}{ak} = a$ moles/L.

(d) As $t \to \infty$, $\dfrac{d[\text{C}]}{dt} = \dfrac{a^2k}{(akt + 1)^2} \to 0$.

(e) As t increases, nearly all of the reactants A and B are converted into product C. In practical terms, the reaction virtually stops.

25. In Example 6, the population function was $n = 2^t\, n_0$. Since we are tripling instead of doubling and the initial population is 400, the population function is $n(t) = 400 \cdot 3^t$. The rate of growth is $n'(t) = 400 \cdot 3^t \cdot \ln 3$, so the rate of growth after 2.5 hours is $n'(2.5) = 400 \cdot 3^{2.5} \cdot \ln 3 \approx 6850$ bacteria/hour.

26. $n = f(t) = \dfrac{a}{1 + be^{-0.7t}} \quad\Rightarrow\quad n' = -\dfrac{a \cdot be^{-0.7t}(-0.7)}{(1 + be^{-0.7t})^2}$ [Reciprocal Rule]. When $t = 0$, $n = 20$ and $n' = 12$.

$f(0) = 20 \quad\Rightarrow\quad 20 = \dfrac{a}{1+b} \quad\Rightarrow\quad a = 20(1+b).\quad f'(0) = 12 \quad\Rightarrow\quad 12 = \dfrac{0.7ab}{(1+b)^2} \quad\Rightarrow\quad 12 = \dfrac{0.7(20)(1+b)b}{(1+b)^2} \quad\Rightarrow$

$\dfrac{12}{14} = \dfrac{b}{1+b} \quad\Rightarrow\quad 6(1+b) = 7b \quad\Rightarrow\quad 6 + 6b = 7b \quad\Rightarrow\quad b = 6$ and $a = 20(1+6) = 140$. For the long run, we let t increase without bound. $\displaystyle\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{140}{1 + 6e^{-0.7t}} = \frac{140}{1 + 6 \cdot 0} = 140$, indicating that the yeast population stabilizes at 140 cells.

27. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \dfrac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \dfrac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \dfrac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \dfrac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.000\,284\,900\,3$, $b \approx 0.522\,433\,122\,43$, $c \approx -6.395\,641\,396$, and $d \approx 1720.586\,081$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad\Rightarrow\quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and $P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

(e) $f(t) = pq^t$ (where $p = 1.43653 \times 10^9$ and $q = 1.01395$) $\quad\Rightarrow\quad f'(t) = pq^t \ln q$ (in millions of people per year)

(f) $f'(20) \approx 26.25$ million/year [much larger than the estimates in part (a) and (d)].

$f'(80) \approx 60.28$ million/year [much smaller than the estimates in parts (a) and (d)].

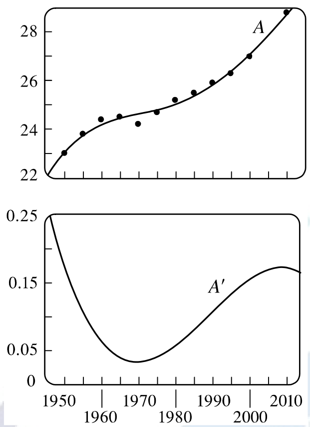
(g) $P'(85) \approx 76.24$ million/year and $f'(85) \approx 64.61$ million/year. The first estimate is probably more accurate.

28. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a \approx -1.199\,781 \times 10^{-6}$, $b \approx 9.545\,853 \times 10^3$, $c \approx -28.478\,550$, $d \approx 37{,}757.105\,467$, and $e \approx -1.877\,031 \times 10^7$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad\Rightarrow\quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$.

(c) Part (b) gives $A'(1990) \approx 0.106$ years of age per year.

(d)



29. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$$v(r) = \frac{3000}{4(0.027)3}(0.01^2 - r^2).\;\; v(0) = 0.\overline{925}\text{ cm/s},\; v(0.005) = 0.69\overline{4}\text{ cm/s},\; v(0.01) = 0.$$

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \;\;\Rightarrow\;\; v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$$v'(r) = -\frac{3000r}{2(0.027)3}.\;\; v'(0) = 0,\; v'(0.005) = -92.\overline{592}\text{ (cm/s)/cm, and } v'(0.01) = -185.\overline{185}\text{ (cm/s)/cm.}$$

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

30. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \;\;\Rightarrow\;\; \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \;\;\Rightarrow\;\; \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \;\;\Rightarrow\;\; \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

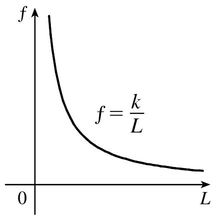
(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\;\Rightarrow\;$ f is increasing $\;\Rightarrow\;$ higher note

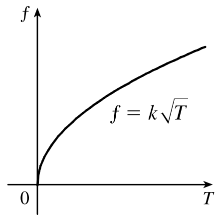
(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\;\Rightarrow\;$ f is increasing $\;\Rightarrow\;$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\;\Rightarrow\;$ f is decreasing $\;\Rightarrow\;$ lower note

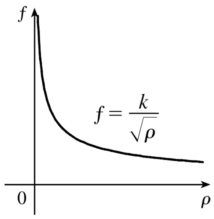
(i)



(ii)



(iii)



31. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \quad \Rightarrow \quad C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is
$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07$. This is close to the marginal cost from part (b).

32. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \quad \Rightarrow \quad C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and
$C'(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

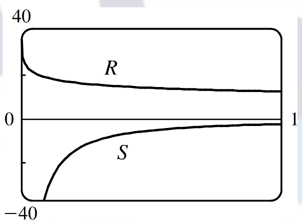
(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$

33. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x)\cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.
$A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\quad \Rightarrow \quad p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow \quad xp'(x) > p(x) \quad \Rightarrow$
$xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0.$

34. (a) $$S = \frac{dR}{dx} = \frac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$$
$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

35. $t = \ln\left(\dfrac{3c + \sqrt{9c^2 - 8c}}{2}\right) = \ln\left(3c + \sqrt{9c^2 - 8c}\right) - \ln 2 \quad \Rightarrow$

$$\frac{dt}{dc} = \frac{1}{3c + \sqrt{9c^2 - 8c}}\,\frac{d}{dc}\left(3c + \sqrt{9c^2 - 8c}\right) - 0 = \frac{3 + \frac{1}{2}(9c^2 - 8c)^{-1/2}(18c - 8)}{3c + \sqrt{9c^2 - 8c}}$$
$$= \frac{3 + \dfrac{9c - 4}{\sqrt{9c^2 - 8c}}}{3c + \sqrt{9c^2 - 8c}} = \frac{3\sqrt{9c^2 - 8c} + 9c - 4}{\sqrt{9c^2 - 8c}\left(3c + \sqrt{9c^2 - 8c}\right)}.$$

This derivative represents the rate of change of duration of dialysis required with respect to the initial urea concentration.

36. $f(r) = 2\sqrt{Dr} \quad \Rightarrow \quad f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \dfrac{D}{\sqrt{Dr}} = \sqrt{\dfrac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

37. $PV = nRT \quad \Rightarrow \quad T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min.}$$

38. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \quad \Rightarrow \quad \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \quad \Rightarrow \quad \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \quad \Rightarrow \quad \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \quad \Rightarrow \quad P = P_c\left(1 - \dfrac{\beta}{r_0}\right)$.

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

39. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \quad \Rightarrow \quad C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \quad \Leftrightarrow \quad 0.5W - 0.01W^2 = 0 \quad \Leftrightarrow \quad 50W - W^2 = 0 \quad \Leftrightarrow$ $W(50 - W) = 0 \quad \Leftrightarrow \quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.8 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.7944$, so $\dfrac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$.

Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. In 20 minutes $\left(\frac{1}{3}\text{ hour}\right)$, there are 100 cells, so $P\left(\frac{1}{3}\right) = 50e^{k/3} = 100 \quad \Rightarrow$ $e^{k/3} = 2 \quad \Rightarrow \quad k/3 = \ln 2 \quad \Rightarrow \quad k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 50e^{(\ln 8)t} = 50 \cdot 8^t$

(c) $P(6) = 50 \cdot 8^6 = 50 \cdot 2^{18} = 13{,}107{,}200$ cells

(d) $\dfrac{dP}{dt} = kP \quad \Rightarrow \quad P'(6) = kP(6) = (\ln 8)P(6) \approx 27{,}255{,}656$ cells/h

(e) $P(t) = 10^6 \quad \Leftrightarrow \quad 50 \cdot 8^t = 1{,}000{,}000 \quad \Leftrightarrow \quad 8^t = 20{,}000 \quad \Leftrightarrow \quad t\ln 8 = \ln 20{,}000 \quad \Leftrightarrow \quad t = \dfrac{\ln 20{,}000}{\ln 8} \approx 4.76$ h

3. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 100e^{kt}$. Now $P(1) = 100e^{k(1)} = 420 \quad \Rightarrow \quad e^k = \frac{420}{100} \quad \Rightarrow \quad k = \ln 4.2$.

So $P(t) = 100e^{(\ln 4.2)t} = 100(4.2)^t$.

(b) $P(3) = 100(4.2)^3 = 7408.8 \approx 7409$ bacteria

(c) $dP/dt = kP \quad \Rightarrow \quad P'(3) = k \cdot P(3) = (\ln 4.2)\big(100(4.2)^3\big)$ [from part (a)] $\approx 10{,}632$ bacteria/h

(d) $P(t) = 100(4.2)^t = 10{,}000 \quad \Rightarrow \quad (4.2)^t = 100 \quad \Rightarrow \quad t = (\ln 100)/(\ln 4.2) \approx 3.2$ hours

4. (a) $y(t) = y(0)e^{kt} \quad \Rightarrow \quad y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get $e^{6k}/e^{2k} = 25{,}600/400 \quad \Rightarrow \quad e^{4k} = 64 \quad \Rightarrow \quad 4k = \ln 2^6 = 6\ln 2 \quad \Rightarrow \quad k = \frac{3}{2}\ln 2 \approx 1.0397$, about 104% per hour.

(b) $400 = y(0)e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{3\ln 2} = 400/\left(e^{\ln 2}\right)^3 = 400/2^3 = 50$.

(c) $y(t) = y(0)e^{kt} = 50e^{(3/2)(\ln 2)t} = 50(e^{\ln 2})^{(3/2)t} \quad \Rightarrow \quad y(t) = 50(2)^{1.5t}$

(d) $y(4.5) = 50(2)^{1.5(4.5)} = 50(2)^{6.75} \approx 5382$ bacteria

(e) $\dfrac{dy}{dt} = ky = \left(\dfrac{3}{2}\ln 2\right)(50(2)^{6.75}) \approx 5596$ bacteria/h

(f) $y(t) = 50{,}000 \quad \Rightarrow \quad 50{,}000 = 50(2)^{1.5t} \quad \Rightarrow \quad 1000 = (2)^{1.5t} \quad \Rightarrow \quad \ln 1000 = 1.5t\ln 2 \quad \Rightarrow$

$t = \dfrac{\ln 1000}{1.5\ln 2} \approx 6.64$ h

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \quad \Rightarrow$ $\frac{980}{790} = e^{k(50)} \quad \Rightarrow \quad \ln\frac{980}{790} = 50k \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \quad \Rightarrow \quad P(1900) = 1650 = 1260e^{k(1900-1850)} \quad \Rightarrow$ $\ln\frac{1650}{1260} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2560 = 1650e^{k(1950-1900)} \quad \Rightarrow$ $\ln\frac{2560}{1650} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1950, we substitute $t - 1950$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1950)e^{k(t-1950)} \quad \Rightarrow$ $P(1960) = 100 = 83e^{k(1960-1950)} \quad \Rightarrow \quad \frac{100}{83} = e^{10k} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{100}{83} \approx 0.0186$. With this model, we estimate $P(1980) = 83e^{k(1980-1950)} = 83e^{30k} \approx 145$ million, which is an underestimate of the actual population of 150 million.

(b) As in part (a), $P(t) = P(1960)e^{k(t-1960)} \quad \Rightarrow \quad P(1980) = 150 = 100e^{20k} \quad \Rightarrow \quad 20k = \ln \frac{150}{100} \quad \Rightarrow$

$k = \frac{1}{20} \ln \frac{3}{2} \approx 0.0203$. Thus, $P(2000) = 100e^{40k} = 225$ million, which is an overestimate of the actual population of 214 million.

(c) As in part (a), $P(t) = P(1980)e^{k(t-1980)} \quad \Rightarrow \quad P(2000) = 214 = 150e^{20k} \quad \Rightarrow \quad 20k = \ln \frac{214}{150} \quad \Rightarrow$

$k = \frac{1}{20} \ln \frac{214}{150} \approx 0.0178$. Thus, $P(2010) = 150e^{30k} \approx 256$, which is an overestimate of the actual population of 243 million.

(d) $P(2020) = 150e^{k(2020-1980)} \approx 305$ million. This estimate will probably be an overestimate since this model gave us an overestimate in part (c) — indicating that k is too large. Creating a model with more recent data would likely result in an improved estimate.

7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \quad \Rightarrow \quad y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

(b) $y(t) = Ce^{-0.0005t} = 0.9C \quad \Rightarrow \quad e^{-0.0005t} = 0.9 \quad \Rightarrow \quad -0.0005t = \ln 0.9 \quad \Rightarrow \quad t = -2000 \ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 50e^{kt}$. Since the half-life is 28 days, $y(28) = 50e^{28k} = 25 \quad \Rightarrow$

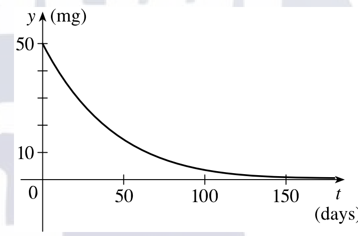
$e^{28k} = \frac{1}{2} \quad \Rightarrow \quad 28k = \ln \frac{1}{2} \quad \Rightarrow \quad k = -(\ln 2)/28$, so $y(t) = 50e^{-(\ln 2)t/28} = 50 \cdot 2^{-t/28}$.

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6$ mg

(c) $y(t) = 2 \quad \Rightarrow \quad 2 = 50 \cdot 2^{-t/28} \quad \Rightarrow \quad \frac{2}{50} = 2^{-t/28} \quad \Rightarrow$

$(-t/28)\ln 2 = \ln \frac{1}{25} \quad \Rightarrow \quad t = \left(-28 \ln \frac{1}{25}\right) / \ln 2 \approx 130$ days

(d)



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$.

$y(30) = 100e^{30k} = \frac{1}{2}(100) \quad \Rightarrow \quad e^{30k} = \frac{1}{2} \quad \Rightarrow \quad k = -(\ln 2)/30 \quad \Rightarrow \quad y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \quad \Rightarrow \quad -(\ln 2)t/30 = \ln \frac{1}{100} \quad \Rightarrow \quad t = -30 \frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$.

$y(1) = Ae^{k} = 0.945A \quad \Rightarrow \quad e^{k} = 0.945 \quad \Rightarrow \quad k = \ln 0.945$.

Then $Ae^{(\ln 0.945)t} = \frac{1}{2}A \quad \Leftrightarrow \quad \ln e^{(\ln 0.945)t} = \ln \frac{1}{2} \quad \Leftrightarrow \quad (\ln 0.945)t = \ln \frac{1}{2} \quad \Leftrightarrow \quad t = -\frac{\ln 2}{\ln 0.945} \approx 12.25$ years.

(b) $Ae^{(\ln 0.945)t} = 0.20A \quad \Leftrightarrow \quad (\ln 0.945)t = \ln \frac{1}{5} \quad \Leftrightarrow \quad t = -\frac{\ln 5}{\ln 0.945} \approx 28.45$ years

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$y(5730) = \frac{1}{2}y(0) \quad \Rightarrow \quad y(0)e^{-k(5730)} = \frac{1}{2}y(0) \quad \Rightarrow \quad e^{-5730k} = \frac{1}{2} \quad \Rightarrow \quad -5730k = \ln \frac{1}{2} \quad \Rightarrow \quad k = -\dfrac{\ln \frac{1}{2}}{5730} = \dfrac{\ln 2}{5730}$.

If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0) \quad \Rightarrow \quad 0.74 = e^{-t(\ln 2)/5730} \quad \Rightarrow \quad \ln 0.74 = -\dfrac{t \ln 2}{5730} \quad \Rightarrow$

$t = -\dfrac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From Exercise 11, we have the model $y(t) = y(0)e^{-kt}$ with $k = (\ln 2)/5730$. Thus, $y(68{,}000{,}000) = y(0)e^{-68{,}000{,}000k} \approx y(0) \cdot 0 = 0$. There would be an undetectable amount of ^{14}C remaining for a 68-million-year-old dinosaur.

Now let $y(t) = 0.1\%\, y(0)$, so $0.001y(0) = y(0)e^{-kt} \Rightarrow 0.001 = e^{-kt} \Rightarrow \ln 0.001 = -kt \Rightarrow$ $t = \dfrac{\ln 0.001}{-k} = \dfrac{\ln 0.001}{-(\ln 2)/5730} \approx 57{,}104$, which is the maximum age of a fossil that we could date using ^{14}C.

13. Let t measure time since a dinosaur died in millions of years, and let $y(t)$ be the amount of ^{40}K in the dinosaur's bones at time t. Then $y(t) = y(0)e^{-kt}$ and k is determined by the half-life: $y(1250) = \frac{1}{2}y(0) \Rightarrow y(0)e^{-k(1250)} = \frac{1}{2}y(0) \Rightarrow$ $e^{-1250k} = \frac{1}{2} \Rightarrow -1250k = \ln\frac{1}{2} \Rightarrow k = -\dfrac{\ln\frac{1}{2}}{1250} = \dfrac{\ln 2}{1250}$. To determine if a dinosaur dating of 68 million years is possible, we find that $y(68) = y(0)e^{-k(68)} \approx 0.963y(0)$, indicating that about 96% of the ^{40}K is remaining, which is clearly detectable. To determine the maximum age of a fossil by using ^{40}K, we solve $y(t) = 0.1\%y(0)$ for t.

$y(0)e^{-kt} = 0.001y(0) \Leftrightarrow e^{-kt} = 0.001 \Leftrightarrow -kt = \ln 0.001 \Leftrightarrow t = \dfrac{\ln 0.001}{-(\ln 2)/1250} \approx 12{,}457$ million, or 12.457 billion years.

14. From the information given, we know that $\dfrac{dy}{dx} = 2y \Rightarrow y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \Rightarrow C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

15. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 75)$. Now let $y = T - 75$, so $y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.

$y(30) = 110e^{30k} = 150 - 75 \Rightarrow e^{30k} = \frac{75}{110} = \frac{15}{22} \Rightarrow k = \frac{1}{30}\ln\frac{15}{22}$, so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62^\circ\text{F}$. Thus, $T(45) \approx 62 + 75 = 137^\circ\text{F}$.

(b) $T(t) = 100 \Rightarrow y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \Rightarrow e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \Rightarrow \frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110} \Rightarrow$ $t = \dfrac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

16. Let $T(t)$ be the temperature of the body t hours after 1:30 PM. Then $T(0) = 32.5$ and $T(1) = 30.3$. Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 20)$. Now let $y = T - 20$, so $y(0) = T(0) - 20 = 32.5 - 20 = 12.5$, so y is a solution to the initial value problem $dy/dt = ky$ with $y(0) = 12.5$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 12.5e^{kt}$.

$y(1) = 30.3 - 20 \Rightarrow 10.3 = 12.5e^{k(1)} \Rightarrow e^k = \frac{10.3}{12.5} \Rightarrow k = \ln\frac{10.3}{12.5}$. The murder occurred when

$y(t) = 37 - 20 \quad \Rightarrow \quad 12.5e^{kt} = 17 \quad \Rightarrow \quad e^{kt} = \frac{17}{12.5} \quad \Rightarrow \quad kt = \ln\frac{17}{12.5} \quad \Rightarrow \quad t = \left(\ln\frac{17}{12.5}\right) / \ln\frac{10.3}{12.5} \approx -1.588 \text{ h}$

≈ -95 minutes. Thus, the murder took place about 95 minutes before 1:30 PM, or 11:55 AM.

17. $\dfrac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $\quad y(0) = T(0) - 20 = 5 - 20 = -15$, so

$y(25) = y(0)e^{25k} = -15e^{25k}$, and $y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \quad \Rightarrow \quad e^{25k} = \frac{2}{3}$. Thus,

$25k = \ln\left(\frac{2}{3}\right)$ and $k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \quad \Rightarrow \quad e^k = \left(\frac{2}{3}\right)^{1/25} \quad \Rightarrow$

$e^{kt} = \left(\frac{2}{3}\right)^{t/25} \quad \Rightarrow \quad y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,^\circ\text{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \quad \Rightarrow \quad 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \quad \Rightarrow \quad \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \quad \Rightarrow$

$(t/25)\ \ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \quad \Rightarrow \quad t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

18. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $\quad y(0) = T(0) - 20 = 95 - 20 = 75$,

so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1^\circ\text{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus,

$-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says

$50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \quad \Rightarrow \quad -t/50 = \ln\left(\frac{2}{3}\right) \quad \Rightarrow \quad t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

19. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \quad \Rightarrow \quad P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$P(1000) = 101.3e^{1000k} = 87.14 \quad \Rightarrow \quad 1000k = \ln\left(\frac{87.14}{101.3}\right) \quad \Rightarrow \quad k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \quad \Rightarrow$

$P(h) = 101.3\, e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\, e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

20. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 1000$, $r = 0.08$, and $t = 3$, we have:

(i) Annually: $n = 1$; $\quad A = 1000\left(1 + \frac{0.08}{1}\right)^{1\cdot 3} = \$1259.71$

(ii) Quarterly: $n = 4$; $\quad A = 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 3} = \$1268.24$

(iii) Monthly: $n = 12$; $\quad A = 1000\left(1 + \frac{0.08}{12}\right)^{12\cdot 3} = \$1270.24$

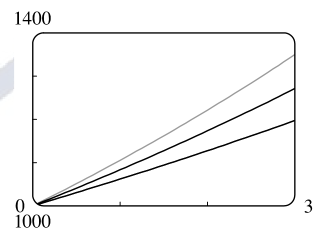
(iv) Weekly: $n = 52$ $\quad A = 1000\left(1 + \frac{0.08}{52}\right)^{52\cdot 3} = \$1271.01$

(v) Daily: $n = 365$; $\quad A = 1000\left(1 + \frac{0.08}{365}\right)^{365\cdot 3} = \$1271.22$

(vi) Hourly: $n = 365 \cdot 24$; $\quad A = 1000\left(1 + \frac{0.08}{365 \cdot 24}\right)^{365\cdot 24\cdot 3} = \$1271.25$

(vii) Continuously: $\quad A = 1000e^{(0.08)3} = \$1271.25$

(b)



$A_{0.10}(3) = \$1349.86$,

$A_{0.08}(3) = \$1271.25$, and

$A_{0.06}(3) = \$1197.22$.

21. (a) Using $A = A_0\left(1+\dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1+\frac{0.05}{1}\right)^{1\cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1+\frac{0.05}{2}\right)^{2\cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1+\frac{0.05}{12}\right)^{12\cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1+\frac{0.05}{52}\right)^{52\cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1+\frac{0.05}{365}\right)^{365\cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

22. (a) $A_0e^{0.06t} = 2A_0 \;\Leftrightarrow\; e^{0.06t} = 2 \;\Leftrightarrow\; 0.06t = \ln 2 \;\Leftrightarrow\; t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A = A_0(1+r)^t$ is r. From part (a), we have $A = A_0e^{0.06t}$. These amounts must be equal, so $(1+r)^t = e^{0.06t} \;\Rightarrow\; 1 + r = e^{0.06} \;\Rightarrow\; r = e^{0.06} - 1 \approx 0.0618 = 6.18\%$, which is the equivalent annual interest rate.

APPLIED PROJECT Controlling Red Blood Cell Loss During Surgery

1. Let $R(t)$ be the volume of RBCs (in liters) at time t (in hours). Since the total volume of blood is 5 L, the concentration of RBCs is $R/5$. The patient bleeds 2 L of blood in 4 hours, so

$$\frac{dR}{dt} = -\frac{2L}{4h}\cdot\frac{R}{5} = -\frac{1}{10}R$$

From Section 3.8, we know that $dR/dt = kR$ has solution $R(t) = R(0)e^{kt}$. In this case, $R(0) = 45\%$ of $5 = \frac{9}{4}$ and $k = -\frac{1}{10}$, so $R(t) = \frac{9}{4}e^{-t/10}$. At the end of the operation, the volume of RBCs is $R(4) = \frac{9}{4}e^{-0.4} \approx 1.51$ L.

2. Let V be the volume of blood that is extracted and replaced with saline solution. Let $R_A(t)$ be the volume of RBCs with the ANH procedure. Then $R_A(0)$ is 45% of $(5 - V)$, or $\frac{9}{20}(5 - V)$, and hence $R_A(t) = \frac{9}{20}(5 - V)e^{-t/10}$. We want $R_A(4) \geq 25\%$ of 5 $\;\Leftrightarrow\; \frac{9}{20}(5-V)e^{-0.4} \geq \frac{5}{4} \;\Leftrightarrow\; 5 - V \geq \frac{25}{9}e^{0.4} \;\Leftrightarrow\; V \leq 5 - \frac{25}{9}e^{0.4} \approx 0.86$ L. To maximize the effect of the ANH procedure, the surgeon should remove 0.86 L of blood and replace it with saline solution.

3. The RBC loss *without* the ANH procedure is $R(0) - R(4) = \frac{9}{4} - \frac{9}{4}e^{-0.4} \approx 0.74$ L. The RBC loss *with* the ANH procedure is $R_A(0) - R_A(4) = \frac{9}{20}(5 - V) - \frac{9}{20}(5 - V)e^{-0.4} = \frac{9}{20}(5 - V)(1 - e^{-0.4})$. Now let $V = 5 - \frac{25}{9}e^{0.4}$ [from Problem 2] to get $R_A(0) - R_A(4) = \frac{9}{20}\left[5 - \left(5 - \frac{25}{9}e^{0.4}\right)\right](1 - e^{0.4}) = \frac{9}{20}\cdot\frac{25}{9}e^{0.4}(1 - e^{0.4}) = \frac{5}{4}(e^{0.4} - 1) \approx 0.61$ L. Thus, the ANH procedure reduces the RBC loss by about $0.74 - 0.61 = 0.13$ L (about 4.4 fluid ounces).

3.9 Related Rates

1. $V = x^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

2. (a) $A = \pi r^2 \quad \Rightarrow \quad \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2/\text{s}$.

4. $A = \ell w \quad \Rightarrow \quad \dfrac{dA}{dt} = \ell \cdot \dfrac{dw}{dt} + w \cdot \dfrac{d\ell}{dt} = 20(3) + 10(8) = 140\text{ cm}^2/\text{s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \quad \Rightarrow \quad \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \quad \Rightarrow \quad 3 = 25\pi\,\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dh}{dt} = \dfrac{3}{25\pi}\text{ m/min}$.

6. $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\,\dfrac{dr}{dt} \quad \Rightarrow \quad \dfrac{dV}{dt} = 4\pi\left(\frac{1}{2}\cdot 80\right)^2(4) = 25{,}600\pi\text{ mm}^3/\text{s}$.

7. $S = 4\pi r^2 \quad \Rightarrow \quad \dfrac{dS}{dt} = 4\pi \cdot 2r\,\dfrac{dr}{dt} \quad \Rightarrow \quad \dfrac{dS}{dt} = 4\pi \cdot 2 \cdot 8 \cdot 2 = 128\pi\text{ cm}^2/\text{min}$.

8. (a) $A = \frac{1}{2}ab\sin\theta \quad \Rightarrow \quad \dfrac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = \frac{1}{2}(2)(3)\left(\cos\frac{\pi}{3}\right)(0.2) = 3\left(\frac{1}{2}\right)(0.2) = 0.3\text{ cm}^2/\text{min}$.

(b) $A = \frac{1}{2}ab\sin\theta \quad \Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}a\left(b\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{db}{dt}\right) = \tfrac{1}{2}(2)\left[3\left(\cos\tfrac{\pi}{3}\right)(0.2) + \left(\sin\tfrac{\pi}{3}\right)(1.5)\right]$$
$$= 3\left(\tfrac{1}{2}\right)(0.2) + \tfrac{1}{2}\sqrt{3}\left(\tfrac{3}{2}\right) = 0.3 + \tfrac{3}{4}\sqrt{3}\text{ cm}^2/\text{min} \quad [\approx 1.6]$$

(c) $A = \frac{1}{2}ab\sin\theta \quad \Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}\,b\sin\theta + a\,\frac{db}{dt}\sin\theta + ab\cos\theta\,\frac{d\theta}{dt}\right) \qquad \text{[by Exercise 3.2.61(a)]}$$
$$= \tfrac{1}{2}\left[(2.5)(3)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(1.5)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(3)\left(\tfrac{1}{2}\right)(0.2)\right]$$
$$= \left(\tfrac{15}{8}\sqrt{3} + \tfrac{3}{4}\sqrt{3} + 0.3\right) = \left(\tfrac{21}{8}\sqrt{3} + 0.3\right)\ \text{cm}^2/\text{min} \quad [\approx 4.85]$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $y = \sqrt{2x+1}$ and $\dfrac{dx}{dt} = 3 \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{2}(2x+1)^{-1/2} \cdot 2 \cdot 3 = \dfrac{3}{\sqrt{2x+1}}$. When $x = 4$, $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{9}} = 1$.

(b) $y = \sqrt{2x+1} \quad \Rightarrow \quad y^2 = 2x+1 \quad \Rightarrow \quad 2x = y^2 - 1 \quad \Rightarrow \quad x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\dfrac{dy}{dt} = 5 \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{dx}{dy}\dfrac{dy}{dt} = y \cdot 5 = 5y$. When $x = 12$, $y = \sqrt{25} = 5$, so $\dfrac{dx}{dt} = 5(5) = 25$.

10. (a) $\dfrac{d}{dt}\left(4x^2 + 9y^2\right) = \dfrac{d}{dt}(36) \quad \Rightarrow \quad 8x\,\dfrac{dx}{dt} + 18y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow$

$4(2)\,\dfrac{dx}{dt} + 9\left(\dfrac{2}{3}\sqrt{5}\right)\left(\dfrac{1}{3}\right) = 0 \quad \Rightarrow \quad 8\,\dfrac{dx}{dt} = -2\sqrt{5} \quad \Rightarrow \quad \dfrac{dx}{dt} = -\dfrac{1}{4}\sqrt{5}$

(b) $4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 4(-2)(3) + 9\left(\dfrac{2}{3}\sqrt{5}\right)\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 6\sqrt{5}\,\dfrac{dy}{dt} = 24 \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{4}{\sqrt{5}}$

11. $\dfrac{d}{dt}\left(x^2 + y^2 + z^2\right) = \dfrac{d}{dt}(9) \quad\Rightarrow\quad 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} + 2z\,\dfrac{dz}{dt} = 0 \quad\Rightarrow\quad x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0.$

If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1\,\dfrac{dz}{dt} = 0 \quad\Rightarrow\quad \dfrac{dz}{dt} = -18.$

12. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \quad\Rightarrow\quad x\,\dfrac{dy}{dt} + y\,\dfrac{dx}{dt} = 0.$ If $\dfrac{dy}{dt} = -3\text{ cm/s}$ and $(x, y) = (4, 2)$, then $4(-3) + 2\,\dfrac{dx}{dt} = 0 \quad\Rightarrow$

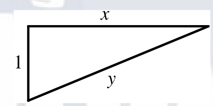
$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station.

If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500\text{ mi/h}$.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



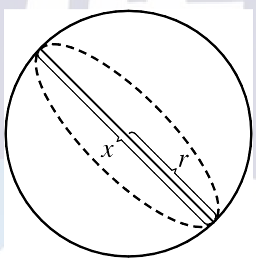
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \quad\Rightarrow\quad 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433\text{ mi/h}$.

14. (a) Given: the rate of decrease of the surface area is $1\text{ cm}^2\text{/min}$. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1\text{ cm}^2\text{/s}$.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and

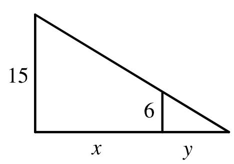
$S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \quad\Rightarrow\quad \dfrac{dS}{dt} = \dfrac{dS}{dx}\dfrac{dx}{dt} = 2\pi x\,\dfrac{dx}{dt}.$

(e) $-1 = \dfrac{dS}{dt} = 2\pi x\,\dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\dfrac{1}{20\pi}$ cm/min.

15. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5\text{ ft/s}$.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x + y)$ when $x = 40$ ft.

(c)

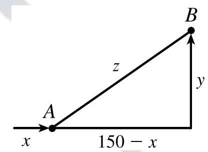


(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \;\Rightarrow\; 15y = 6x + 6y \;\Rightarrow\; 9y = 6x \;\Rightarrow\; y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \dfrac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

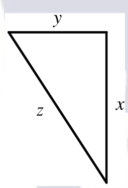
(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(c) 

(d) $z^2 = (150-x)^2 + y^2 \;\Rightarrow\; 2z\,\dfrac{dz}{dt} = 2(150-x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \;\Rightarrow\; z = \sqrt{(150-140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x-150)\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

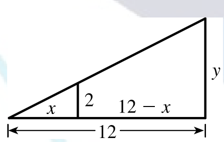
17. 

We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \;\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} \;\Rightarrow\; z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} \;\Rightarrow\; \dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right)$.

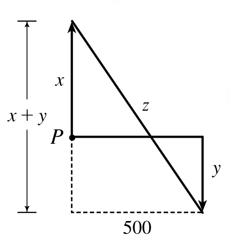
After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \;\Rightarrow\; z = \sqrt{120^2 + 50^2} = 130$,

so $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

18. 

We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \;\Rightarrow\; y = \dfrac{24}{x} \;\Rightarrow$

$\dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19. 

We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \;\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

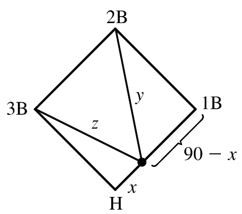
$x = (4\text{ ft/s})(20\text{ min})(60\text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \;\Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$\dfrac{dz}{dt} = \dfrac{x+y}{z}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = \dfrac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \dfrac{837}{\sqrt{8674}} \approx 8.99$ ft/s.

20. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a)



$y^2 = (90 - x)^2 + 90^2 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2(90 - x)\left(-\dfrac{dx}{dt}\right)$. When $x = 45$,

$y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\dfrac{dy}{dt} = \dfrac{90 - x}{y}\left(-\dfrac{dx}{dt}\right) = \dfrac{45}{45\sqrt{5}}(-24) = -\dfrac{24}{\sqrt{5}}$,

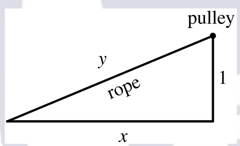
so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

$z^2 = x^2 + 90^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

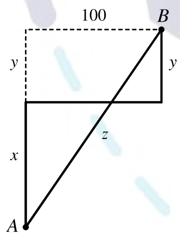
21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \quad \Rightarrow \quad \frac{1}{2}b = 10 \quad \Rightarrow$ $b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \quad \Rightarrow \quad 4 = 20 + 10\,\dfrac{db}{dt} \quad \Rightarrow \quad \dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

22.



Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \quad \Rightarrow$ $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

23.



We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x + y)^2 + 100^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x + y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad \Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$\dfrac{dz}{dt} = \dfrac{x + y}{z}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = \dfrac{140 + 100}{260}(35 + 25) = \dfrac{720}{13} \approx 55.4$ km/h.

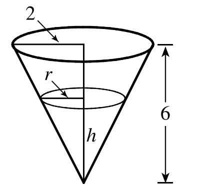
24. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2\sin(\pi x/2)]^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 4 \cdot 2\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right) \cdot \dfrac{\pi}{2}\dfrac{dx}{dt} \quad \Rightarrow \quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + 2\pi\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right)\dfrac{dx}{dt}$. When

$(x, y) = \left(\dfrac{1}{3}, 1\right)$, $z = \sqrt{\left(\dfrac{1}{3}\right)^2 + 1^2} = \sqrt{\dfrac{10}{9}} = \dfrac{1}{3}\sqrt{10}$, so $\dfrac{1}{3}\sqrt{10}\,\dfrac{dz}{dt} = \dfrac{1}{3}\sqrt{10} + 2\pi\,\sin\dfrac{\pi}{6}\,\cos\dfrac{\pi}{6} \cdot \sqrt{10} \quad \Rightarrow$

$\dfrac{1}{3}\dfrac{dz}{dt} = \dfrac{1}{3} + 2\pi\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\sqrt{3}\right) \quad \Rightarrow \quad \dfrac{dz}{dt} = 1 + \dfrac{3\sqrt{3}\,\pi}{2}$ cm/s.

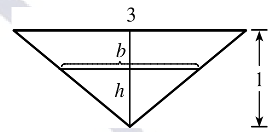
25. If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where $V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \Rightarrow r = \dfrac{1}{3}h \Rightarrow$ $V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \dfrac{dV}{dt} = \dfrac{\pi}{9}h^2\dfrac{dh}{dt}$. When $h = 200$ cm, $\dfrac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \dfrac{\pi}{9}(200)^2(20) \Rightarrow C = 10{,}000 + \dfrac{800{,}000}{9}\pi \approx 289{,}253\ \text{cm}^3/\text{min}$.

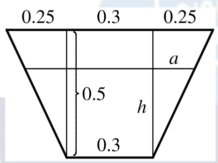


26. By similar triangles, $\dfrac{3}{1} = \dfrac{b}{h}$, so $b = 3h$. The trough has volume

$V = \frac{1}{2}bh(10) = 5(3h)h = 15h^2 \Rightarrow 12 = \dfrac{dV}{dt} = 30h\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{2}{5h}$.

When $h = \frac{1}{2}$, $\dfrac{dh}{dt} = \dfrac{2}{5\cdot\frac{1}{2}} = \dfrac{4}{5}$ ft/min.



27. 

The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$.

By similar triangles, $\dfrac{a}{h} = \dfrac{0.25}{0.5} = \dfrac{1}{2}$, so $2a = h \Rightarrow V = 5(0.6 + h)h = 3h + 5h^2$.

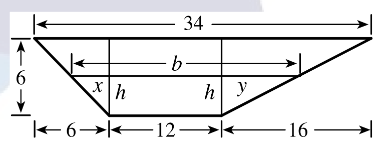
Now $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \Rightarrow 0.2 = (3 + 10h)\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{0.2}{3 + 10h}$. When $h = 0.3$,

$\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10(0.3)} = \dfrac{0.2}{6}$ m/min $= \dfrac{1}{30}$ m/min or $\dfrac{10}{3}$ cm/min.

28. The figure is drawn without the top 3 feet.

$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles,

$\dfrac{x}{h} = \dfrac{6}{6}$ and $\dfrac{y}{h} = \dfrac{16}{6} = \dfrac{8}{3}$, so $b = x + 12 + y = h + 12 + \dfrac{8h}{3} = 12 + \dfrac{11h}{3}$.



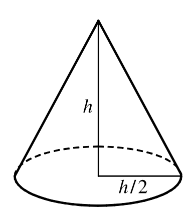
Thus, $V = 10\left(24 + \dfrac{11h}{3}\right)h = 240h + \dfrac{110h^2}{3}$ and so $0.8 = \dfrac{dV}{dt} = \left(240 + \dfrac{220}{3}h\right)\dfrac{dh}{dt}$.

When $h = 5$, $\dfrac{dh}{dt} = \dfrac{0.8}{240 + 5(220/3)} = \dfrac{3}{2275} \approx 0.00132$ ft/min.

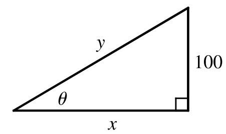
29. We are given that $\dfrac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi h^3}{12} \Rightarrow$

$\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \Rightarrow 30 = \dfrac{\pi h^2}{4}\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{120}{\pi h^2}$.

When $h = 10$ ft, $\dfrac{dh}{dt} = \dfrac{120}{10^2\pi} = \dfrac{6}{5\pi} \approx 0.38$ ft/min.



30. We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \quad\Rightarrow\quad x = 100\cot\theta \quad\Rightarrow$



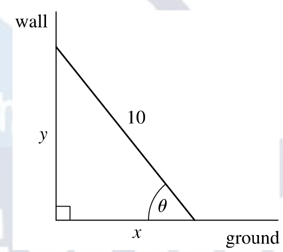
$\dfrac{dx}{dt} = -100\csc^2\theta\,\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \quad\Rightarrow$

$\dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}\,s^2$.

$\dfrac{dA}{dt} = \frac{1}{4}\sqrt{3}\cdot 2s\,\dfrac{ds}{dt} = \frac{1}{4}\sqrt{3}\cdot 2(30)(10) = 150\sqrt{3}\ \text{cm}^2/\text{min}.$

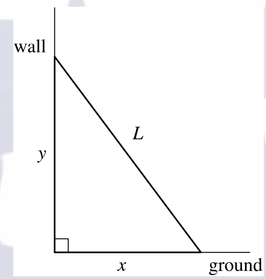
32. $\cos\theta = \dfrac{x}{10} \quad\Rightarrow\quad -\sin\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{10}\,\dfrac{dx}{dt}$. From Example 2, $\dfrac{dx}{dt} = 1$ and



when $x = 6$, $y = 8$, so $\sin\theta = \dfrac{8}{10}$.

Thus, $-\dfrac{8}{10}\,\dfrac{d\theta}{dt} = \dfrac{1}{10}(1) \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{1}{8}$ rad/s.

33. From the figure and given information, we have $x^2 + y^2 = L^2$, $\dfrac{dy}{dt} = -0.15\ \text{m/s}$, and



$\dfrac{dx}{dt} = 0.2\ \text{m/s}$ when $x = 3$ m. Differentiating implicitly with respect to t, we get

$x^2 + y^2 = L^2 \quad\Rightarrow\quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Substituting the given

information gives us $y(-0.15) = -3(0.2) \quad\Rightarrow\quad y = 4$ m. Thus, $3^2 + 4^2 = L^2 \quad\Rightarrow$

$L^2 = 25 \quad\Rightarrow\quad L = 5$ m.

34. According to the model in Example 2, $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt} \to -\infty$ as $y \to 0$, which doesn't make physical sense. For example, the model predicts that for sufficiently small y, the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y. What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*, 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics*, 53 (3), March 1985, pages 276–277.

35. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so

$\dfrac{dA}{dt} = \dfrac{1}{2}r^2\,\dfrac{d\theta}{dt}$. The minute hand rotates through $360^\circ = 2\pi$ radians each hour, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2(2\pi) = \pi r^2\ \text{cm}^2/\text{h}$. This answer makes sense because the minute hand sweeps through the full area of a circle, πr^2, each hour.

36. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18{,}000\pi\ \text{cm}^3$.

If the basin is half full, then $V = \pi\left(rh^2 - \frac{1}{3}h^3\right) \Rightarrow 9000\pi = \pi\left(30h^2 - \frac{1}{3}h^3\right) \Rightarrow \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \Rightarrow$

$h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$$V = \pi\left(30h^2 - \tfrac{1}{3}h^3\right) \Rightarrow \frac{dV}{dt} = \pi\left(60h\,\frac{dh}{dt} - h^2\,\frac{dh}{dt}\right) \Rightarrow \left(2\,\frac{\text{L}}{\text{min}}\right)\left(1000\,\frac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2)\,\frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = \frac{2000}{\pi(60H - H^2)} \approx 0.804\ \text{cm/min}.$$

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\,\frac{dV}{dt} + V\,\frac{dP}{dt} = 0 \Rightarrow$

$\frac{dV}{dt} = -\frac{V}{P}\,\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

38. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4}\,\frac{dV}{dt} + V^{1.4}\,\frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V^{1.4}}{P \cdot 1.4V^{0.4}}\,\frac{dP}{dt} = -\frac{V}{1.4P}\,\frac{dP}{dt}$.

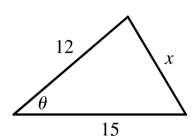
When $V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, so we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36\ \text{cm}^3$/min.

39. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ with respect to t, we have $-\frac{1}{R^2}\,\frac{dR}{dt} = -\frac{1}{R_1^2}\,\frac{dR_1}{dt} - \frac{1}{R_2^2}\,\frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\,\frac{dR_1}{dt} + \frac{1}{R_2^2}\,\frac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\frac{dR}{dt} = \frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2)\right] = \frac{107}{810} \approx 0.132\ \Omega/\text{s}$.

40. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\begin{aligned}\frac{dB}{dt} &= \frac{dB}{dW}\,\frac{dW}{dL}\,\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)(0.12 \cdot 2.53 \cdot L^{1.53})\left(\frac{20 - 15}{10{,}000{,}000}\right) \\ &= \left[0.007 \cdot \tfrac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right](0.12 \cdot 2.53 \cdot 18^{1.53})\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8}\ \text{g/yr}\end{aligned}$$

41. We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

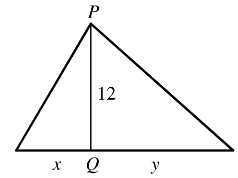


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\,\frac{dx}{dt} = 360\sin\theta\,\frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{180\sin\theta}{x}\,\frac{d\theta}{dt}$. When $\theta = 60°$,

$$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21},\ \text{so}\ \frac{dx}{dt} = \frac{180\sin 60°}{3\sqrt{21}}\,\frac{\pi}{90} = \frac{\pi\sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\,\pi}{21} \approx 0.396\ \text{m/min}.$$

42. Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$.



Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get

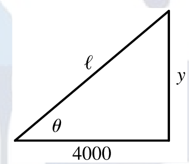
$$\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt} = 0, \text{ so } \frac{dy}{dt} = -\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}.$$

Now when $x = -5$, $39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2+12^2} = 13 + \sqrt{y^2+12^2} \quad \Leftrightarrow \quad \sqrt{y^2+12^2} = 26$, and

$y = \sqrt{26^2 - 12^2} = \sqrt{532}$. So when $x = -5$, $\frac{dy}{dt} = -\frac{(-5)(26)}{\sqrt{532}\,(13)}(-2) = -\frac{10}{\sqrt{133}} \approx -0.87$ ft/s.

So cart B is moving towards Q at about 0.87 ft/s.

43. (a) By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t,



we obtain $2y\,\frac{dy}{dt} = 2\ell\,\frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft,

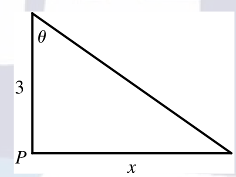
$\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft

and $\frac{d\ell}{dt} = \frac{y}{\ell}\frac{dy}{dt} = \frac{3000}{5000}(600) = \frac{1800}{5} = 360$ ft/s.

(b) Here $\tan\theta = \frac{y}{4000} \quad \Rightarrow \quad \frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{4000}\right) \quad \Rightarrow \quad \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4000}\frac{dy}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dy}{dt}$. When

$y = 3000$ ft, $\frac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}$, so $\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

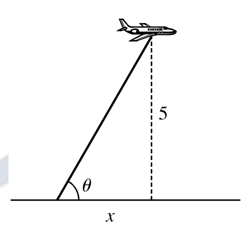
44. We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad \Rightarrow$



$\frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

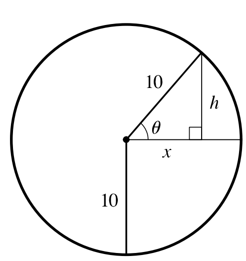
and $\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.

45. $\cot\theta = \frac{x}{5} \quad \Rightarrow \quad -\csc^2\theta\,\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt} \quad \Rightarrow \quad -\left(\csc\frac{\pi}{3}\right)^2\left(-\frac{\pi}{6}\right) = \frac{1}{5}\frac{dx}{dt} \quad \Rightarrow$



$\frac{dx}{dt} = \frac{5\pi}{6}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$

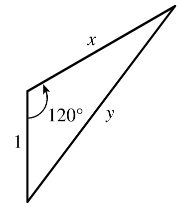
46. We are given that $\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{2 \text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when



$h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \frac{h}{10} \quad \Rightarrow$

$h = 10\sin\theta$, so $\frac{dh}{dt} = 10\cos\theta\,\frac{d\theta}{dt} = 10\left(\frac{8}{10}\right)\pi = 8\pi$ m/min.

47. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,

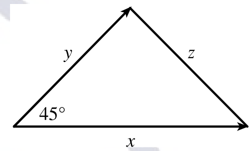


$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\tfrac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} + \dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\,\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad\Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.

48. We are given that $\dfrac{dx}{dt} = 3$ mi/h and $\dfrac{dy}{dt} = 2$ mi/h. By the Law of Cosines,



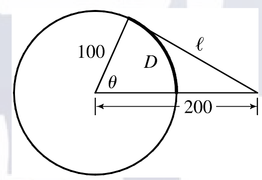
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}\,xy \quad\Rightarrow$

$2z\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} - \sqrt{2}\,x\,\dfrac{dy}{dt} - \sqrt{2}\,y\,\dfrac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,

we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \quad\Rightarrow\quad z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \quad\Rightarrow\quad z = \dfrac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$$\frac{dz}{dt} = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\,\frac{13 - 6\sqrt{2}}{2} = \sqrt{13 - 6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

49. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,

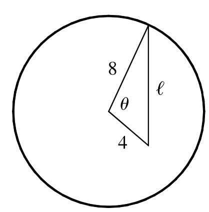


$\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\,\dfrac{d\ell}{dt} = -40{,}000(-\sin\theta)\,\dfrac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \dfrac{1}{100}D \quad\Rightarrow\quad \dfrac{d\theta}{dt} = \dfrac{1}{100}\,\dfrac{dD}{dt} = \dfrac{7}{100}$. To substitute into the expression for $\dfrac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad\Leftrightarrow$

$\cos\theta = \frac{1}{4} \quad\Rightarrow\quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\,\dfrac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad\Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

50. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta$ $(\star)$.



Differentiating implicitly with respect to t, we get $2\ell\,\dfrac{d\ell}{dt} = -64(-\sin\theta)\dfrac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use ($\star$) to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\,\dfrac{d\ell}{dt} = 64\sin\dfrac{\pi}{6}\left(-\dfrac{11\pi}{6}\right) \;\Rightarrow\; \dfrac{d\ell}{dt} = \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6.$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

3.10 Linear Approximations and Differentials

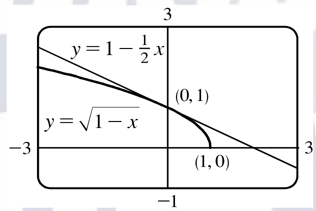
1. $f(x) = x^3 - x^2 + 3 \;\Rightarrow\; f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,
$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23.$

2. $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$. Thus,
$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi.$

3. $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Thus,
$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1.$

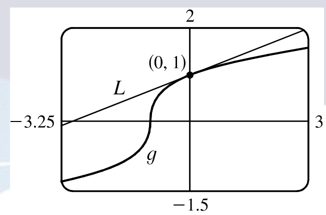
4. $f(x) = 2^x \;\Rightarrow\; f'(x) = 2^x \ln 2$, so $f(0) = 1$ and $f'(0) = \ln 2$. Thus, $L(x) = f(0) + f'(0)(x - 0) = 1 + (\ln 2)x$.

5. $f(x) = \sqrt{1 - x} \;\Rightarrow\; f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

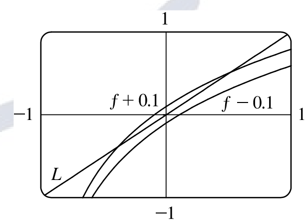


Therefore,
$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x.$
So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$
and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

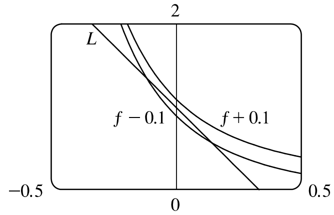
6. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \;\Rightarrow\; g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and $g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.
So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,
and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.



7. $f(x) = \ln(1 + x) \;\Rightarrow\; f'(x) = \dfrac{1}{1 + x}$, so $f(0) = 0$ and $f'(0) = 1$.
Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x) = x$. We need
$\ln(1 + x) - 0.1 < x < \ln(1 + x) + 0.1$, which is true when
$-0.383 < x < 0.516$.

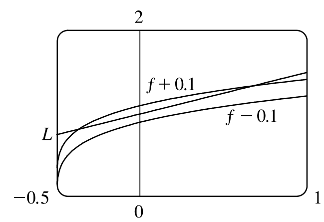


8. $f(x) = (1 + x)^{-3} \;\Rightarrow\; f'(x) = -3(1 + x)^{-4}$, so $f(0) = 1$ and $f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need
$(1 + x)^{-3} - 0.1 < 1 - 3x < (1 + x)^{-3} + 0.1$, which is true when
$-0.116 < x < 0.144$.



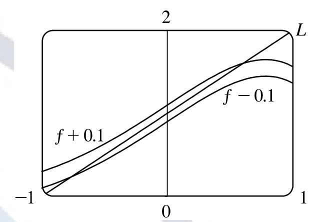
9. $f(x) = \sqrt[4]{1+2x} \quad \Rightarrow \quad f'(x) = \frac{1}{4}(1+2x)^{-3/4}(2) = \frac{1}{2}(1+2x)^{-3/4}$, so $f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$.

We need $\sqrt[4]{1+2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1+2x} + 0.1$, which is true when $-0.368 < x < 0.677$.



10. $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$, so $f(0) = 1$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + x$.

We need $e^x \cos x - 0.1 < 1 + x < e^x \cos x + 0.1$, which is true when $-0.762 < x < 0.607$.



11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = xe^{-4x}$,

$f'(x) = xe^{-4x}(-4) + e^{-4x} \cdot 1 = e^{-4x}(-4x+1)$, so $dy = (1-4x)e^{-4x}dx$.

(b) For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

12. (a) For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2) - (1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

(b) For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

13. (a) For $y = f(t) = \tan\sqrt{t}$, $f'(t) = \sec^2\sqrt{t} \cdot \dfrac{1}{2}t^{-1/2} = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}$, so $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$.

(b) For $y = f(v) = \dfrac{1-v^2}{1+v^2}$,

$$f'(v) = \frac{(1+v^2)(-2v) - (1-v^2)(2v)}{(1+v^2)^2} = \frac{-2v[(1+v^2)+(1-v^2)]}{(1+v^2)^2} = \frac{-2v(2)}{(1+v^2)^2} = \frac{-4v}{(1+v^2)^2},$$

so $dy = \dfrac{-4v}{(1+v^2)^2}\,dv$.

14. (a) For $y = f(\theta) = \ln(\sin\theta)$, $f'(\theta) = \dfrac{1}{\sin\theta}\cos\theta = \cot\theta$, so $dy = \cot\theta\,d\theta$.

(b) For $y = f(x) = \dfrac{e^x}{1-e^x}$, $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x) - (-e^x)]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$, so

$dy = \dfrac{e^x}{(1-e^x)^2}\,dx$.

15. (a) $y = e^{x/10} \quad \Rightarrow \quad dy = e^{x/10} \cdot \frac{1}{10}\,dx = \frac{1}{10}e^{x/10}dx$

(b) $x = 0$ and $dx = 0.1 \quad \Rightarrow \quad dy = \frac{1}{10}e^{0/10}(0.1) = 0.01$.

16. (a) $y = \cos\pi x \quad \Rightarrow \quad dy = -\sin\pi x \cdot \pi\,dx = -\pi\sin\pi x\,dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad \Rightarrow \quad dy = -\pi\sin\frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054$.

17. (a) $y = \sqrt{3 + x^2} \quad \Rightarrow \quad dy = \frac{1}{2}(3 + x^2)^{-1/2}(2x)\,dx = \frac{x}{\sqrt{3 + x^2}}\,dx$

(b) $x = 1$ and $dx = -0.1 \quad \Rightarrow \quad dy = \frac{1}{\sqrt{3 + 1^2}}(-0.1) = \frac{1}{2}(-0.1) = -0.05.$

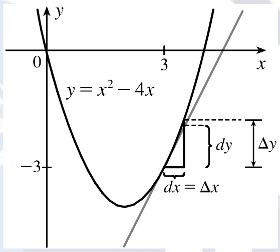
18. (a) $y = \frac{x + 1}{x - 1} \quad \Rightarrow \quad dy = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\,dx = \frac{-2}{(x - 1)^2}\,dx$

(b) $x = 2$ and $dx = 0.05 \quad \Rightarrow \quad dy = \frac{-2}{(2 - 1)^2}(0.05) = -2(0.05) = -0.1.$

19. $y = f(x) = x^2 - 4x,\ x = 3,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

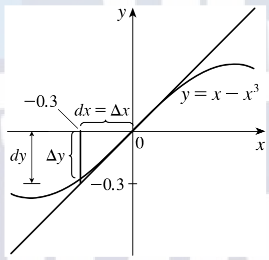
$dy = f'(x)\,dx = (2x - 4)\,dx = (6 - 4)(0.5) = 1$



20. $y = f(x) = x - x^3,\ x = 0,\ \Delta x = -0.3 \quad \Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

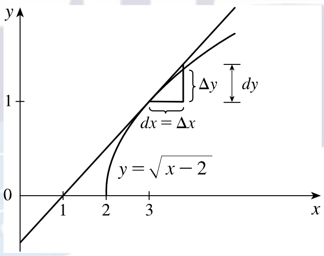
$dy = f'(x)\,dx = (1 - 3x^2)\,dx = (1 - 0)(-0.3) = -0.3$



21. $y = f(x) = \sqrt{x - 2},\ x = 3,\ \Delta x = 0.8 \quad \Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

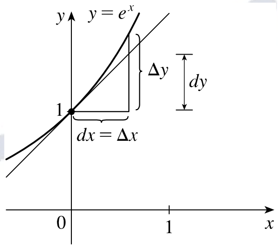
$dy = f'(x)\,dx = \frac{1}{2\sqrt{x - 2}}\,dx = \frac{1}{2(1)}(0.8) = 0.4$



22. $y = f(x) = e^x,\ x = 0,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(0.5) - f(0) = \sqrt{e} - 1 \ [\approx 0.65]$

$dy = e^x\,dx = e^0(0.5) = 0.5$



23. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$.

24. $y = f(x) = 1/x \quad \Rightarrow \quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

25. $y = f(x) = \sqrt[3]{x} \quad \Rightarrow \quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

26. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

27. $y = f(x) = e^x \quad \Rightarrow \quad dy = e^x\,dx$. When $x = 0$ and $dx = 0.1$, $dy = e^0(0.1) = 0.1$, so $e^{0.1} = f(0.1) \approx f(0) + dy = 1 + 0.1 = 1.1$.

28. $y = f(x) = \cos x \quad \Rightarrow \quad dy = -\sin x\,dx$. When $x = 30°\ [\pi/6]$ and $dx = -1°\ [-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29° = f(29°) \approx f(30°) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

29. $y = f(x) = \sec x \quad \Rightarrow \quad f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

30. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

31. $y = f(x) = 1/x \quad \Rightarrow \quad f'(x) = -1/x^2$, so $f(10) = 0.1$ and $f'(10) = -0.01$. The linear approximation of f at 10 is $f(10) + f'(10)(x - 10) = 0.1 - 0.01(x - 10)$. Now $f(9.98) = 1/9.98 \approx 0.1 - 0.01(-0.02) = 0.1 + 0.0002 = 0.1002$, so the approximation is reasonable.

32. (a) $f(x) = (x - 1)^2 \quad \Rightarrow \quad f'(x) = 2(x - 1)$, so $f(0) = 1$ and $f'(0) = -2$.

Thus, $f(x) \approx L_f(x) = f(0) + f'(0)(x - 0) = 1 - 2x$.

$g(x) = e^{-2x} \quad \Rightarrow \quad g'(x) = -2e^{-2x}$, so $g(0) = 1$ and $g'(0) = -2$.

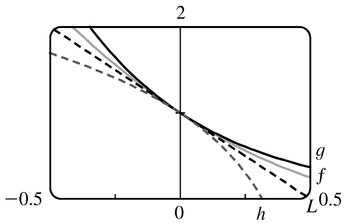
Thus, $g(x) \approx L_g(x) = g(0) + g'(0)(x - 0) = 1 - 2x$.

$h(x) = 1 + \ln(1 - 2x) \quad \Rightarrow \quad h'(x) = \dfrac{-2}{1 - 2x}$, so $h(0) = 1$ and $h'(0) = -2$.

Thus, $h(x) \approx L_h(x) = h(0) + h'(0)(x - 0) = 1 - 2x$.

Notice that $L_f = L_g = L_h$. This happens because f, g, and h have the same function values and the same derivative values at $a = 0$.

(b) The linear approximation appears to be the best for the function f since it is closer to f for a larger domain than it is to g and h. The approximation looks worst for h since h moves away from L faster than f and g do.



33. (a) If x is the edge length, then $V = x^3 \quad\Rightarrow\quad dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error $=$ relative error $\times\, 100\% = 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \quad\Rightarrow\quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error $=$ relative error $\times\, 100\% = 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

34. (a) $A = \pi r^2 \quad\Rightarrow\quad dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

35. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \quad\Rightarrow\quad S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \quad\Rightarrow\quad dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012 = 1.2\%$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \quad\Rightarrow\quad dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3.

The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018 = 1.8\%$.

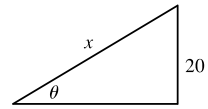
36. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \quad\Rightarrow\quad dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05$ cm $= 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2$ m^3.

37. (a) $V = \pi r^2 h \quad\Rightarrow\quad \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$$\Delta V - dV = [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r = \pi(\Delta r)^2 h.$$

38. (a) $\sin\theta = \dfrac{20}{x} \quad\Rightarrow\quad x = 20\csc\theta \quad\Rightarrow$



$$dx = 20(-\csc\theta\,\cot\theta)\,d\theta = -20\csc 30^\circ\,\cot 30^\circ\,(\pm 1^\circ)$$

$$= -20(2)\left(\sqrt{3}\right)\left(\pm\frac{\pi}{180}\right) = \pm\frac{2\sqrt{3}}{9}\pi$$

So the maximum error is about $\pm\frac{2}{9}\sqrt{3}\,\pi \approx \pm 1.21$ cm.

(b) The relative error is $\dfrac{\Delta x}{x} \approx \dfrac{dx}{x} = \dfrac{\pm\frac{2}{9}\sqrt{3}\,\pi}{20(2)} = \pm\dfrac{\sqrt{3}}{180}\,\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

39. $V = RI \quad\Rightarrow\quad I = \dfrac{V}{R} \quad\Rightarrow\quad dI = -\dfrac{V}{R^2}\,dR$. The relative error in calculating I is $\dfrac{\Delta I}{I} \approx \dfrac{dI}{I} = \dfrac{-(V/R^2)\,dR}{V/R} = -\dfrac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

40. $F = kR^4 \quad\Rightarrow\quad dF = 4kR^3\,dR \quad\Rightarrow\quad \dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

41. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

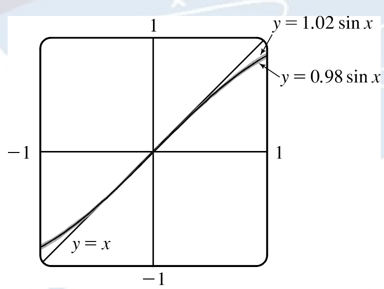
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

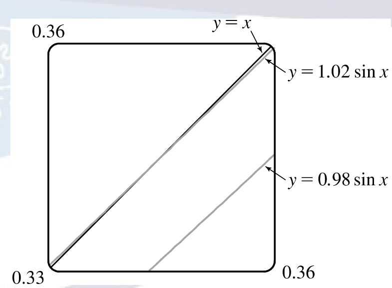
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

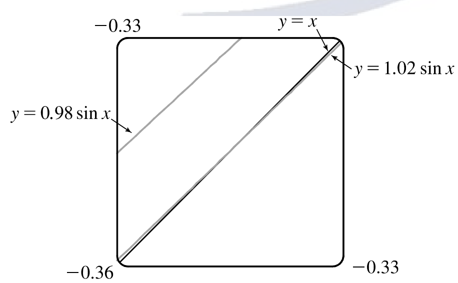
(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

42. (a) $f(x) = \sin x \quad\Rightarrow\quad f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)







[continued]

We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad \Leftrightarrow \quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad \Leftrightarrow$$

$$\begin{cases} -0.02 \sin x < x - \sin x < 0.02 \sin x & \text{if } \sin x > 0 \\ -0.02 \sin x > x - \sin x > 0.02 \sin x & \text{if } \sin x < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 0.98 \sin x < x < 1.02 \sin x & \text{if } \sin x > 0 \\ 1.02 \sin x < x < 0.98 \sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02 \sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

43. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

44. (a) $g'(x) = \sqrt{x^2 + 5} \quad \Rightarrow \quad g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

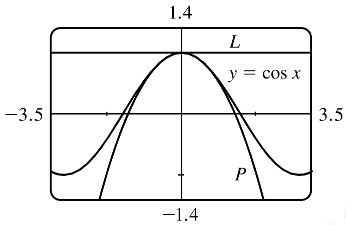
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & \qquad f(x) &= \cos x \\ P'(x) &= B + 2Cx & \qquad f'(x) &= -\sin x \\ P''(x) &= 2C & \qquad f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

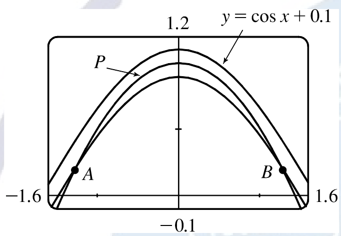
$$\begin{aligned} P(0) &= f(0): & A &= \cos 0 = 1 \\ P'(0) &= f'(0): & B &= -\sin 0 = 0 \\ P''(0) &= f''(0): & 2C &= -\cos 0 = -1 \quad \Rightarrow \quad C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \quad \Leftrightarrow \quad -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \quad \Leftrightarrow$ $0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \quad \Leftrightarrow \quad \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \quad \Leftrightarrow \quad \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1.$



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

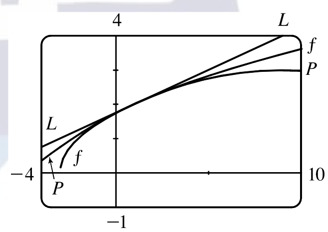
3. If $P(x) = A + B(x - a) + C(x - a)^2$, then $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a): \qquad A = f(a)$$
$$P'(a) = f'(a): \qquad B = f'(a)$$
$$P''(a) = f''(a): \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

Thus, $P(x) = A + B(x - a) + C(x - a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$.

4. From Example 3.10.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x + 3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x + 3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.



From Problem 3, the quadratic approximation $P(x)$ is

$\sqrt{x + 3} \approx f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = 2 + \frac{1}{4}(x - 1) - \frac{1}{64}(x - 1)^2$.

The figure shows the function $f(x) = \sqrt{x + 3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373 . . .	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118 . . .	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015 . . .	2.04937500

5. $T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots + nc_n(x - a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a^2) + \cdots + (n - 1)nc_n(x - a)^{n-2}$ and so

$T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2 \cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2 \cdot 3 \cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot kc_k = k!\, c_k \quad \Rightarrow$ $c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

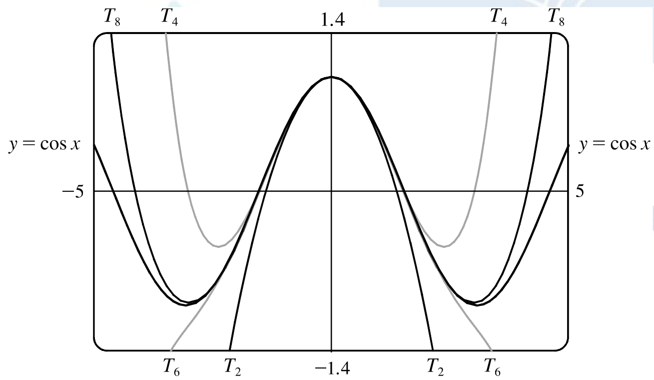
6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \dfrac{x^2}{2!}$, $T_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$, and $T_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3.11 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\cosh(\ln 5) = \frac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \frac{1}{2}\left(5 + (e^{\ln 5})^{-1}\right) = \frac{1}{2}(5 + 5^{-1}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}$

(b) $\cosh 5 = \frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$

4. (a) $\sinh 4 = \frac{1}{2}(e^4 - e^{-4}) \approx 27.28992$

(b) $\sinh(\ln 4) = \frac{1}{2}(e^{\ln 4} - e^{-\ln 4}) = \frac{1}{2}\left(4 - (e^{\ln 4})^{-1}\right) = \frac{1}{2}(4 - 4^{-1}) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^{-x} - e^x) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11.
$$\begin{aligned}\sinh x\, \cosh y + \cosh x\, \sinh y &= \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)\end{aligned}$$

12.
$$\begin{aligned}\cosh x\, \cosh y + \sinh x\, \sinh y &= \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})\right] \\ &= \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)\end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14.
$$\begin{aligned}\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x\, \cosh y + \cosh x\, \sinh y}{\cosh x\, \cosh y + \sinh x\, \sinh y} = \frac{\dfrac{\sinh x\, \cosh y}{\cosh x\, \cosh y} + \dfrac{\cosh x\, \sinh y}{\cosh x\, \cosh y}}{\dfrac{\cosh x\, \cosh y}{\cosh x\, \cosh y} + \dfrac{\sinh x\, \sinh y}{\cosh x\, \cosh y}} \\ &= \frac{\tanh x + \tanh y}{1 + \tanh x\, \tanh y}\end{aligned}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x+x) = \sinh x\, \cosh x + \cosh x\, \sinh x = 2\sinh x\, \cosh x$.

16. Putting $y = x$ in the result from Exercise 12, we have

$\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2 - 1)/x}{(x^2 + 1)/x} = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})} = \dfrac{e^x}{e^{-x}} = e^{2x}$

Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx.$

20. $\coth x = \dfrac{1}{\tanh x} \Rightarrow \coth x = \dfrac{1}{\tanh x} = \dfrac{1}{12/13} = \dfrac{13}{12}.$

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} \Rightarrow \operatorname{sech} x = \frac{5}{13}$ [sech, like cosh, is positive].

$\cosh x = \dfrac{1}{\operatorname{sech} x} \Rightarrow \cosh x = \dfrac{1}{5/13} = \dfrac{13}{5}.$

$\tanh x = \dfrac{\sinh x}{\cosh x} \Rightarrow \sinh x = \tanh x \cosh x \Rightarrow \sinh x = \dfrac{12}{13} \cdot \dfrac{13}{5} = \dfrac{12}{5}.$

$\operatorname{csch} x = \dfrac{1}{\sinh x} \Rightarrow \operatorname{csch} x = \dfrac{1}{12/5} = \dfrac{5}{12}.$

21. $\operatorname{sech} x = \dfrac{1}{\cosh x} \Rightarrow \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}.$

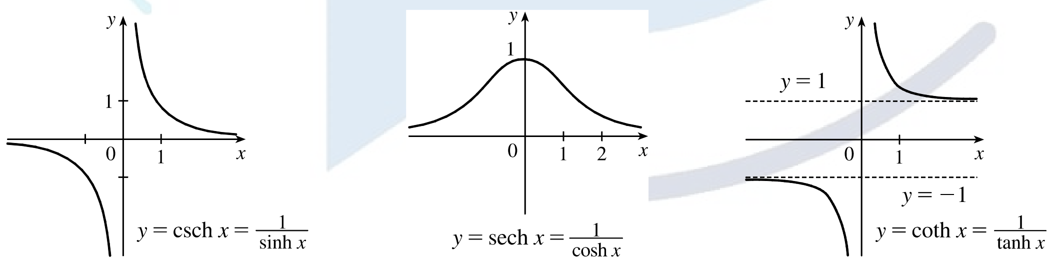
$\cosh^2 x - \sinh^2 x = 1 \Rightarrow \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \Rightarrow \sinh x = \frac{4}{3}$ [because $x > 0$].

$\operatorname{csch} x = \dfrac{1}{\sinh x} \Rightarrow \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}.$

$\tanh x = \dfrac{\sinh x}{\cosh x} \Rightarrow \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}.$

$\coth x = \dfrac{1}{\tanh x} \Rightarrow \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}.$

22. (a)



23. (a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \dfrac{1 - 0}{1 + 0} = 1$

(b) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^x}{e^x} = \lim\limits_{x\to-\infty} \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{0 - 1}{0 + 1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim\limits_{x\to 0^+} \coth x = \lim\limits_{x\to 0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim\limits_{x\to 0^-} \coth x = \lim\limits_{x\to 0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

(j) $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2e^x} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{2} = \dfrac{1-0}{2} = \dfrac{1}{2}$

24. (a) $\dfrac{d}{dx}(\cosh x) = \dfrac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\dfrac{d}{dx}(\tanh x) = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh x\,\cosh x - \sinh x\,\sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}(\operatorname{csch} x) = \dfrac{d}{dx}\left(\dfrac{1}{\sinh x}\right) = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x\,\coth x$

(d) $\dfrac{d}{dx}(\operatorname{sech} x) = \dfrac{d}{dx}\left(\dfrac{1}{\cosh x}\right) = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x\,\tanh x$

(e) $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left(\dfrac{\cosh x}{\sinh x}\right) = \dfrac{\sinh x\,\sinh x - \cosh x\,\cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$]

$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow\; y = \ln\!\left(x + \sqrt{1 + x^2}\right)$.

26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \ge 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9,

$e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \;\Rightarrow\; y = \ln\!\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

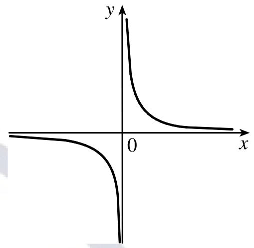
27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1 - x) \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\frac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right).$$

28. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



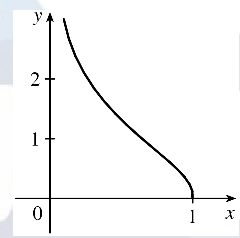
(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2 \quad \Rightarrow$

$x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2 + 1}}{x}$. But $e^y > 0$, so for $x > 0$,

$e^y = \dfrac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2 + 1}}{x}$. Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

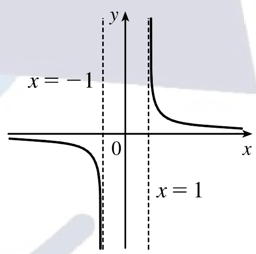
$x\,(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$. But $y > 0 \quad \Rightarrow \quad e^y > 1$.

This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x \quad \Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow$

$1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$.

Thus, $e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)e^y = (x + 1)e^{-y} \quad \Rightarrow \quad e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow$

$2y = \ln \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \quad \Rightarrow \quad \sinh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ [since $\sinh y \geq 0$ for $y \geq 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \Rightarrow -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$. By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2+1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2+1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2+1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \Rightarrow -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1-x^2}}$. [Note that $y > 0$ and so $\tanh y > 0$.]

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1-\coth^2 y} = \dfrac{1}{1-x^2}$ by Exercise 13.

30. $f(x) = e^x \cosh x \overset{\text{PR}}{\Rightarrow} f'(x) = e^x \sinh x + (\cosh x)e^x = e^x(\sinh x + \cosh x)$, or, using Exercise 9, $e^x(e^x) = e^{2x}$.

31. $f(x) = \tanh\sqrt{x} \Rightarrow f'(x) = \operatorname{sech}^2\sqrt{x}\,\dfrac{d}{dx}\sqrt{x} = \operatorname{sech}^2\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

32. $g(x) = \sinh^2 x = (\sinh x)^2 \Rightarrow g'(x) = 2(\sinh x)^1 \dfrac{d}{dx}(\sinh x) = 2\sinh x \cosh x$, or, using Exercise 15, $\sinh 2x$.

33. $h(x) = \sinh(x^2) \Rightarrow h'(x) = \cosh(x^2)\dfrac{d}{dx}(x^2) = 2x\cosh(x^2)$

34. $F(t) = \ln(\sinh t) \Rightarrow F'(t) = \dfrac{1}{\sinh t}\dfrac{d}{dt}\sinh t = \dfrac{1}{\sinh t}\cosh t = \coth t$

35. $G(t) = \sinh(\ln t) \Rightarrow G'(t) = \cosh(\ln t)\dfrac{d}{dt}\ln t = \dfrac{1}{2}\left(e^{\ln t} + e^{-\ln t}\right)\left(\dfrac{1}{t}\right) = \dfrac{1}{2t}\left(t + \dfrac{1}{t}\right) = \dfrac{1}{2t}\left(\dfrac{t^2+1}{t}\right) = \dfrac{t^2+1}{2t^2}$

Or: $G(t) = \sinh(\ln t) = \dfrac{1}{2}(e^{\ln t} - e^{-\ln t}) = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right) \Rightarrow G'(t) = \dfrac{1}{2}\left(1 + \dfrac{1}{t^2}\right) = \dfrac{t^2+1}{2t^2}$

36. $y = \operatorname{sech} x(1 + \ln\operatorname{sech} x) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} y' &= \operatorname{sech} x\,\frac{d}{dx}(1+\ln\operatorname{sech} x) + (1+\ln\operatorname{sech} x)\frac{d}{dx}\operatorname{sech} x \\ &= \operatorname{sech} x\left(\frac{-\operatorname{sech} x\tanh x}{\operatorname{sech} x}\right) + (1+\ln\operatorname{sech} x)(-\operatorname{sech} x\tanh x) \\ &= -\operatorname{sech} x\tanh x\,[1 + (1+\ln\operatorname{sech} x)] = -\operatorname{sech} x\tanh x\,(2+\ln\operatorname{sech} x) \end{aligned}$$

37. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x}\cdot\sinh 3x\cdot 3 = 3e^{\cosh 3x}\sinh 3x$

38. $f(t) = \dfrac{1+\sinh t}{1-\sinh t} \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(t) &= \frac{(1-\sinh t)\cosh t - (1+\sinh t)(-\cosh t)}{(1-\sinh t)^2} = \frac{\cosh t - \sinh t\cosh t + \cosh t + \sinh t\cosh t}{(1-\sinh t)^2} \\ &= \frac{2\cosh t}{(1-\sinh t)^2} \end{aligned}$$

39. $g(t) = t\coth\sqrt{t^2+1} \overset{\text{PR}}{\Rightarrow}$

$$g'(t) = t\left[-\operatorname{csch}^2\sqrt{t^2+1}\left(\tfrac{1}{2}(t^2+1)^{-1/2}\cdot 2t\right)\right] + \left(\coth\sqrt{t^2+1}\right)(1) = \coth\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}\operatorname{csch}^2\sqrt{t^2+1}$$

40. $y = \sinh^{-1}(\tan x) \;\Rightarrow\; y' = \dfrac{1}{\sqrt{1+(\tan x)^2}}\dfrac{d}{dx}(\tan x) = \dfrac{\sec^2 x}{\sqrt{\sec^2 x}} = \dfrac{|\sec^2 x|}{|\sec x|} = |\sec x|$

41. $y = \cosh^{-1}\sqrt{x} \;\Rightarrow\; y' = \dfrac{1}{\sqrt{(\sqrt{x})^2 - 1}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{\sqrt{x-1}}\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x(x-1)}}$

42. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \;\Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

43. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \;\Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

44. $y = \operatorname{sech}^{-1}(e^{-x}) \;\Rightarrow\; y' = -\dfrac{1}{e^{-x}\sqrt{1-(e^{-x})^2}}\dfrac{d}{dx}(e^{-x}) = -\dfrac{1}{e^{-x}\sqrt{1-e^{-2x}}}(-e^{-x}) = \dfrac{1}{\sqrt{1-e^{-2x}}}$

45. $y = \coth^{-1}(\sec x) \;\Rightarrow$

$$y' = \frac{1}{1-(\sec x)^2}\frac{d}{dx}(\sec x) = \frac{\sec x\,\tan x}{1-\sec^2 x} = \frac{\sec x\,\tan x}{1-(\tan^2 x+1)} = \frac{\sec x\,\tan x}{-\tan^2 x}$$

$$= -\frac{\sec x}{\tan x} = -\frac{1/\cos x}{\sin x/\cos x} = -\frac{1}{\sin x} = -\csc x$$

46. $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \dfrac{\cosh x+\sinh x}{\cosh x-\sinh x} = \dfrac{e^x}{e^{-x}}$ [by Exercises 9 and 10] $= e^{2x}$, so

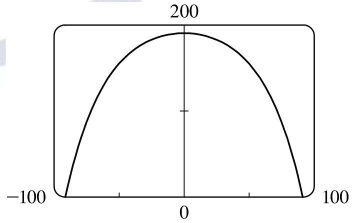
$\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \sqrt[4]{e^{2x}} = e^{x/2}$. Thus, $\dfrac{d}{dx}\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \dfrac{d}{dx}(e^{x/2}) = \dfrac{1}{2}e^{x/2}$.

47. $\dfrac{d}{dx}\arctan(\tanh x) = \dfrac{1}{1+(\tanh x)^2}\dfrac{d}{dx}(\tanh x) = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \dfrac{1/\cosh^2 x}{1+(\sinh^2 x)/\cosh^2 x}$

$$= \frac{1}{\cosh^2 x+\sinh^2 x} = \frac{1}{\cosh 2x} \text{ [by Exercise 16] } = \operatorname{sech} 2x$$

48. (a) Let $a = 0.03291765$. A graph of the central curve,

$y = f(x) = 211.49 - 20.96\cosh ax$, is shown.



(b) $f(0) = 211.49 - 20.96\cosh 0 = 211.49 - 20.96(1) = 190.53$ m.

(c) $y = 100 \;\Rightarrow\; 100 = 211.49 - 20.96\cosh ax \;\Rightarrow$

$20.96\cosh ax = 111.49 \;\Rightarrow\; \cosh ax = \dfrac{111.49}{20.96} \;\Rightarrow$

$ax = \pm\cosh^{-1}\dfrac{111.49}{20.96} \;\Rightarrow\; x = \pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96} \approx \pm 71.56$ m. The points are approximately $(\pm 71.56, 100)$.

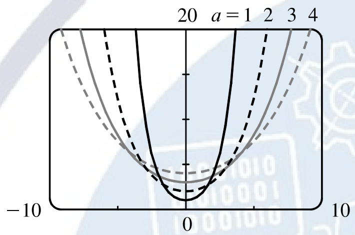
(d) $f(x) = 211.49 - 20.96\cosh ax \quad \Rightarrow \quad f'(x) = -20.96\sinh ax \cdot a.$

$f'\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right) = -20.96a\sinh\left[a\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right)\right] = -20.96a\sinh\left(\pm\cosh^{-1}\frac{111.49}{20.96}\right) \approx \mp 3.6.$

So the slope at $(71.56, 100)$ is about -3.6 and the slope at $(-71.56, 100)$ is about 3.6.

49. As the depth d of the water gets large, the fraction $\frac{2\pi d}{L}$ gets large, and from Figure 3 or Exercise 23(a), $\tanh\left(\frac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)} \approx \sqrt{\frac{gL}{2\pi}(1)} = \sqrt{\frac{gL}{2\pi}}$.

50.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

51. (a) $y = 20\cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66°$. Thus, the angle between the line and the pole is $\theta = 90° - \alpha \approx 70.34°$.

52. We differentiate the function twice, then substitute into the differential equation: $y = \frac{T}{\rho g}\cosh\frac{\rho g x}{T} \quad \Rightarrow$

$\frac{dy}{dx} = \frac{T}{\rho g}\sinh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \sinh\frac{\rho g x}{T} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}$. We evaluate the two sides separately: LHS $= \frac{d^2y}{dx^2} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}$ and RHS $= \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T}\sqrt{1 + \sinh^2\frac{\rho g x}{T}} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}$, by the identity proved in Example 1(a).

53. (a) From Exercise 52, the shape of the cable is given by $y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$. The shape is symmetric about the y-axis, so the lowest point is $(0, f(0)) = \left(0, \frac{T}{\rho g}\right)$ and the poles are at $x = \pm 100$. We want to find T when the lowest point is 60 m, so $\frac{T}{\rho g} = 60 \quad \Rightarrow \quad T = 60\rho g = (60\text{ m})(2\text{ kg/m})(9.8\text{ m/s}^2) = 1176\ \frac{\text{kg-m}}{\text{s}^2}$, or 1176 N (newtons).

The height of each pole is $f(100) = \frac{T}{\rho g}\cosh\left(\frac{\rho g \cdot 100}{T}\right) = 60\cosh\left(\frac{100}{60}\right) \approx 164.50$ m.

(b) If the tension is doubled from T to $2T$, then the low point is doubled since $\frac{T}{\rho g} = 60 \quad \Rightarrow \quad \frac{2T}{\rho g} = 120$. The height of the poles is now $f(100) = \frac{2T}{\rho g}\cosh\left(\frac{\rho g \cdot 100}{2T}\right) = 120\cosh\left(\frac{100}{120}\right) \approx 164.13$ m, just a slight decrease.

54. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} \sqrt{\dfrac{mg}{k}} \tanh\left(t\sqrt{\dfrac{gk}{m}}\right) = \sqrt{\dfrac{mg}{k}} \lim\limits_{t\to\infty} \tanh\left(t\sqrt{\dfrac{gk}{m}}\right) = \sqrt{\dfrac{mg}{k}} \cdot 1 \quad \left[\begin{array}{c}\text{as } t\to\infty,\\ t\sqrt{gk/m}\to\infty\end{array}\right] \quad = \sqrt{\dfrac{mg}{k}}$

(b) Belly-to-earth: $g = 9.8$, $k = 0.515$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.515}} \approx 33.79$ m/s.

Feet-first: $g = 9.8$, $k = 0.067$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.067}} \approx 93.68$ m/s.

55. (a) $y = A\sinh mx + B\cosh mx \quad\Rightarrow\quad y' = mA\cosh mx + mB\sinh mx \quad\Rightarrow$

$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \quad\Rightarrow\quad 6 = y'(0) = 3A \quad\Rightarrow\quad A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

56. $\cosh x = \cosh[\ln(\sec\theta + \tan\theta)] = \dfrac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{1}{\sec\theta+\tan\theta}\right]$

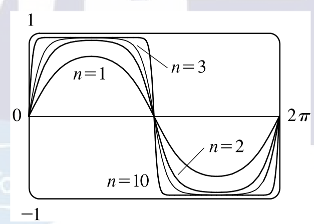
$= \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta-\tan\theta}{\sec^2\theta - \tan^2\theta}\right]$

$= \frac{1}{2}(\sec\theta + \tan\theta + \sec\theta - \tan\theta) = \sec\theta$

57. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \quad\Rightarrow\quad x = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1+\sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$.

58. $f_n(x) = \tanh(n\sin x)$, where n is a positive integer. Note that $f_n(x+2\pi) = f_n(x)$; that is, f_n is periodic with period 2π. Also, from Figure 3, $-1 < \tanh x < 1$, so we can choose a viewing rectangle of $[0, 2\pi] \times [-1, 1]$. From the graph, we see that $f_n(x)$ becomes more rectangular looking as n increases. As n becomes large, the graph of f_n approaches the graph of $y = 1$ on the intervals $(2k\pi, (2k+1)\pi)$ and $y = -1$ on the intervals $((2k-1)\pi, 2k\pi)$.



59. If $ae^x + be^{-x} = \alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\frac{a}{b} = \pm e^{2\beta} \quad\Rightarrow\quad (\star) \quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad\Rightarrow\quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us

$e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad\Rightarrow\quad \alpha^2 = \pm 4ab \quad\Rightarrow\quad \alpha = 2\sqrt{\pm ab}$.

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

3 Review

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

2. False. See the warning before the Product Rule.

3. True. This is the Chain Rule.

4. True. $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{d}{dx}[f(x)]^{1/2} = \dfrac{1}{2}[f(x)]^{-1/2}\,f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

5. False. $\dfrac{d}{dx}\,f(\sqrt{x}\,) = f'(\sqrt{x}\,)\cdot \frac{1}{2}x^{-1/2} = \dfrac{f'(\sqrt{x}\,)}{2\sqrt{x}}$, which is not $\dfrac{f'(x)}{2\sqrt{x}}$.

6. False. $y = e^2$ is a constant, so $y' = 0$, not $2e$.

7. False. $\dfrac{d}{dx}(10^x) = 10^x \ln 10$, which is not equal to $x10^{x-1}$.

8. False. ln 10 is a constant, so its derivative, $\dfrac{d}{dx}(\ln 10)$, is 0, not $\frac{1}{10}$.

9. True. $\dfrac{d}{dx}(\tan^2 x) = 2\,\tan x\,\sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2\,\sec x\,(\sec x\,\tan x) = 2\,\tan x\,\sec^2 x$.

Or: $\dfrac{d}{dx}(\sec^2 x) = \dfrac{d}{dx}(1 + \tan^2 x) = \dfrac{d}{dx}(\tan^2 x)$.

10. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$.

So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

11. True. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$, which is a polynomial.

12. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

13. True. If $r(x) = \dfrac{p(x)}{q(x)}$, then $r'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

14. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

15. True. $g(x) = x^5 \quad\Rightarrow\quad g'(x) = 5x^4 \quad\Rightarrow\quad g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,

$$\lim_{x\to 2}\frac{g(x) - g(2)}{x - 2} = g'(2) = 5(2)^4 = 80.$$

EXERCISES

1. $y = (x^2 + x^3)^4 \quad \Rightarrow \quad y' = 4(x^2 + x^3)^3(2x + 3x^2) = 4(x^2)^3(1 + x)^3 x(2 + 3x) = 4x^7(x + 1)^3(3x + 2)$

2. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \quad \Rightarrow \quad y' = -\dfrac{1}{2}x^{-3/2} + \dfrac{3}{5}x^{-8/5}$ or $\dfrac{3}{5x\sqrt[5]{x^3}} - \dfrac{1}{2x\sqrt{x}}$ or $\dfrac{1}{10}x^{-8/5}(-5x^{1/10} + 6)$

3. $y = \dfrac{x^2 - x + 2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \quad \Rightarrow \quad y' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2} - x^{-3/2} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

4. $y = \dfrac{\tan x}{1 + \cos x} \quad \Rightarrow \quad y' = \dfrac{(1 + \cos x)\sec^2 x - \tan x(-\sin x)}{(1 + \cos x)^2} = \dfrac{(1 + \cos x)\sec^2 x + \tan x\ \sin x}{(1 + \cos x)^2}$

5. $y = x^2 \sin \pi x \quad \Rightarrow \quad y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2 \sin \pi x)$

6. $y = x \cos^{-1} x \quad \Rightarrow \quad y' = x\left(-\dfrac{1}{\sqrt{1 - x^2}}\right) + (\cos^{-1} x)(1) = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}}$

7. $y = \dfrac{t^4 - 1}{t^4 + 1} \quad \Rightarrow \quad y' = \dfrac{(t^4 + 1)4t^3 - (t^4 - 1)4t^3}{(t^4 + 1)^2} = \dfrac{4t^3[(t^4 + 1) - (t^4 - 1)]}{(t^4 + 1)^2} = \dfrac{8t^3}{(t^4 + 1)^2}$

8. $\dfrac{d}{dx}(xe^y) = \dfrac{d}{dx}(y \sin x) \quad \Rightarrow \quad xe^y y' + e^y \cdot 1 = y \cos x + \sin x \cdot y' \quad \Rightarrow \quad xe^y y' - \sin x \cdot y' = y \cos x - e^y \quad \Rightarrow$

$(xe^y - \sin x)y' = y \cos x - e^y \quad \Rightarrow \quad y' = \dfrac{y \cos x - e^y}{xe^y - \sin x}$

9. $y = \ln(x \ln x) \quad \Rightarrow \quad y' = \dfrac{1}{x \ln x}(x \ln x)' = \dfrac{1}{x \ln x}\left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) = \dfrac{1 + \ln x}{x \ln x}$

Another method: $y = \ln(x \ln x) = \ln x + \ln \ln x \quad \Rightarrow \quad y' = \dfrac{1}{x} + \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{\ln x + 1}{x \ln x}$

10. $y = e^{mx} \cos nx \quad \Rightarrow$

$y' = e^{mx}(\cos nx)' + \cos nx\,(e^{mx})' = e^{mx}(-\sin nx \cdot n) + \cos nx\,(e^{mx} \cdot m) = e^{mx}(m \cos nx - n \sin nx)$

11. $y = \sqrt{x} \cos \sqrt{x} \quad \Rightarrow$

$y' = \sqrt{x}\,\left(\cos \sqrt{x}\right)' + \cos \sqrt{x}\,\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin \sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos \sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$

$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\,\sin \sqrt{x} + \cos \sqrt{x}\right) = \dfrac{\cos \sqrt{x} - \sqrt{x}\,\sin \sqrt{x}}{2\sqrt{x}}$

12. $y = (\arcsin 2x)^2 \quad \Rightarrow \quad y' = 2(\arcsin 2x) \cdot (\arcsin 2x)' = 2 \arcsin 2x \cdot \dfrac{1}{\sqrt{1 - (2x)^2}} \cdot 2 = \dfrac{4 \arcsin 2x}{\sqrt{1 - 4x^2}}$

13. $y = \dfrac{e^{1/x}}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2(e^{1/x})' - e^{1/x}\,(x^2)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1 + 2x)}{x^4}$

14. $y = \ln \sec x \quad \Rightarrow \quad y' = \dfrac{1}{\sec x}\,\dfrac{d}{dx}\,(\sec x) = \dfrac{1}{\sec x}\,(\sec x\ \tan x) = \tan x$

15. $\frac{d}{dx}(y + x\cos y) = \frac{d}{dx}(x^2 y) \Rightarrow y' + x(-\sin y \cdot y') + \cos y \cdot 1 = x^2 y' + y \cdot 2x \Rightarrow$

$y' - x\sin y \cdot y' - x^2 y' = 2xy - \cos y \Rightarrow (1 - x\sin y - x^2)y' = 2xy - \cos y \Rightarrow y' = \dfrac{2xy - \cos y}{1 - x\sin y - x^2}$

16. $y = \left(\dfrac{u-1}{u^2+u+1}\right)^4 \Rightarrow$

$$y' = 4\left(\frac{u-1}{u^2+u+1}\right)^3 \frac{d}{du}\left(\frac{u-1}{u^2+u+1}\right) = 4\left(\frac{u-1}{u^2+u+1}\right)^3 \frac{(u^2+u+1)(1)-(u-1)(2u+1)}{(u^2+u+1)^2}$$

$$= \frac{4(u-1)^3}{(u^2+u+1)^3}\,\frac{u^2+u+1-2u^2+u+1}{(u^2+u+1)^2} = \frac{4(u-1)^3(-u^2+2u+2)}{(u^2+u+1)^5}$$

17. $y = \sqrt{\arctan x} \Rightarrow y' = \frac{1}{2}(\arctan x)^{-1/2}\,\frac{d}{dx}(\arctan x) = \dfrac{1}{2\sqrt{\arctan x}\,(1+x^2)}$

18. $y = \cot(\csc x) \Rightarrow y' = -\csc^2(\csc x)\,\frac{d}{dx}(\csc x) = -\csc^2(\csc x)\cdot(-\csc x\cot x) = \csc^2(\csc x)\,\csc x\,\cot x$

19. $y = \tan\left(\dfrac{t}{1+t^2}\right) \Rightarrow$

$$y' = \sec^2\left(\frac{t}{1+t^2}\right)\frac{d}{dt}\left(\frac{t}{1+t^2}\right) = \sec^2\left(\frac{t}{1+t^2}\right)\cdot\frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}\sec^2\left(\frac{t}{1+t^2}\right)$$

20. $y = e^{x\sec x} \Rightarrow y' = e^{x\sec x}\,\frac{d}{dx}(x\sec x) = e^{x\sec x}(x\sec x\tan x + \sec x\cdot 1) = \sec x\, e^{x\sec x}(x\tan x + 1)$

21. $y = 3^{x\ln x} \Rightarrow y' = 3^{x\ln x}(\ln 3)\,\frac{d}{dx}(x\ln x) = 3^{x\ln x}(\ln 3)\left(x\cdot\frac{1}{x} + \ln x\cdot 1\right) = 3^{x\ln x}(\ln 3)(1 + \ln x)$

22. $y = \sec(1+x^2) \Rightarrow y' = 2x\sec(1+x^2)\tan(1+x^2)$

23. $y = (1 - x^{-1})^{-1} \Rightarrow$

$y' = -1(1-x^{-1})^{-2}[-(-1x^{-2})] = -(1-1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

24. $y = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}} = \left(x+\sqrt{x}\right)^{-1/3} \Rightarrow y' = -\frac{1}{3}\left(x+\sqrt{x}\right)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

25. $\sin(xy) = x^2 - y \Rightarrow \cos(xy)(xy' + y\cdot 1) = 2x - y' \Rightarrow x\cos(xy)y' + y' = 2x - y\cos(xy) \Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \Rightarrow y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

26. $y = \sqrt{\sin\sqrt{x}} \Rightarrow y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

27. $y = \log_5(1+2x) \Rightarrow y' = \dfrac{1}{(1+2x)\ln 5}\,\frac{d}{dx}(1+2x) = \dfrac{2}{(1+2x)\ln 5}$

28. $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x = x\ln\cos x \Rightarrow \dfrac{y'}{y} = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \Rightarrow$

$y' = (\cos x)^x(\ln\cos x - x\tan x)$

29. $y = \ln \sin x - \frac{1}{2}\sin^2 x \quad \Rightarrow \quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \cot x - \sin x \, \cos x$

30. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \quad \Rightarrow$

$$\ln y = \ln \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} = \ln(x^2+1)^4 - \ln[(2x+1)^3(3x-1)^5] = 4\ln(x^2+1) - [\ln(2x+1)^3 + \ln(3x-1)^5]$$
$$= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \quad \Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2+1} \cdot 2x - 3 \cdot \frac{1}{2x+1} \cdot 2 - 5 \cdot \frac{1}{3x-1} \cdot 3 \quad \Rightarrow \quad y' = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right).$$

[The answer could be simplified to $y' = -\dfrac{(x^2+56x+9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}$, but this is unnecessary.]

31. $y = x \tan^{-1}(4x) \quad \Rightarrow \quad y' = x \cdot \dfrac{1}{1+(4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x) \quad \Rightarrow \quad y' = e^{\cos x}(-\sin x) + [-\sin(e^x) \cdot e^x] = -\sin x \, e^{\cos x} - e^x \sin(e^x)$

33. $y = \ln|\sec 5x + \tan 5x| \quad \Rightarrow$

$$y' = \frac{1}{\sec 5x + \tan 5x}(\sec 5x \, \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \frac{5 \sec 5x \, (\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5 \sec 5x$$

34. $y = 10^{\tan \pi\theta} \quad \Rightarrow \quad y' = 10^{\tan \pi\theta} \cdot \ln 10 \cdot \sec^2 \pi\theta \cdot \pi = \pi(\ln 10)10^{\tan \pi\theta} \sec^2 \pi\theta$

35. $y = \cot(3x^2+5) \quad \Rightarrow \quad y' = -\csc^2(3x^2+5)(6x) = -6x\csc^2(3x^2+5)$

36. $y = \sqrt{t \ln(t^4)} \quad \Rightarrow$

$$y' = \frac{1}{2}[t\ln(t^4)]^{-1/2} \frac{d}{dt}[t \ln(t^4)] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot \left[1 \cdot \ln(t^4) + t \cdot \frac{1}{t^4} \cdot 4t^3\right] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot [\ln(t^4) + 4] = \frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}}$$

Or: Since y is only defined for $t > 0$, we can write $y = \sqrt{t \cdot 4 \ln t} = 2\sqrt{t \ln t}$. Then

$y' = 2 \cdot \dfrac{1}{2\sqrt{t\ln t}} \cdot \left(1 \cdot \ln t + t \cdot \dfrac{1}{t}\right) = \dfrac{\ln t + 1}{\sqrt{t \ln t}}$. This agrees with our first answer since

$$\frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}} = \frac{4\ln t + 4}{2\sqrt{t \cdot 4 \ln t}} = \frac{4(\ln t + 1)}{2 \cdot 2\sqrt{t \ln t}} = \frac{\ln t + 1}{\sqrt{t \ln t}}.$$

37. $y = \sin\left(\tan\sqrt{1+x^3}\right) \quad \Rightarrow \quad y' = \cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\left[3x^2/\left(2\sqrt{1+x^3}\right)\right]$

38. $y = \arctan\left(\arcsin\sqrt{x}\right) \quad \Rightarrow \quad y' = \dfrac{1}{1+\left(\arcsin\sqrt{x}\right)^2} \cdot \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

39. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

40. $xe^y = y - 1 \quad \Rightarrow \quad xe^y y' + e^y = y' \quad \Rightarrow \quad e^y = y' - xe^y y' \quad \Rightarrow \quad y' = e^y/(1 - xe^y)$

41. $y = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \quad\Rightarrow\quad \dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow$

$y' = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\left[\dfrac{1}{2(x+1)} - \dfrac{5}{2-x} - \dfrac{7}{x+3}\right]$ or $y' = \dfrac{(2-x)^4(3x^2-55x-52)}{2\sqrt{x+1}\,(x+3)^8}$.

42. $y = \dfrac{(x+\lambda)^4}{x^4+\lambda^4} \quad\Rightarrow\quad y' = \dfrac{(x^4+\lambda^4)(4)(x+\lambda)^3 - (x+\lambda)^4(4x^3)}{(x^4+\lambda^4)^2} = \dfrac{4(x+\lambda)^3(\lambda^4-\lambda x^3)}{(x^4+\lambda^4)^2}$

43. $y = x\sinh(x^2) \quad\Rightarrow\quad y' = x\cosh(x^2)\cdot 2x + \sinh(x^2)\cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

44. $y = (\sin mx)/x \quad\Rightarrow\quad y' = (mx\cos mx - \sin mx)/x^2$

45. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

46. $y = \ln\left|\dfrac{x^2-4}{2x+5}\right| = \ln|x^2-4| - \ln|2x+5| \quad\Rightarrow\quad y' = \dfrac{2x}{x^2-4} - \dfrac{2}{2x+5}$ or $\dfrac{2(x+1)(x+4)}{(x+2)(x-2)(2x+5)}$

47. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sinh x)^2-1}}\cdot\cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

48. $y = x\tanh^{-1}\sqrt{x} \quad\Rightarrow\quad y' = \tanh^{-1}\sqrt{x} + x\,\dfrac{1}{1-\left(\sqrt{x}\right)^2}\,\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

49. $y = \cos\left(e^{\sqrt{\tan 3x}}\right) \quad\Rightarrow$

$$y' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\cdot\tfrac{1}{2}(\tan 3x)^{-1/2}\cdot\sec^2(3x)\cdot 3$$
$$= \frac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

50. $y = \sin^2\left(\cos\sqrt{\sin\pi x}\right) = \left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]^2 \quad\Rightarrow$

$$y' = 2\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(\cos\sqrt{\sin\pi x}\right)'$$
$$= 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(-\sin\sqrt{\sin\pi x}\right)\left(\sqrt{\sin\pi x}\right)'$$
$$= -2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\cdot\tfrac{1}{2}(\sin\pi x)^{-1/2}(\sin\pi x)'$$
$$= \frac{-\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}}{\sqrt{\sin\pi x}}\cdot\cos\pi x\cdot\pi$$
$$= \frac{-\pi\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\,\cos\pi x}{\sqrt{\sin\pi x}}$$

51. $f(t) = \sqrt{4t+1} \quad\Rightarrow\quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad\Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

52. $g(\theta) = \theta \sin\theta \quad\Rightarrow\quad g'(\theta) = \theta\cos\theta + \sin\theta \cdot 1 \quad\Rightarrow\quad g''(\theta) = \theta(-\sin\theta) + \cos\theta \cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

53. $x^6 + y^6 = 1 \quad\Rightarrow\quad 6x^5 + 6y^5 y' = 0 \quad\Rightarrow\quad y' = -x^5/y^5 \quad\Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4 y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6 + x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

54. $f(x) = (2-x)^{-1} \quad\Rightarrow\quad f'(x) = (2-x)^{-2} \quad\Rightarrow\quad f''(x) = 2(2-x)^{-3} \quad\Rightarrow\quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad\Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

55. We first show it is true for $n = 1$: $f(x) = xe^x \quad\Rightarrow\quad f'(x) = xe^x + e^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = (x+k)e^x + e^x = [(x+k)+1]e^x = [x + (k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. $$\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t \cdot \frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\displaystyle\lim_{t\to 0}\frac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$$

57. $y = 4\sin^2 x \quad\Rightarrow\quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

58. $y = \dfrac{x^2 - 1}{x^2 + 1} \quad\Rightarrow\quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0, -1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x - 0)$, or $y = -1$.

59. $y = \sqrt{1 + 4\sin x} \quad\Rightarrow\quad y' = \frac{1}{2}(1 + 4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1 + 4\sin x}}$.

At $(0, 1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

60. $x^2 + 4xy + y^2 = 13 \quad\Rightarrow\quad 2x + 4(xy' + y\cdot 1) + 2yy' = 0 \quad\Rightarrow\quad x + 2xy' + 2y + yy' = 0 \quad\Rightarrow$

$2xy' + yy' = -x - 2y \quad\Rightarrow\quad y'(2x + y) = -x - 2y \quad\Rightarrow\quad y' = \dfrac{-x - 2y}{2x + y}$.

At $(2, 1)$, $y' = \dfrac{-2-2}{4+1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

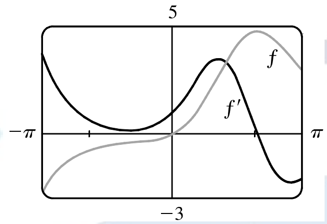
The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

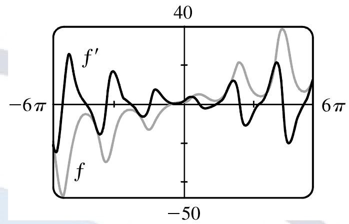
61. $y = (2 + x)e^{-x} \quad\Rightarrow\quad y' = (2 + x)(-e^{-x}) + e^{-x}\cdot 1 = e^{-x}[-(2 + x) + 1] = e^{-x}(-x - 1)$.

At $(0, 2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

The slope of the normal line is 1, so an equation of the normal line is $y - 2 = 1(x - 0)$, or $y = x + 2$.

62. $f(x) = xe^{\sin x} \quad \Rightarrow \quad f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





63. (a) $f(x) = x\sqrt{5-x} \quad \Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}} = \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}}$$

$$= \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

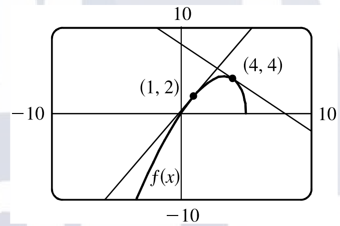
(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

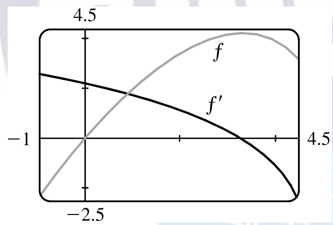
At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



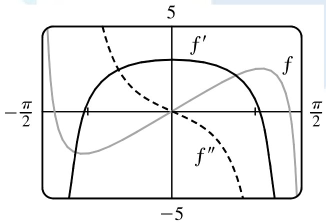
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

64. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

65. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

66. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

67. $f(x) = (x-a)(x-b)(x-c) \Rightarrow f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

Or: $f(x) = (x-a)(x-b)(x-c) \Rightarrow \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \Rightarrow$

$\dfrac{f'(x)}{f(x)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$

68. (a) $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \Leftrightarrow \sin 2x = 2\sin x \cos x$

(b) $\sin(x+a) = \sin x \cos a + \cos x \sin a \Rightarrow \cos(x+a) = \cos x \cos a - \sin x \sin a$.

69. (a) $S(x) = f(x) + g(x) \Rightarrow S'(x) = f'(x) + g'(x) \Rightarrow S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)\,g(x) \Rightarrow P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)f'(2) = 1(4) + 1(2) = 4 + 2 = 6$

(c) $Q(x) = \dfrac{f(x)}{g(x)} \Rightarrow Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \Rightarrow$

$Q'(1) = \dfrac{g(1)\,f'(1) - f(1)\,g'(1)}{[g(1)]^2} = \dfrac{3(3) - 2(1)}{3^2} = \dfrac{9-2}{9} = \dfrac{7}{9}$

(d) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))\,g'(x) \Rightarrow C'(2) = f'(g(2))\,g'(2) = f'(1)\cdot 4 = 3\cdot 4 = 12$

70. (a) $P(x) = f(x)\,g(x) \Rightarrow P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \Rightarrow Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \Rightarrow$

$Q'(2) = \dfrac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

71. $f(x) = x^2 g(x) \Rightarrow f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

72. $f(x) = g(x^2) \Rightarrow f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

73. $f(x) = [g(x)]^2 \Rightarrow f'(x) = 2[g(x)]\cdot g'(x) = 2g(x)\,g'(x)$

74. $f(x) = g(g(x)) \Rightarrow f'(x) = g'(g(x))\,g'(x)$

75. $f(x) = g(e^x) \Rightarrow f'(x) = g'(e^x)\,e^x$

76. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)} g'(x)$

77. $f(x) = \ln|g(x)| \Rightarrow f'(x) = \dfrac{1}{g(x)} g'(x) = \dfrac{g'(x)}{g(x)}$

78. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x) \cdot \dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

79. $h(x) = \dfrac{f(x)\,g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$h'(x) = \frac{[f(x) + g(x)]\,[f(x)\,g'(x) + g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2}$$

$$= \frac{[f(x)]^2\,g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [\,g(x)]^2\,f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x) + g(x)]^2}$$

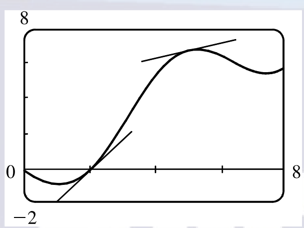
$$= \frac{f'(x)\,[\,g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2}$$

80. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

81. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$h'(x) = f'(g(\sin 4x)) \cdot \dfrac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \dfrac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$

82. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$,

so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$.

So $f'(2) > f'(5)$, as predicted in part (c).

83. $y = [\ln(x + 4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x + 4)]^1 \cdot \dfrac{1}{x + 4} \cdot 1 = 2\,\dfrac{\ln(x + 4)}{x + 4}$ and $y' = 0 \quad \Leftrightarrow \quad \ln(x + 4) = 0 \quad \Leftrightarrow$

$x + 4 = e^0 \quad \Rightarrow \quad x + 4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

84. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow \quad x = \ln\frac{1}{4} = -\ln 4$.

Since $y = e^x$, the y-coordinate is $\frac{1}{4}$ and the point of tangency is $\left(-\ln 4, \frac{1}{4}\right)$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\dfrac{d}{dx}\,e^x\right|_{x=a} = e^a$. Thus, an equation of the tangent line is

$y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0 - a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent line at the point $(a, e^a) = (1, e)$ is $y - e = e(x - 1)$ or $y = ex$.

85. $y = f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad \Rightarrow \quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad \Rightarrow \quad 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

86. (a) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} [K(e^{-at} - e^{-bt})] = K \lim_{t\to\infty} (e^{-at} - e^{-bt}) = K(0-0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K(e^{-at} - e^{-bt}) \quad \Rightarrow \quad C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$

(c) $C'(t) = 0 \quad \Leftrightarrow \quad be^{-bt} = ae^{-at} \quad \Leftrightarrow \quad \dfrac{b}{a} = e^{(-a+b)t} \quad \Leftrightarrow \quad \ln\dfrac{b}{a} = (b-a)t \quad \Leftrightarrow \quad t = \dfrac{\ln(b/a)}{b-a}$

87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \quad \Rightarrow$

$v(t) = s'(t) = A\{e^{-ct}[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\} = -Ae^{-ct}[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)] \quad \Rightarrow$

$$\begin{aligned} a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)](-ce^{-ct})\} \\ &= -Ae^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c^2\cos(\omega t+\delta)] \\ &= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t+\delta) - 2c\omega\sin(\omega t+\delta)] = Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)] \end{aligned}$$

88. (a) $x = \sqrt{b^2 + c^2t^2} \quad \Rightarrow \quad v(t) = x' = \left[1/\left(2\sqrt{b^2+c^2t^2}\right)\right]2c^2t = c^2t/\sqrt{b^2+c^2t^2} \quad \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2+c^2t^2} - c^2t\left(c^2t/\sqrt{b^2+c^2t^2}\right)}{b^2+c^2t^2} = \frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}$$

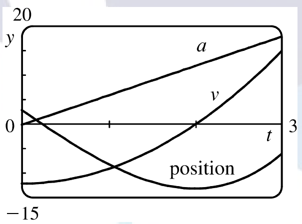
(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

89. (a) $y = t^3 - 12t + 3 \quad \Rightarrow \quad v(t) = y' = 3t^2 - 12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,
Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

90. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dr = \frac{2}{3}\pi rh \quad [h \text{ constant}]$

91. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1 + \sqrt{x}\right) = x + x^{3/2} \quad \Rightarrow \quad \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

92. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

93. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \quad\Rightarrow\quad y(0.5) = 200e^{0.5k} = 360 \quad\Rightarrow\quad e^{0.5k} = 1.8 \quad\Rightarrow\quad 0.5k = \ln 1.8 \quad\Rightarrow$

$k = 2\ln 1.8 = \ln(1.8)^2 = \ln 3.24 \quad\Rightarrow\quad y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ bacteria

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ bacteria per hour

(d) $200(3.24)^t = 10{,}000 \quad\Rightarrow\quad (3.24)^t = 50 \quad\Rightarrow\quad t\ln 3.24 = \ln 50 \quad\Rightarrow\quad t = \ln 50/\ln 3.24 \approx 3.33$ hours

94. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(5.24) = 100e^{5.24k} = \frac{1}{2}\cdot 100 \quad\Rightarrow$

$e^{5.24k} = \frac{1}{2} \quad\Rightarrow\quad 5.24k = -\ln 2 \quad\Rightarrow\quad k = -\frac{1}{5.24}\ln 2 \quad\Rightarrow\quad y(t) = 100e^{-(\ln 2)t/5.24} = 100\cdot 2^{-t/5.24}$. Thus,

$y(20) = 100\cdot 2^{-20/5.24} \approx 7.1$ mg.

(b) $100\cdot 2^{-t/5.24} = 1 \quad\Rightarrow\quad 2^{-t/5.24} = \dfrac{1}{100} \quad\Rightarrow\quad -\dfrac{t}{5.24}\ln 2 = \ln\dfrac{1}{100} \quad\Rightarrow\quad t = 5.24\,\dfrac{\ln 100}{\ln 2} \approx 34.8$ years

95. (a) $C'(t) = -kC(t) \quad\Rightarrow\quad C(t) = C(0)e^{-kt}$ by Theorem 3.8.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \quad\Rightarrow \ln\frac{1}{2} = -30k \quad\Rightarrow$

$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \quad\Rightarrow\quad \ln 0.1 = -t(\ln 2)/30 \quad\Rightarrow\quad t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

96. (a) If $y = u - 20$, $u(0) = 80 \quad\Rightarrow\quad y(0) = 80 - 20 = 60$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 60$.

So the solution is $y(t) = 60e^{kt}$. Now $y(0.5) = 60e^{k(0.5)} = 60 - 20 \quad\Rightarrow\quad e^{0.5k} = \frac{40}{60} = \frac{2}{3} \quad\Rightarrow\quad k = 2\ln\frac{2}{3} = \ln\frac{4}{9}$,

so $y(t) = 60e^{(\ln 4/9)t} = 60(\frac{4}{9})^t$. Thus, $y(1) = 60(\frac{4}{9})^1 = \frac{80}{3} = 26\frac{2}{3}$ °C and $u(1) = 46\frac{2}{3}$ °C.

(b) $u(t) = 40 \quad\Rightarrow\quad y(t) = 20.\quad y(t) = 60\left(\dfrac{4}{9}\right)^t = 20 \quad\Rightarrow\quad \left(\dfrac{4}{9}\right)^t = \dfrac{1}{3} \quad\Rightarrow\quad t\ln\dfrac{4}{9} = \ln\dfrac{1}{3} \quad\Rightarrow\quad t = \dfrac{\ln\frac{1}{3}}{\ln\frac{4}{9}} \approx 1.35$ h

or 81.3 min.

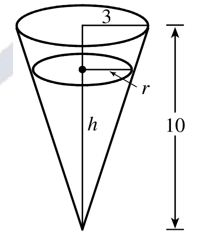
97. If x = edge length, then $V = x^3 \quad\Rightarrow\quad dV/dt = 3x^2\,dx/dt = 10 \quad\Rightarrow\quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad\Rightarrow$

$dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm²/min.

98. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad\Rightarrow\quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$2 = \dfrac{dV}{dt} = \dfrac{9\pi}{100}h^2\,\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{200}{9\pi h^2} = \dfrac{200}{9\pi\,(5)^2} = \dfrac{8}{9\pi}$ cm/s

when $h = 5$.

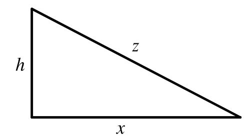


99. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \quad\Rightarrow$

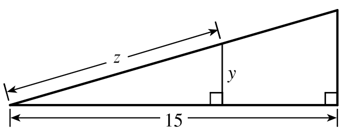
$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \quad\Rightarrow\quad z = \sqrt{45^2 + 60^2} = 75$,

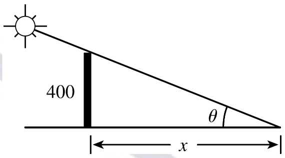
so $\dfrac{dz}{dt} = \dfrac{1}{75}[15(45) + 5(60)] = 13$ ft/s.



100. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}} \Rightarrow$ $y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



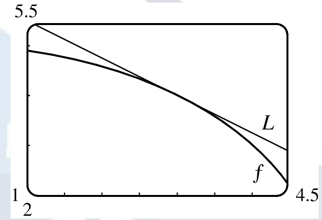
101. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \Rightarrow$ $x = 400\cot\theta \Rightarrow \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$, $\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.



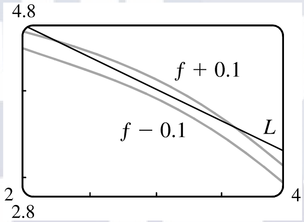
102. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3 is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

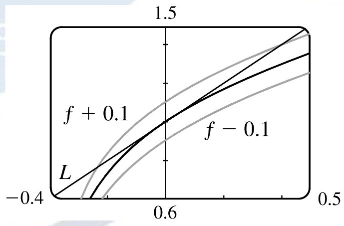


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



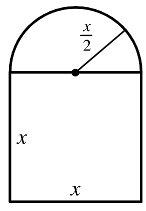
103. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \Rightarrow f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph, it appears that this is true when $-0.235 < x < 0.401$.



104. $y = x^3 - 2x^2 + 1 \Rightarrow dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

105. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \Rightarrow dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\text{ cm}^2$.



106. $\lim_{x\to 1} \dfrac{x^{17}-1}{x-1} = \left[\dfrac{d}{dx}\,x^{17}\right]_{x=1} = 17(1)^{16} = 17$

107. $\lim_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = \left[\dfrac{d}{dx}\,\sqrt[4]{x}\right]_{x=16} = \dfrac{1}{4}x^{-3/4}\Big|_{x=16} = \dfrac{1}{4\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{32}$

108. $\lim_{\theta\to \pi/3} \dfrac{\cos\theta - 0.5}{\theta-\pi/3} = \left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3} = -\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$

109.
$$\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} = \lim_{x\to 0} \frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$
$$= \lim_{x\to 0} \frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} = \lim_{x\to 0} \frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}$$
$$= \lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0} \frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3 \lim_{x\to 0} \frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= 1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)} = \frac{1}{4}$$

110. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \quad\Rightarrow$ $f'(g(x))\,g'(x) = 1 \quad\Rightarrow\quad g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1+x^2}$.

111. $\dfrac{d}{dx}\,[f(2x)] = x^2 \quad\Rightarrow\quad f'(2x)\cdot 2 = x^2 \quad\Rightarrow\quad f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

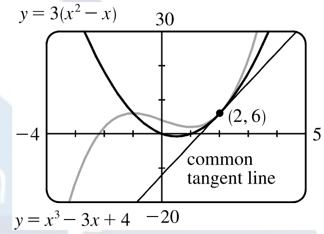
112. Let (b,c) be on the curve, that is, $b^{2/3}+c^{2/3} = a^{2/3}$. Now $x^{2/3}+y^{2/3} = a^{2/3} \quad\Rightarrow\quad \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$, so at (b,c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y-c = -(c/b)^{1/3}(x-b)$ or $y = -(c/b)^{1/3}x + (c+b^{2/3}c^{1/3})$. Setting $y=0$, we find that the x-intercept is $b^{1/3}c^{2/3}+b = b^{1/3}(c^{2/3}+b^{2/3}) = b^{1/3}a^{2/3}$ and setting $x=0$ we find that the y-intercept is $c+b^{2/3}c^{1/3} = c^{1/3}(c^{2/3}+b^{2/3}) = c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is
$$\sqrt{(b^{1/3}a^{2/3})^2+(c^{1/3}a^{2/3})^2} = \sqrt{b^{2/3}a^{4/3}+c^{2/3}a^{4/3}} = \sqrt{(b^{2/3}+c^{2/3})a^{4/3}}$$
$$= \sqrt{a^{2/3}a^{4/3}} = \sqrt{a^2} = a = \text{constant}$$

PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad \Rightarrow \quad a = \frac{\sqrt{3}}{2}$.

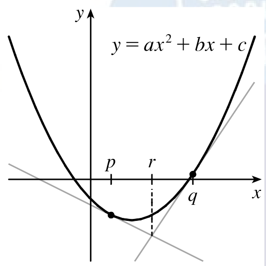
Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \quad \Rightarrow \quad y' = 3x^2 - 3$, and $y = 3(x^2 - x) \quad \Rightarrow \quad y' = 6x - 3$.



The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.

3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or

$$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$[(2aq + b) - (2ap + b)]x - aq^2 + ap^2 = 0$$
$$(2aq - 2ap)x = aq^2 - ap^2$$
$$2a(q - p)x = a(q^2 - p^2)$$
$$x = \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2}$$

Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} = \frac{\sin^2 x}{1 + \dfrac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \dfrac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x}$$

$$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad \text{[factor sum of cubes]} \quad = \frac{(\sin x + \cos x)(\sin^2 x - \sin x \, \cos x + \cos^2 x)}{\sin x + \cos x}$$

$$= \sin^2 x - \sin x \, \cos x + \cos^2 x = 1 - \sin x \, \cos x = 1 - \tfrac{1}{2}(2 \sin x \, \cos x) = 1 - \tfrac{1}{2} \sin 2x$$

Thus, $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = \dfrac{d}{dx}\left(1 - \tfrac{1}{2} \sin 2x\right) = -\tfrac{1}{2} \cos 2x \cdot 2 = -\cos 2x$.

5. Using $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, we recognize the given expression, $f(x) = \lim\limits_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, as $g'(x)$ with $g(x) = \sec x$. Now $f'(\frac{\pi}{4}) = g''(\frac{\pi}{4})$, so we will find $g''(x)$. $g'(x) = \sec x \tan x \quad \Rightarrow$

$g''(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x(\sec^2 x + \tan^2 x)$, so $g''(\frac{\pi}{4}) = \sqrt{2}(\sqrt{2}^2 + 1^2) = \sqrt{2}(2 + 1) = 3\sqrt{2}$.

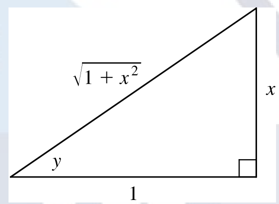
6. Using $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$, we see that for the given equation, $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{ax + b} - 2}{x} = \dfrac{5}{12}$, we have $f(x) = \sqrt[3]{ax + b}$, $f(0) = 2$, and $f'(0) = \frac{5}{12}$. Now $f(0) = 2 \quad \Leftrightarrow \quad \sqrt[3]{b} = 2 \quad \Leftrightarrow \quad b = 8$. Also $f'(x) = \frac{1}{3}(ax + b)^{-2/3} \cdot a$, so $f'(0) = \frac{5}{12}$ $\Leftrightarrow \quad \frac{1}{3}(8)^{-2/3} \cdot a = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(\frac{1}{4})a = \frac{5}{12} \quad \Leftrightarrow \quad a = 5$.

7. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1 + x^2}}$. Using this fact we have that

$\sin(\tan^{-1}(\sinh x)) = \dfrac{\sinh x}{\sqrt{1 + \sinh^2 x}} = \dfrac{\sinh x}{\cosh x} = \tanh x$.

Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.



8. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \quad \Rightarrow \quad a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \quad \Rightarrow \quad x_0^2 - 200x_0 + 5000 = 0 \quad \Rightarrow$

$x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right] \quad \Rightarrow \quad x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

9. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\,\cos x - 4\cos^3 x\,\sin x = 4\sin x\,\cos x\left(\sin^2 x - \cos^2 x\right)x \\ &= -4\sin x\,\cos x\,\cos 2x = -2\sin 2x\,\cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1\end{aligned}$$

[continued]

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x + \cos^4 x\right) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right]\\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right)\\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\,\cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4}\cdot 4^n\cos\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\dfrac{\pi}{2}\right)$.

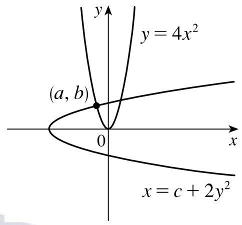
10. $$\begin{aligned}\lim_{x\to a}\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} &= \lim_{x\to a}\left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}\cdot\frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}\cdot\left(\sqrt{x} + \sqrt{a}\right)\right]\\ &= \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\cdot\lim_{x\to a}\left(\sqrt{x} + \sqrt{a}\right) = f'(a)\cdot\left(\sqrt{a} + \sqrt{a}\right) = 2\sqrt{a}\,f'(a)\end{aligned}$$

11. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \dfrac{1}{2}$.

We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \;\Leftrightarrow\; x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \;\Leftrightarrow\; x^2 + x^4 - 2ax^2 + a^2 = 1 \;\Leftrightarrow\; x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \;\Leftrightarrow\; 1 - 4a + 4a^2 - 4a^2 + 4 = 0 \;\Leftrightarrow\; 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

12. See the figure. The parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles at the point (a, b) if and only if (a, b) satisfies both equations and the tangent lines at (a, b) are perpendicular. $y = 4x^2 \Rightarrow y' = 8x$ and $x = c + 2y^2 \Rightarrow 1 = 4y\,y' \Rightarrow y' = \dfrac{1}{4y}$, so at (a, b) we must

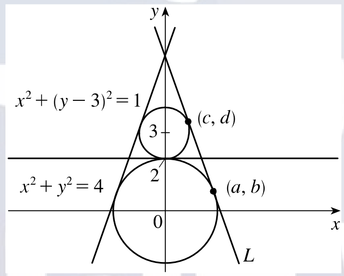


have $8a = -\dfrac{1}{1/(4b)} \Rightarrow 8a = -4b \Rightarrow b = -2a$. Since (a, b) is on both parabolas, we have **(1)** $b = 4a^2$ and **(2)** $a = c + 2b^2$. Substituting $-2a$ for b in **(1)** gives us $-2a = 4a^2 \Rightarrow 4a^2 + 2a = 0 \Rightarrow 2a(2a + 1) = 0 \Rightarrow a = 0$ or $a = -\frac{1}{2}$.

If $a = 0$, then $b = 0$ and $c = 0$, and the tangent lines at $(0, 0)$ are $y = 0$ and $x = 0$.

If $a = -\frac{1}{2}$, then $b = -2\left(-\frac{1}{2}\right) = 1$ and $-\frac{1}{2} = c + 2(1)^2 \Rightarrow c = -\frac{5}{2}$, and the tangent lines at $\left(-\frac{1}{2}, 1\right)$ are $y - 1 = -4\left(x + \frac{1}{2}\right)$ [or $y = -4x - 1$] and $y - 1 = \frac{1}{4}\left(x + \frac{1}{2}\right)$ $\left[\text{or } y = \frac{1}{4}x + \frac{9}{8}\right]$.

13. See the figure. Clearly, the line $y = 2$ is tangent to both circles at the point $(0, 2)$. We'll look for a tangent line L through the points (a, b) and (c, d), and if such a line exists, then its reflection through the y-axis is another such line. The slope of L is the same at (a, b) and (c, d). Find those slopes: $x^2 + y^2 = 4 \Rightarrow$ $2x + 2y\,y' = 0 \Rightarrow y' = -\dfrac{x}{y}$ $\left[= -\dfrac{a}{b}\right]$ and $x^2 + (y - 3)^2 = 1 \Rightarrow$ $2x + 2(y - 3)y' = 0 \Rightarrow y' = -\dfrac{x}{y - 3}$ $\left[= -\dfrac{c}{d - 3}\right]$.



Now an equation for L can be written using either point-slope pair, so we get $y - b = -\dfrac{a}{b}(x - a)$ $\left[\text{or } y = -\dfrac{a}{b}x + \dfrac{a^2}{b} + b\right]$ and $y - d = -\dfrac{c}{d - 3}(x - c)$ $\left[\text{or } y = -\dfrac{c}{d - 3}\,x + \dfrac{c^2}{d - 3} + d\right]$. The slopes are equal, so $-\dfrac{a}{b} = -\dfrac{c}{d - 3} \Leftrightarrow$ $d - 3 = \dfrac{bc}{a}$. Since (c, d) is a solution of $x^2 + (y - 3)^2 = 1$, we have $c^2 + (d - 3)^2 = 1$, so $c^2 + \left(\dfrac{bc}{a}\right)^2 = 1 \Rightarrow$ $a^2c^2 + b^2c^2 = a^2 \Rightarrow c^2(a^2 + b^2) = a^2 \Rightarrow 4c^2 = a^2$ [since (a, b) is a solution of $x^2 + y^2 = 4$] $\Rightarrow a = 2c$.

Now $d - 3 = \dfrac{bc}{a} \Rightarrow d = 3 + \dfrac{bc}{2c}$, so $d = 3 + \dfrac{b}{2}$. The y-intercepts are equal, so $\dfrac{a^2}{b} + b = \dfrac{c^2}{d - 3} + d \Leftrightarrow$ $\dfrac{a^2}{b} + b = \dfrac{(a/2)^2}{b/2} + \left(3 + \dfrac{b}{2}\right) \Leftrightarrow \left[\dfrac{a^2}{b} + b = \dfrac{a^2}{2b} + 3 + \dfrac{b}{2}\right](2b) \Leftrightarrow 2a^2 + 2b^2 = a^2 + 6b + b^2 \Leftrightarrow$ $a^2 + b^2 = 6b \Leftrightarrow 4 = 6b \Leftrightarrow b = \frac{2}{3}$. It follows that $d = 3 + \dfrac{b}{2} = \dfrac{10}{3}$, $a^2 = 4 - b^2 = 4 - \frac{4}{9} = \frac{32}{9} \Rightarrow a = \frac{4}{3}\sqrt{2}$, and $c^2 = 1 - (d - 3)^2 = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} \Rightarrow c = \frac{2}{3}\sqrt{2}$. Thus, L has equation $y - \dfrac{2}{3} = -\dfrac{(4/3)\sqrt{2}}{2/3}\left(x - \dfrac{4}{3}\sqrt{2}\right) \Leftrightarrow$ $y - \frac{2}{3} = -2\sqrt{2}\left(x - \frac{4}{3}\sqrt{2}\right) \Leftrightarrow y = -2\sqrt{2}\,x + 6$. Its reflection has equation $y = 2\sqrt{2}\,x + 6$.

[continued]

In summary, there are three lines tangent to both circles: $y = 2$ touches at $(0, 2)$, L touches at $\left(\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$, and its reflection through the y-axis touches at $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$.

14. $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x} = \dfrac{x^{45}(x+1) + 2}{x+1} = \dfrac{x^{45}(x+1)}{x+1} + \dfrac{2}{x+1} = x^{45} + 2(x+1)^{-1}$, so

$f^{(46)}(x) = (x^{45})^{(46)} + 2\left[(x+1)^{-1}\right]^{(46)}$. The forty-sixth derivative of any forty-fifth degree polynomial is 0, so $(x^{45})^{46} = 0$. Thus, $f^{(46)}(x) = 2\left[(-1)(-2)(-3)\cdots(-46)(x+1)^{-47}\right] = 2(46!)(x+1)^{-47}$ and $f^{(46)}(3) = 2(46!)(4)^{-47}$ or $(46!)2^{-93}$.

15. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2 \text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \dfrac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$. When $\theta = \dfrac{\pi}{3}$, we have

$\sin\alpha = \dfrac{1}{3}\sin\theta = \dfrac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\dfrac{\sqrt{3}}{6}\right)^2} = \sqrt{\dfrac{11}{12}}$ and $\dfrac{d\alpha}{dt} = \dfrac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \dfrac{2\pi}{\sqrt{11/12}} = \dfrac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56$ rad/s.

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \quad \Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \quad \Rightarrow \quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad \Rightarrow$

$|OP| = \frac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm

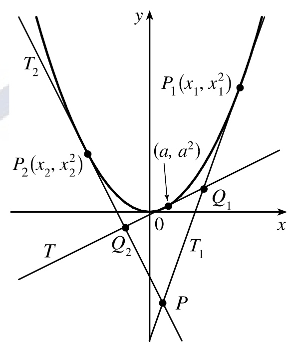
[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8+\cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8+\cos^2\theta}}\right) \text{ cm/s.}$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

16. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$. The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \quad \Rightarrow \quad x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$. Therefore,



$|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and

$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$.

So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

17. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\dfrac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \quad\Rightarrow\quad \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$. So the statement is true for $n = 1$.

Assume it is true for $n = k$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)\\ &= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]\end{aligned}$$

But

$\sin[bx+(k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta) = \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$.

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx+k\theta)+b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)] = r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)].$$

Therefore, the statement is true for all n by mathematical induction.

18. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form

$\displaystyle\lim_{x\to\pi}\frac{f(x)-f(\pi)}{x-\pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1\cdot e^0 = -1$.

19. It seems from the figure that as P approaches the point $(0,2)$ from the right, $x_T\to\infty$ and $y_T\to 2^+$. As P approaches the point $(3,0)$ from the left, it appears that $x_T\to 3^+$ and $y_T\to\infty$. So we guess that $x_T\in(3,\infty)$ and $y_T\in(2,\infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N\in(0,3)$, and $y_N\in(-\infty,0)$ or $(-2,0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \quad\Rightarrow\quad \dfrac{2x}{9}+\dfrac{2y}{4}y'=0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0,y_0) on the ellipse, an equation of the tangent line is $y-y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x-x_0)$ or $4x_0x+9y_0y = 4x_0^2+9y_0^2$. This can be written as $\dfrac{x_0x}{9}+\dfrac{y_0y}{4} = \dfrac{x_0^2}{9}+\dfrac{y_0^2}{4}=1$, because (x_0,y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9}+\dfrac{y_0y}{4}=1$.

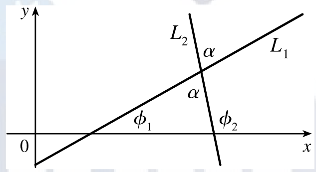
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9}=1 \quad\Leftrightarrow\quad x_T=\dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4}=1 \quad\Leftrightarrow\quad y_T=\dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0,3)$, x_T takes on all values in $(3,\infty)$, and as y_0 takes on all values in $(0,2)$, y_T takes on all values in $(2,\infty)$. At the point (x_0,y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0,y_0)}=\dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(x-x_0)$. So the x-intercept x_N for the normal line is given by $0-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(x_N-x_0) \Rightarrow$ $x_N=-\dfrac{4x_0}{9}+x_0=\dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N-y_0=\dfrac{9}{4}\dfrac{y_0}{x_0}(0-x_0) \Rightarrow y_N=-\dfrac{9y_0}{4}+y_0=-\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0,3)$, x_N takes on all values in $\left(0,\frac{5}{3}\right)$, and as y_0 takes on all values in $(0,2)$, y_N takes on all values in $\left(-\frac{5}{2},0\right)$.

20. $\displaystyle\lim_{x\to 0}\frac{\sin(3+x)^2-\sin 9}{x}=f'(3)$ where $f(x)=\sin x^2$. Now $f'(x)=(\cos x^2)(2x)$, so $f'(3)=6\cos 9$.

21. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1=\tan\,\phi_1$ and $m_2=\tan\,\phi_2$. The triangle in the figure shows that $\phi_1+\alpha+(180^\circ-\phi_2)=180^\circ$ and so $\alpha=\phi_2-\phi_1$. Therefore, using the identity for $\tan(x-y)$, we have $\tan\alpha=\tan(\phi_2-\phi_1)=\dfrac{\tan\phi_2-\tan\phi_1}{1+\tan\phi_2\tan\phi_1}$ and so $\tan\,\alpha=\dfrac{m_2-m_1}{1+m_1m_2}$.



(b) (i) The parabolas intersect when $x^2=(x-2)^2 \Rightarrow x=1$. If $y=x^2$, then $y'=2x$, so the slope of the tangent to $y=x^2$ at $(1,1)$ is $m_1=2(1)=2$. If $y=(x-2)^2$, then $y'=2(x-2)$, so the slope of the tangent to $y=(x-2)^2$ at $(1,1)$ is $m_2=2(1-2)=-2$. Therefore, $\tan\alpha=\dfrac{m_2-m_1}{1+m_1m_2}=\dfrac{-2-2}{1+2(-2)}=\dfrac{4}{3}$ and so $\alpha=\tan^{-1}\left(\frac{4}{3}\right)\approx 53^\circ$ [or 127°].

(ii) $x^2-y^2=3$ and $x^2-4x+y^2+3=0$ intersect when $x^2-4x+(x^2-3)+3=0 \Leftrightarrow 2x(x-2)=0 \Rightarrow$ $x=0$ or 2, but 0 is extraneous. If $x=2$, then $y=\pm 1$. If $x^2-y^2=3$ then $2x-2yy'=0 \Rightarrow y'=x/y$ and $x^2-4x+y^2+3=0 \Rightarrow 2x-4+2yy'=0 \Rightarrow y'=\dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1=2$ and $m_2=0$, so $\tan\alpha=\frac{0-2}{1+2\cdot 0}=-2 \Rightarrow \alpha\approx 117^\circ$. At $(2,-1)$ the slopes are $m_1=-2$ and $m_2=0$, so $\tan\alpha=\dfrac{0-(-2)}{1+(-2)(0)}=2 \Rightarrow \alpha\approx 63^\circ$ [or 117°].

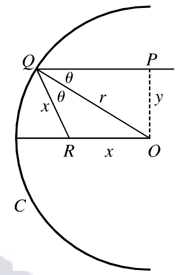
22. $y^2=4px \Rightarrow 2yy'=4p \Rightarrow y'=2p/y \Rightarrow$ slope of tangent at $P(x_1,y_1)$ is $m_1=2p/y_1$. The slope of FP is $m_2=\dfrac{y_1}{x_1-p}$, so by the formula from Problem 19(a),

$$\tan\alpha=\frac{\dfrac{y_1}{x_1-p}-\dfrac{2p}{y_1}}{1+\left(\dfrac{2p}{y_1}\right)\left(\dfrac{y_1}{x_1-p}\right)}\cdot\frac{y_1\,(x_1-p)}{y_1\,(x_1-p)}=\frac{y_1^2-2p(x_1-p)}{y_1(x_1-p)+2py_1}=\frac{4px_1-2px_1+2p^2}{x_1y_1-py_1+2py_1}$$

$$=\frac{2p(p+x_1)}{y_1(p+x_1)}=\frac{2p}{y_1}=\text{slope of tangent at }P=\tan\beta$$

Since $0\leq\alpha,\beta\leq\frac{\pi}{2}$, this proves that $\alpha=\beta$.

23. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



24. $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(x)-0}{g(x)-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}} = \frac{\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\displaystyle\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)}$$

25. $$\lim_{x\to 0}\frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin a\,\cos 2x + \cos a\,\sin 2x - 2\sin a\,\cos x - 2\cos a\,\sin x + \sin a}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2}$$

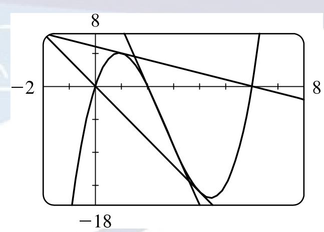
$$= \lim_{x\to 0}\frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\,\cos x - 2\sin x)}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2}$$

$$= \lim_{x\to 0}\frac{2(\cos x - 1)[\sin a\,\cos x + \cos a\,\sin x](\cos x + 1)}{x^2(\cos x + 1)}$$

$$= \lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x + 1)} = -2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\sin(a+x)}{\cos x + 1} = -2(1)^2\,\frac{\sin(a+0)}{\cos 0 + 1} = -\sin a$$

26. (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \;\Rightarrow\; f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \;\Rightarrow\; b = 6$ and the tangent has equation $y = -x + 6$.



Similarly, at $x = \dfrac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \dfrac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.

(b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or $f'(x) = 3x^2 - 2(a+b+c)x + ab + ac + bc$. Using the `Simplify` command, we get $f'\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{4}$ and $f\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \dfrac{a+b}{2}$

is $y = -\dfrac{(a-b)^2}{4}\left(x - \dfrac{a+b}{2}\right) - \dfrac{(a-b)^2}{8}(a+b-2c)$. To find the x-intercept, let $y = 0$ and use the `Solve` command. The result is $x = c$.

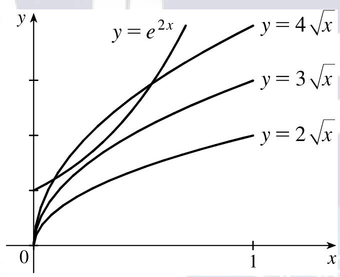
Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file `DifferentiationApplications`. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#4$. The form in expression #4 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

```
#1:  (x - a)·(x - b)·(x - c)

#2:  LOAD(C:\Program Files\TI Education\Derive 6\Math\DifferentiationApplications.mth

                                             a + b
#3:  TANGENT ((x - a)·(x - b)·(x - c), x, ───────)
                                               2

                                              2            2
                                            (a  - 2·a·b + b )·(c-x)
#4:                                         ───────────────────────
                                                       4
```

27. 

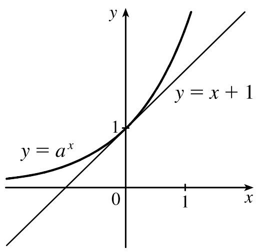
Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $[k > 0]$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

$$f(a) = g(a) \quad \Rightarrow \quad e^{2a} = k\sqrt{a} \quad (\star)$$

and $$f'(a) = g'(a) \quad \Rightarrow \quad 2e^{2a} = \frac{k}{2\sqrt{a}} \quad \Rightarrow \quad e^{2a} = \frac{k}{4\sqrt{a}}.$$

So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \quad \Rightarrow \quad \left(\sqrt{a}\right)^2 = \dfrac{k}{4k} \quad \Rightarrow \quad a = \frac{1}{4}$. From $(\star)$, $e^{2(1/4)} = k\sqrt{1/4} \quad \Rightarrow$ $k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

28. We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$. [To see this analytically, note that $a^x \geq 1 + x \quad \Rightarrow \quad a^x - 1 \geq x \quad \Rightarrow \quad \dfrac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim\limits_{x \to 0^+} \dfrac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \quad \Rightarrow \quad \dfrac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim\limits_{x \to 0^-} \dfrac{a^x - 1}{x} \leq 1$.



[continued]

Since $1 \le f'(0) \le 1$, we must have $f'(0) = 1$.] But $f'(x) = a^x \ln a \quad \Rightarrow \quad f'(0) = \ln a$, so we have $\ln a = 1 \quad \Leftrightarrow \quad a = e$.

Another method: The inequality certainly holds for $x \le -1$, so consider $x > -1$, $x \ne 0$. Then $a^x \ge 1 + x \quad \Rightarrow$ $a \ge (1+x)^{1/x}$ for $x > 0 \quad \Rightarrow \quad a \ge \lim\limits_{x \to 0^+} (1+x)^{1/x} = e$, by Equation 3.6.5. Also, $a^x \ge 1 + x \quad \Rightarrow \quad a \le (1+x)^{1/x}$ for $x < 0 \quad \Rightarrow \quad a \le \lim\limits_{x \to 0^-} (1+x)^{1/x} = e$. So since $e \le a \le e$, we must have $a = e$.

29. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \arctan \dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1}$$

$$= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak-1)$.

So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a + \cos x)}$. But $ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$,

so $y' = 1/(a + \cos x)$.

30. Suppose that $y = mx + c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1 \quad \Leftrightarrow \quad (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ must be 0; that is,

$$0 = (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2$$

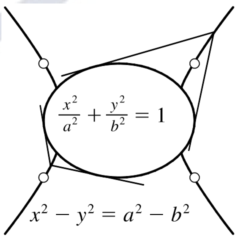
$$= 4a^2b^2(a^2m^2 + b^2 - c^2)$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

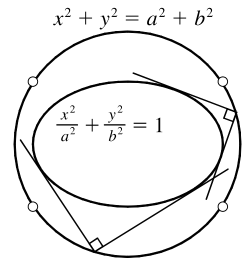
Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \quad \Leftrightarrow \quad m^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}m + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0.$$

(a) Suppose that the two tangent lines from the point (α, β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2 + \dfrac{2\alpha\beta}{a^2 - \alpha^2}z + \dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = 0$. This implies that $(z - m)\left(z - \dfrac{1}{m}\right) = 0 \quad \Leftrightarrow$ $z^2 - \left(m + \dfrac{1}{m}\right)z + m\left(\dfrac{1}{m}\right) = 0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = m\left(\dfrac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.



(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\dfrac{1}{m}$, then m and $-\dfrac{1}{m}$ are roots of the quadratic equation, and so $(z-m)\left(z+\dfrac{1}{m}\right)=0$, and equating the constant terms as in part (a), we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2}=-1$, and hence $b^2-\beta^2=\alpha^2-a^2$. So the point (α, β) lies on the circle $x^2+y^2=a^2+b^2$.



31. $y=x^4-2x^2-x \quad\Rightarrow\quad y'=4x^3-4x-1$. The equation of the tangent line at $x=a$ is $y-(a^4-2a^2-a)=(4a^3-4a-1)(x-a)$ or $y=(4a^3-4a-1)x+(-3a^4+2a^2)$ and similarly for $x=b$. So if at $x=a$ and $x=b$ we have the same tangent line, then $4a^3-4a-1=4b^3-4b-1$ and $-3a^4+2a^2=-3b^4+2b^2$. The first equation gives $a^3-b^3=a-b \quad\Rightarrow\quad (a-b)(a^2+ab+b^2)=(a-b)$. Assuming $a\neq b$, we have $1=a^2+ab+b^2$. The second equation gives $3(a^4-b^4)=2(a^2-b^2) \quad\Rightarrow\quad 3(a^2-b^2)(a^2+b^2)=2(a^2-b^2)$ which is true if $a=-b$. Substituting into $1=a^2+ab+b^2$ gives $1=a^2-a^2+a^2 \quad\Rightarrow\quad a=\pm 1$ so that $a=1$ and $b=-1$ or vice versa. Thus, the points $(1,-2)$ and $(-1,0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2-b^2\neq 0$. Then $3(a^2-b^2)(a^2+b^2)=2(a^2-b^2)$ gives $3(a^2+b^2)=2$ or $a^2+b^2=\frac{2}{3}$. Thus, $ab=(a^2+ab+b^2)-(a^2+b^2)=1-\dfrac{2}{3}=\dfrac{1}{3}$, so $b=\dfrac{1}{3a}$. Hence, $a^2+\dfrac{1}{9a^2}=\dfrac{2}{3}$, so $9a^4+1=6a^2 \quad\Rightarrow$ $0=9a^4-6a^2+1=(3a^2-1)^2$. So $3a^2-1=0 \quad\Rightarrow\quad a^2=\dfrac{1}{3} \quad\Rightarrow\quad b^2=\dfrac{1}{9a^2}=\dfrac{1}{3}=a^2$, contradicting our assumption that $a^2\neq b^2$.

32. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1=0$) then by symmetry $a_2=-a_3$, so $a_1+a_2+a_3=0$. So we can assume that none of the a_i is 0.

The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\dfrac{1}{2a_i}$ and its equation is $y-a_i^2=-\dfrac{1}{2a_i}(x-a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2):

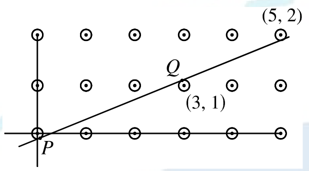
$$y=a_1^2-\frac{1}{2a_1}(x-a_1)=a_2^2-\frac{1}{2a_2}(x-a_2) \quad\Rightarrow\quad x\left(\frac{1}{2a_2}-\frac{1}{2a_1}\right)=a_2^2-a_1^2 \quad\Rightarrow$$

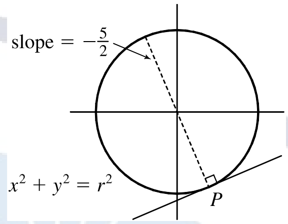
$$x\left(\frac{a_1-a_2}{2a_1a_2}\right)=(-a_1-a_2)(a_1+a_2) \quad\Leftrightarrow\quad x=-2a_1a_2(a_1+a_2) \quad \textbf{(1)}.$$

Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x=-2a_1a_3(a_1+a_3)$ **(2)**.

Equating **(1)** and **(2)** gives $a_2(a_1+a_2)=a_3(a_1+a_3) \quad\Leftrightarrow\quad a_1(a_2-a_3)=a_3^2-a_2^2=-(a_2+a_3)(a_2-a_3) \quad\Leftrightarrow$ $a_1=-(a_2+a_3) \quad\Leftrightarrow\quad a_1+a_2+a_3=0$.

33. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.

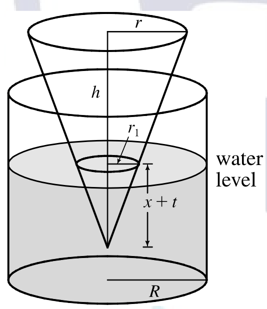




To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \quad \Rightarrow \quad x^2 + \frac{25}{4}x^2 = r^2 \quad \Rightarrow \quad x^2 = \frac{4}{29}r^2 \quad \Rightarrow \quad x = \frac{2}{\sqrt{29}}r,\ y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r,\ y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so $m_{PQ} = \dfrac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \dfrac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \dfrac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \dfrac{2}{5} \quad \Rightarrow$

$5\sqrt{29} + 50r = 6\sqrt{29} - 8r \quad \Leftrightarrow \quad 58r = \sqrt{29} \quad \Leftrightarrow \quad r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

34.



Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + 1t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged).

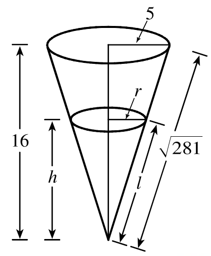
Using similar triangles, $\dfrac{r_1}{x+t} = \dfrac{r}{h} \quad \Rightarrow \quad r_1 = \dfrac{r}{h}(x+t)$.

volume of water and cone at time t	=	original volume of water	+	volume of submerged part of cone
$\pi R^2(H+x)$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi r_1^2(x+t)$
$\pi R^2 H + \pi R^2 x$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi \dfrac{r^2}{h^2}(x+t)^3$
$3h^2R^2x$	=	$r^2(x+t)^3$		

Differentiating implicitly with respect to t gives us $3h^2R^2\dfrac{dx}{dt} = r^2\left[3(x+t)^2\dfrac{dx}{dt} + 3(x+t)^2\dfrac{dt}{dt}\right] \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \quad \Rightarrow \quad \left.\dfrac{dx}{dt}\right|_{x+t=h} = \dfrac{r^2h^2}{h^2R^2 - r^2h^2} = \dfrac{r^2}{R^2 - r^2}$. Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

35.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \quad \Rightarrow \quad r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\,\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16} \quad \Leftrightarrow$ $l = \dfrac{5}{8}\sqrt{281}$, we get $\dfrac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\dfrac{25}{8}\cdot\dfrac{5}{8}\sqrt{281} \quad \Leftrightarrow \quad \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256+375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256+375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\frac{25}{8}\cdot\frac{5\sqrt{281}}{8} = \frac{256+375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$$

4 ☐ APPLICATIONS OF DIFFERENTIATION

4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. (a) The Extreme Value Theorem

   (b) See the Closed Interval Method.

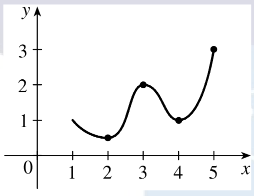
3. Absolute maximum at s, absolute minimum at r, local maximum at c, local minima at b and r, neither a maximum nor a minimum at a and d.

4. Absolute maximum at r; absolute minimum at a; local maxima at b and r; local minimum at d; neither a maximum nor a minimum at c and s.

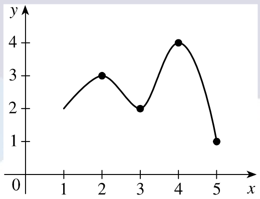
5. Absolute maximum value is $f(4) = 5$; there is no absolute minimum value; local maximum values are $f(4) = 5$ and $f(6) = 4$; local minimum values are $f(2) = 2$ and $f(1) = f(5) = 3$.

6. There is no absolute maximum value; absolute minimum value is $g(4) = 1$; local maximum values are $g(3) = 4$ and $g(6) = 3$; local minimum values are $g(2) = 2$ and $g(4) = 1$.

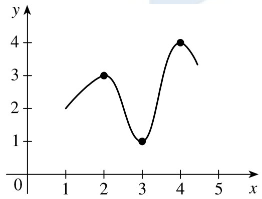
7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



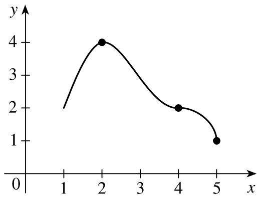
8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3



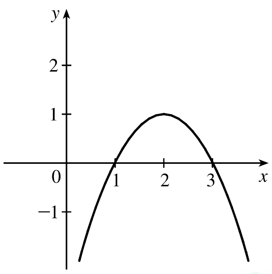
9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2



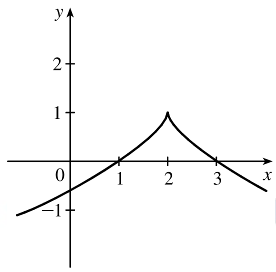
10. Absolute maximum at 2, absolute minimum at 5, 4 is a critial number but there is no local maximum or minimum there.



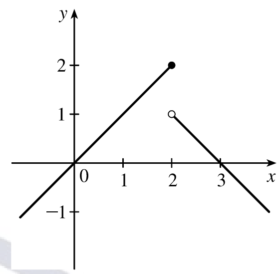
11. (a)



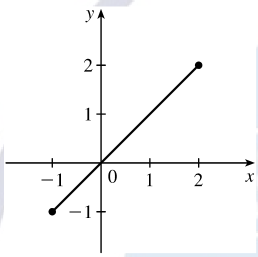
(b)



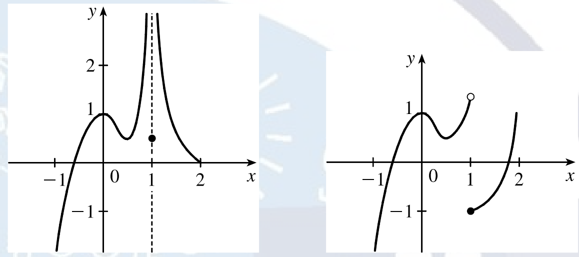
(c)



12. (a) Note that a local maximum cannot occur at an endpoint.

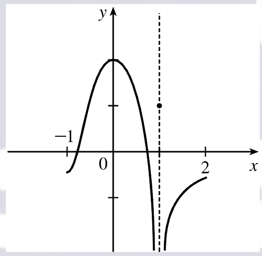


(b)

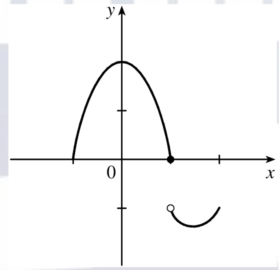


Note: By the Extreme Value Theorem, f must *not* be continuous.

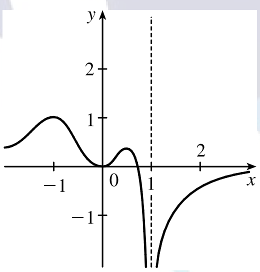
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



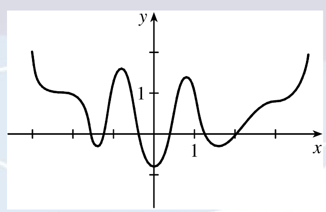
(b)



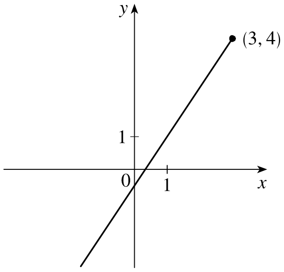
14. (a)



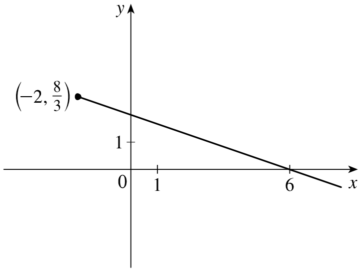
(b)



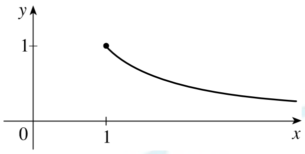
15. $f(x) = \frac{1}{2}(3x - 1)$, $x \leq 3$. Absolute maximum $f(3) = 4$; no local maximum. No absolute or local minimum.



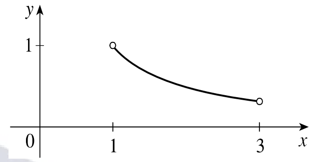
16. $f(x) = 2 - \frac{1}{3}x$, $x \geq -2$. Absolute maximum $f(-2) = \frac{8}{3}$; no local maximum. No absolute or local minimum.



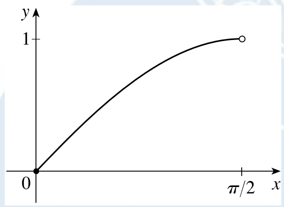
17. $f(x) = 1/x$, $x \geq 1$. Absolute maximum $f(1) = 1$; no local maximum. No absolute or local minimum.



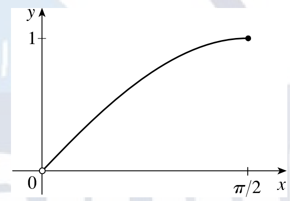
18. $f(x) = 1/x$, $1 < x < 3$. No absolute or local maximum. No absolute or local minimum.



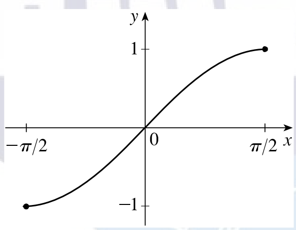
19. $f(x) = \sin x$, $0 \leq x < \pi/2$. No absolute or local maximum. Absolute minimum $f(0) = 0$; no local minimum.



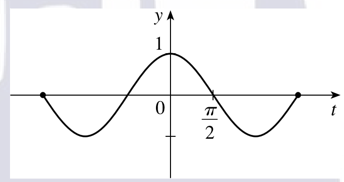
20. $f(x) = \sin x$, $0 < x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. No absolute or local minimum.



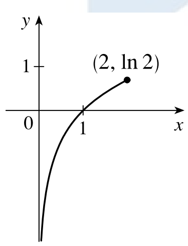
21. $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. Absolute minimum $f\left(-\frac{\pi}{2}\right) = -1$; no local minimum.



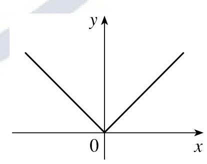
22. $f(t) = \cos t$, $-\frac{3\pi}{2} \leq t \leq \frac{3\pi}{2}$. Absolute and local maximum $f(0) = 1$; absolute and local minima $f(\pm\pi, -1)$.



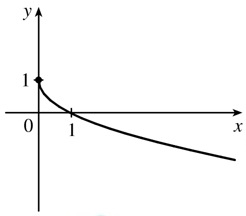
23. $f(x) = \ln x$, $0 < x \leq 2$. Absolute maximum $f(2) = \ln 2 \approx 0.69$; no local maximum. No absolute or local minimum.



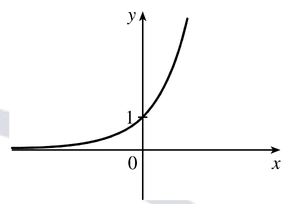
24. $f(x) = | x |$. No absolute or local maximum. Absolute and local minimum $f(0) = 0$.



25. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.



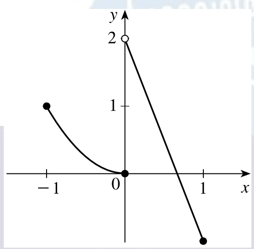
26. $f(x) = e^x$. No absolute or local maximum or minimum value.



27. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x \le 0 \\ 2 - 3x & \text{if } 0 < x \le 1 \end{cases}$

No absolute or local maximum.

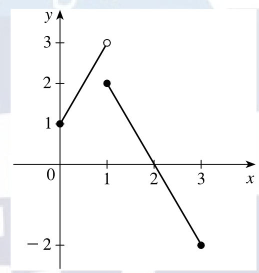
Absolute minimum $f(1) = -1$; no local minimum.



28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \le x < 1 \\ 4 - 2x & \text{if } 1 \le x \le 3 \end{cases}$

No absolute or local maximum.

Absolute minimum $f(3) = -2$; no local minimum.



29. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2 \quad \Rightarrow \quad f'(x) = \frac{1}{3} - x. \quad f'(x) = 0 \quad \Rightarrow \quad x = \frac{1}{3}$. This is the only critical number.

30. $f(x) = x^3 + 6x^2 - 15x \quad \Rightarrow \quad f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x + 5)(x - 1)$.

$f'(x) = 0 \quad \Rightarrow \quad x = -5, 1$. These are the only critical numbers.

31. $f(x) = 2x^3 - 3x^2 - 36x \quad \Rightarrow \quad f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x + 2)(x - 3)$.

$f'(x) = 0 \quad \Leftrightarrow \quad x = -2, 3$. These are the only critical numbers.

32. $f(x) = 2x^3 + x^2 + 2x \quad \Rightarrow \quad f'(x) = 6x^2 + 2x + 2 = 2(3x^2 + x + 1)$. Using the quadratic formula, $f'(x) = 0 \quad \Leftrightarrow$

$x = \dfrac{-1 \pm \sqrt{-11}}{6}$. Since the discrimininant, -11, is negative, there are no real soutions, and hence, there are no critical numbers.

33. $g(t) = t^4 + t^3 + t^2 + 1 \quad \Rightarrow \quad g'(t) = 4t^3 + 3t^2 + 2t = t(4t^2 + 3t + 2)$. Using the quadratic formula, we see that $4t^2 + 3t + 2 = 0$ has no real solution (its discriminant is negative), so $g'(t) = 0$ only if $t = 0$. Hence, the only critical number is 0.

34. $g(t) = |3t - 4| = \begin{cases} 3t - 4 & \text{if } 3t - 4 \ge 0 \\ -(3t - 4) & \text{if } 3t - 4 < 0 \end{cases} = \begin{cases} 3t - 4 & \text{if } t \ge \frac{4}{3} \\ 4 - 3t & \text{if } t < \frac{4}{3} \end{cases}$

$g'(t) = \begin{cases} 3 & \text{if } t > \frac{4}{3} \\ -3 & \text{if } t < \frac{4}{3} \end{cases}$ and $g'(t)$ does not exist at $t = \frac{4}{3}$, so $t = \frac{4}{3}$ is a critical number.

35. $g(y) = \dfrac{y-1}{y^2-y+1} \Rightarrow$

$g'(y) = \dfrac{(y^2-y+1)(1)-(y-1)(2y-1)}{(y^2-y+1)^2} = \dfrac{y^2-y+1-(2y^2-3y+1)}{(y^2-y+1)^2} = \dfrac{-y^2+2y}{(y^2-y+1)^2} = \dfrac{y(2-y)}{(y^2-y+1)^2}$.

$g'(y) = 0 \Rightarrow y = 0, 2$. The expression $y^2 - y + 1$ is never equal to 0, so $g'(y)$ exists for all real numbers. The critical numbers are 0 and 2.

36. $h(p) = \dfrac{p-1}{p^2+4} \Rightarrow h'(p) = \dfrac{(p^2+4)(1)-(p-1)(2p)}{(p^2+4)^2} = \dfrac{p^2+4-2p^2+2p}{(p^2+4)^2} = \dfrac{-p^2+2p+4}{(p^2+4)^2}$.

$h'(p) = 0 \Rightarrow p = \dfrac{-2 \pm \sqrt{4+16}}{-2} = 1 \pm \sqrt{5}$. The critical numbers are $1 \pm \sqrt{5}$. [$h'(p)$ exists for all real numbers.]

37. $h(t) = t^{3/4} - 2t^{1/4} \Rightarrow h'(t) = \frac{3}{4}t^{-1/4} - \frac{2}{4}t^{-3/4} = \frac{1}{4}t^{-3/4}(3t^{1/2} - 2) = \dfrac{3\sqrt{t}-2}{4\sqrt[4]{t^3}}$.

$h'(t) = 0 \Rightarrow 3\sqrt{t} = 2 \Rightarrow \sqrt{t} = \frac{2}{3} \Rightarrow t = \frac{4}{9}$. $h'(t)$ does not exist at $t = 0$, so the critical numbers are 0 and $\frac{4}{9}$.

38. $g(x) = \sqrt[3]{4-x^2} = (4-x^2)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(4-x^2)^{-2/3}(-2x) = \dfrac{-2x}{3(4-x^2)^{2/3}}$. $g'(x) = 0 \Rightarrow x = 0$. $g'(\pm 2)$ do not exist. Thus, the three critical numbers are -2, 0, and 2.

39. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$

$F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \frac{4}{5}x^{-1/5} = \frac{1}{5}x^{-1/5}(x-4)[5 \cdot x \cdot 2 + (x-4) \cdot 4]$

$= \dfrac{(x-4)(14x-16)}{5x^{1/5}} = \dfrac{2(x-4)(7x-8)}{5x^{1/5}}$

$F'(x) = 0 \Rightarrow x = 4, \frac{8}{7}$. $F'(0)$ does not exist. Thus, the three critical numbers are 0, $\frac{8}{7}$, and 4.

40. $g(\theta) = 4\theta - \tan\theta \Rightarrow g'(\theta) = 4 - \sec^2\theta$. $g'(\theta) = 0 \Rightarrow \sec^2\theta = 4 \Rightarrow \sec\theta = \pm 2 \Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow$ $\theta = \frac{\pi}{3} + 2n\pi$, $\frac{5\pi}{3} + 2n\pi$, $\frac{2\pi}{3} + 2n\pi$, and $\frac{4\pi}{3} + 2n\pi$ are critical numbers.

Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g.

41. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta - 1) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ [n an integer] or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

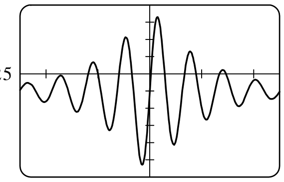
42. $h(t) = 3t - \arcsin t \Rightarrow h'(t) = 3 - \dfrac{1}{\sqrt{1-t^2}}$. $h'(t) = 0 \Rightarrow 3 = \dfrac{1}{\sqrt{1-t^2}} \Rightarrow \sqrt{1-t^2} = \frac{1}{3} \Rightarrow$ $1 - t^2 = \frac{1}{9} \Rightarrow t^2 = \frac{8}{9} \Rightarrow t = \pm\frac{2}{3}\sqrt{2} \approx \pm 0.94$, both in the domain of h, which is $[-1, 1]$.

43. $f(x) = x^2 e^{-3x} \Rightarrow f'(x) = x^2(-3e^{-3x}) + e^{-3x}(2x) = xe^{-3x}(-3x+2)$. $f'(x) = 0 \Rightarrow x = 0, \frac{2}{3}$ [e^{-3x} is never equal to 0]. $f'(x)$ always exists, so the critical numbers are 0 and $\frac{2}{3}$.

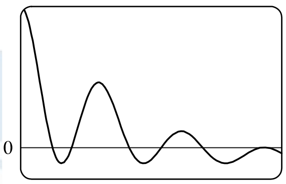
44. $f(x) = x^{-2}\ln x \Rightarrow f'(x) = x^{-2}(1/x) + (\ln x)(-2x^{-3}) = x^{-3} - 2x^{-3}\ln x = x^{-3}(1 - 2\ln x) = \dfrac{1-2\ln x}{x^3}$.

$f'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} \approx 1.65$. $f'(0)$ does not exist, but 0 is not in the domain of f, so the only critical number is $\sqrt{e}$.

45. The graph of $f'(x) = 5e^{-0.1|x|} \sin x - 1$ has 10 zeros and exists everywhere, so f has 10 critical numbers.



46. A graph of $f'(x) = \dfrac{100\cos^2 x}{10 + x^2} - 1$ is shown. There are 7 zeros between 0 and 10, and 7 more zeros since f' is an even function. f' exists everywhere, so f has 14 critical numbers.



47. $f(x) = 12 + 4x - x^2,\ [0, 5]$. $f'(x) = 4 - 2x = 0 \Leftrightarrow x = 2$. $f(0) = 12$, $f(2) = 16$, and $f(5) = 7$. So $f(2) = 16$ is the absolute maximum value and $f(5) = 7$ is the absolute minimum value.

48. $f(x) = 5 + 54x - 2x^3,\ [0, 4]$. $f'(x) = 54 - 6x^2 = 6(9 - x^2) = 6(3 + x)(3 - x) = 0 \Leftrightarrow x = -3, 3$. $f(0) = 5$, $f(3) = 113$, and $f(4) = 93$. So $f(3) = 113$ is the absolute maximum value and $f(0) = 5$ is the absolute minimum value.

49. $f(x) = 2x^3 - 3x^2 - 12x + 1,\ [-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \Leftrightarrow x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

50. $x^3 - 6x^2 + 5,\ [-3, 5]$. $f'(x) = 3x^2 - 12x = 3x(x - 4) = 0 \Leftrightarrow x = 0, 4$. $f(-3) = -76$, $f(0) = 5$, $f(4) = -27$, and $f(5) = -20$. So $f(0) = 5$ is the absolute maximum value and $f(-3) = -76$ is the absolute minimum value.

51. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1,\ [-2, 3]$. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x + 1)(x - 2) = 0 \Leftrightarrow x = -1, 0, 2$. $f(-2) = 33$, $f(-1) = -4$, $f(0) = 1$, $f(2) = -31$, and $f(3) = 28$. So $f(-2) = 33$ is the absolute maximum value and $f(2) = -31$ is the absolute minimum value.

52. $f(t) = (t^2 - 4)^3,\ [-2, 3]$. $f'(t) = 3(t^2 - 4)^2(2t) = 6t(t + 2)^2(t - 2)^2 = 0 \Leftrightarrow t = -2, 0, 2$. $f(\pm 2) = 0$, $f(0) = -64$, and $f(3) = 5^3 = 125$. So $f(3) = 125$ is the absolute maximum value and $f(0) = -64$ is the absolute minimum value.

53. $f(x) = x + \dfrac{1}{x},\ [0.2, 4]$. $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x + 1)(x - 1)}{x^2} = 0 \Leftrightarrow x = \pm 1$, but $x = -1$ is not in the given interval, $[0.2, 4]$. $f'(x)$ does not exist when $x = 0$, but 0 is not in the given interval, so 1 is the only critical nuumber. $f(0.2) = 5.2$, $f(1) = 2$, and $f(4) = 4.25$. So $f(0.2) = 5.2$ is the absolute maximum value and $f(1) = 2$ is the absolute minimum value.

54. $f(x) = \dfrac{x}{x^2 - x + 1},\ [0, 3]$.

$$f'(x) = \frac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2} = \frac{(1 + x)(1 - x)}{(x^2 - x + 1)^2} = 0 \Leftrightarrow$$

$x = \pm 1$, but $x = -1$ is not in the given interval, $[0, 3]$. $f(0) = 0$, $f(1) = 1$, and $f(3) = \frac{3}{7}$. So $f(1) = 1$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

55. $f(t) = t - \sqrt[3]{t}$, $[-1, 4]$. $f'(t) = 1 - \frac{1}{3}t^{-2/3} = 1 - \dfrac{1}{3t^{2/3}}$. $f'(t) = 0 \iff 1 = \dfrac{1}{3t^{2/3}} \iff t^{2/3} = \dfrac{1}{3} \iff$ $t = \pm\left(\dfrac{1}{3}\right)^{3/2} = \pm\sqrt{\dfrac{1}{27}} = \pm\dfrac{1}{3\sqrt{3}} = \pm\dfrac{\sqrt{3}}{9}$. $f'(t)$ does not exist when $t = 0$. $f(-1) = 0$, $f(0) = 0$,

$f\left(\dfrac{-1}{3\sqrt{3}}\right) = \dfrac{-1}{3\sqrt{3}} - \dfrac{-1}{\sqrt{3}} = \dfrac{-1+3}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9} \approx 0.3849$, $f\left(\dfrac{1}{3\sqrt{3}}\right) = \dfrac{1}{3\sqrt{3}} - \dfrac{1}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{9}$, and

$f(4) = 4 - \sqrt[3]{4} \approx 2.413$. So $f(4) = 4 - \sqrt[3]{4}$ is the absolute maximum value and $f\left(\dfrac{\sqrt{3}}{9}\right) = -\dfrac{2\sqrt{3}}{9}$ is the absolute minimum value.

56. $f(t) = \dfrac{\sqrt{t}}{1+t^2}$, $[0, 2]$. $f'(t) = \dfrac{(1+t^2)(1/(2\sqrt{t})) - \sqrt{t}\,(2t)}{(1+t^2)^2} = \dfrac{(1+t^2) - 2\sqrt{t}\,\sqrt{t}\,(2t)}{2\sqrt{t}\,(1+t^2)^2} = \dfrac{1-3t^2}{2\sqrt{t}\,(1+t^2)^2}$.

$f'(t) = 0 \iff 1 - 3t^2 = 0 \iff t^2 = \dfrac{1}{3} \iff t = \pm\dfrac{1}{\sqrt{3}}$, but $t = -\dfrac{1}{\sqrt{3}}$ is not in the given interval, $[0, 2]$. $f'(t)$ does not exist when $t = 0$, which is an endpoint. $f(0) = 0$, $f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{1/\sqrt[4]{3}}{1+1/3} = \dfrac{3^{-1/4}}{4/3} = \dfrac{3^{3/4}}{4} \approx 0.570$, and

$f(2) = \dfrac{\sqrt{2}}{5} \approx 0.283$. So $f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{3^{3/4}}{4}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

57. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$.

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1)$.

$f'(t) = 0 \Rightarrow \sin t = \frac{1}{2}$ or $\sin t = -1 \Rightarrow t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

58. $f(t) = t + \cot(t/2)$, $[\pi/4, 7\pi/4]$. $f'(t) = 1 - \csc^2(t/2) \cdot \frac{1}{2}$.

$f'(t) = 0 \Rightarrow \frac{1}{2}\csc^2(t/2) = 1 \Rightarrow \csc^2(t/2) = 2 \Rightarrow \csc(t/2) = \pm\sqrt{2} \Rightarrow \frac{1}{2}t = \frac{\pi}{4}$ or $\frac{1}{2}t = \frac{3\pi}{4}$

$\left[\frac{\pi}{4} \le t \le \frac{7\pi}{4} \Rightarrow \frac{\pi}{8} \le \frac{1}{2}t \le \frac{7\pi}{8}\right.$ and $\csc(t/2) \ne -\sqrt{2}$ in the last interval$\left.\right] \Rightarrow t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.

$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \cot\frac{\pi}{8} \approx 3.20$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cot\frac{\pi}{4} = \frac{\pi}{2} + 1 \approx 2.57$, $f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} + \cot\frac{3\pi}{4} = \frac{3\pi}{2} - 1 \approx 3.71$, and

$f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} + \cot\frac{7\pi}{8} \approx 3.08$. So $f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$ is the absolute maximum value and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ is the absolute minimum value.

59. $f(x) = x^{-2}\ln x$, $\left[\frac{1}{2}, 4\right]$. $f'(x) = x^{-2} \cdot \dfrac{1}{x} + (\ln x)(-2x^{-3}) = x^{-3} - 2x^{-3}\ln x = x^{-3}(1 - 2\ln x) = \dfrac{1 - 2\ln x}{x^3}$.

$f'(x) = 0 \iff 1 - 2\ln x = 0 \iff 2\ln x = 1 \iff \ln x = \frac{1}{2} \iff x = e^{1/2} \approx 1.65$. $f'(x)$ does not exist when $x = 0$, which is not in the given interval, $\left[\frac{1}{2}, 4\right]$. $f(\frac{1}{2}) = \dfrac{\ln 1/2}{(1/2)^2} = \dfrac{\ln 1 - \ln 2}{1/4} = -4\ln 2 \approx -2.773$,

$f\left(e^{1/2}\right) = \dfrac{\ln e^{1/2}}{(e^{1/2})^2} = \dfrac{1/2}{e} = \dfrac{1}{2e} \approx 0.184$, and $f(4) = \dfrac{\ln 4}{4^2} = \dfrac{\ln 4}{16} \approx 0.087$. So $f(e^{1/2}) = \dfrac{1}{2e}$ is the absolute maximum value and $f\left(\frac{1}{2}\right) = -4\ln 2$ is the absolute minimum value.

60. $f(x) = xe^{x/2}$, $[-3, 1]$. $f'(x) = xe^{x/2}\left(\frac{1}{2}\right) + e^{x/2}(1) = e^{x/2}\left(\frac{1}{2}x + 1\right)$. $f'(x) = 0 \Leftrightarrow \frac{1}{2}x + 1 = 0 \Leftrightarrow x = -2$. $f(-3) = -3e^{-3/2} \approx -0.669$, $f(-2) = -2e^{-1} \approx -0.736$, and $f(1) = e^{1/2} \approx 1.649$. So $f(1) = e^{1/2}$ is the absolute maximum value and $f(-2) = -2/e$ is the absolute minimum value.

61. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$. $f'(x) = \dfrac{1}{x^2 + x + 1} \cdot (2x + 1) = 0 \Leftrightarrow x = -\frac{1}{2}$. Since $x^2 + x + 1 > 0$ for all x, the domain of f and f' is $\mathbb{R}$. $f(-1) = \ln 1 = 0$, $f\left(-\frac{1}{2}\right) = \ln\frac{3}{4} \approx -0.29$, and $f(1) = \ln 3 \approx 1.10$. So $f(1) = \ln 3 \approx 1.10$ is the absolute maximum value and $f\left(-\frac{1}{2}\right) = \ln\frac{3}{4} \approx -0.29$ is the absolute minimum value.

62. $f(x) = x - 2\tan^{-1}x$, $[0, 4]$. $f'(x) = 1 - 2 \cdot \dfrac{1}{1 + x^2} = 0 \Leftrightarrow 1 = \dfrac{2}{1 + x^2} \Leftrightarrow 1 + x^2 = 2 \Leftrightarrow x^2 = 1 \Leftrightarrow$ $x = \pm 1$. $f(0) = 0$, $f(1) = 1 - \frac{\pi}{2} \approx -0.57$, and $f(4) = 4 - 2\tan^{-1}4 \approx 1.35$. So $f(4) = 4 - 2\tan^{-1}4$ is the absolute maximum value and $f(1) = 1 - \frac{\pi}{2}$ is the absolute minimum value.

63. $f(x) = x^a(1 - x)^b$, $0 \le x \le 1$, $a > 0$, $b > 0$.

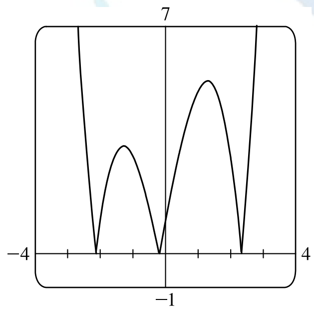
$$\begin{aligned} f'(x) &= x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}[x \cdot b(-1) + (1-x) \cdot a] \\ &= x^{a-1}(1-x)^{b-1}(a - ax - bx) \end{aligned}$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0, 1)$, $f'(x) = 0 \Leftrightarrow x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a\left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

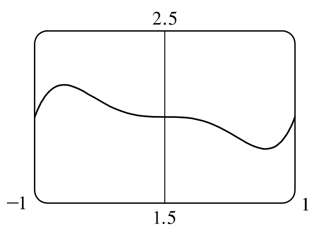
So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

64.



The graph of $f(x) = \left|1 + 5x - x^3\right|$ indicates that $f'(x) = 0$ at $x \approx \pm 1.3$ and that $f'(x)$ does not exist at $x \approx -2.1$, -0.2, and 2.3. Those five values of x are the critical numbers of f.

65. (a)

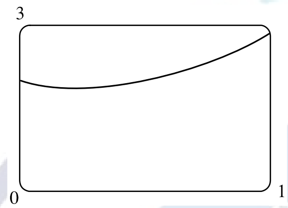


From the graph, it appears that the absolute maximum value is about $f(-0.77) = 2.19$, and the absolute minimum value is about $f(0.77) = 1.81$.

(b) $f(x) = x^5 - x^3 + 2 \quad\Rightarrow\quad f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. So $f'(x) = 0 \quad\Rightarrow\quad x = 0, \pm\sqrt{\frac{3}{5}}$.

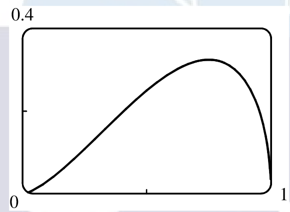
$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \left(-\sqrt{\tfrac{3}{5}}\right)^5 - \left(-\sqrt{\tfrac{3}{5}}\right)^3 + 2 = -\left(\tfrac{3}{5}\right)^2\sqrt{\tfrac{3}{5}} + \tfrac{3}{5}\sqrt{\tfrac{3}{5}} + 2$$
$$= \left(\tfrac{3}{5} - \tfrac{9}{25}\right)\sqrt{\tfrac{3}{5}} + 2 = \tfrac{6}{25}\sqrt{\tfrac{3}{5}} + 2 \quad \text{(maximum)}$$

and similarly, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (minimum).

66. (a) 

From the graph, it appears that the absolute maximum value is about $f(1) = 2.85$, and the absolute minimum value is about $f(0.23) = 1.89$.

(b) $f(x) = e^x + e^{-2x} \quad\Rightarrow\quad f'(x) = e^x - 2e^{-2x} = e^{-2x}(e^{3x} - 2)$. So $f'(x) = 0 \quad\Leftrightarrow\quad e^{3x} = 2 \quad\Leftrightarrow\quad 3x = \ln 2 \quad\Leftrightarrow$ $x = \frac{1}{3}\ln 2$ [≈ 0.23]. $f\left(\frac{1}{3}\ln 2\right) = (e^{\ln 2})^{1/3} + (e^{\ln 2})^{-2/3} = 2^{1/3} + 2^{-2/3}$ [≈ 1.89], the minimum value. $f(1) = e^1 + e^{-2}$ [≈ 2.85], the maximum.

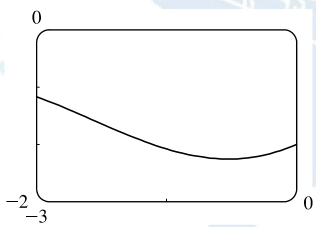
67. (a) 

From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad\Rightarrow\quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad\Rightarrow\quad 3x - 4x^2 = 0 \quad\Rightarrow\quad x(3 - 4x) = 0 \quad\Rightarrow\quad x = 0$ or $\frac{3}{4}$.

$f(0) = f(1) = 0$ (minimum), and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$ (maximum).

68. (a) 

From the graph, it appears that the absolute maximum value is about $f(-2) = -1.17$, and the absolute minimum value is about $f(-0.52) = -2.26$.

(b) $f(x) = x - 2\cos x \quad\Rightarrow\quad f'(x) = 1 + 2\sin x$. So $f'(x) = 0 \quad\Rightarrow\quad \sin x = -\frac{1}{2} \quad\Rightarrow\quad x = -\frac{\pi}{6}$ on $[-2, 0]$.

$f(-2) = -2 - 2\cos(-2)$ (maximum) and $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \sqrt{3}$ (minimum).

69. Let $a = 1.35$ and $b = -2.802$. Then $C(t) = ate^{bt} \quad\Rightarrow\quad C'(t) = a(te^{bt} \cdot b + e^{bt} \cdot 1) = ae^{bt}(bt + 1)$. $C'(t) = 0 \quad\Leftrightarrow$ $bt + 1 = 0 \quad\Leftrightarrow\quad t = -\dfrac{1}{b} \approx 0.36$ h. $C(0) = 0$, $C(-1/b) = -\dfrac{a}{b}e^{-1} = -\dfrac{a}{be} \approx 0.177$, and $C(3) = 3ae^{3b} \approx 0.0009$. The maximum average BAC during the first three hours is about 0.177 mg/mL and it occurs at approximately 0.36 h (21.4 min).

70. $C(t) = 8(e^{-0.4t} - e^{-0.6t}) \Rightarrow C'(t) = 8(-0.4e^{-0.4t} + 0.6e^{-0.6t})$. $C'(t) = 0 \Leftrightarrow 0.6e^{-0.6t} = 0.4e^{-0.4t} \Leftrightarrow$ $\frac{0.6}{0.4} = e^{-0.4t+0.6t} \Leftrightarrow \frac{3}{2} = e^{0.2t} \Leftrightarrow 0.2t = \ln\frac{3}{2} \Leftrightarrow t = 5\ln\frac{3}{2} \approx 2.027$ h. $C(0) = 8(1-1) = 0$,

$C(5\ln\frac{3}{2}) = 8(e^{-2\ln 3/2} - e^{-3\ln 3/2}) = 8\left[\left(\frac{3}{2}\right)^{-2} - \left(\frac{3}{2}\right)^{-3}\right] = 8\left(\frac{4}{9} - \frac{8}{27}\right) = \frac{32}{27} \approx 1.185$, and

$C(12) = 8(e^{-4.8} - e^{-7.2}) \approx 0.060$. The maximum concentration of the antibiotic during the first 12 hours is $\frac{32}{27}$ μg/mL.

71. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V

$\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \Rightarrow V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$.

Setting this equal to 0 and using the quadratic formula to find T, we get

$T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4\cdot 0.0002037\cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested

in the region $0°\text{C} \le T \le 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°$C: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$;

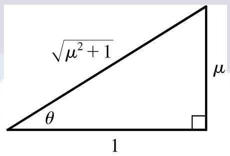
$\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at

about $3.9665°$C.

72. $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \Rightarrow \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta+\cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta+\cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \Rightarrow \mu\cos\theta - \sin\theta = 0 \Rightarrow \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$F = \dfrac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \dfrac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \dfrac{W\tan\theta\,\cos\theta}{\sin^2\theta + \cos^2\theta} = \dfrac{W\sin\theta}{1} = W\sin\theta$.



If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$.

We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.

Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$.

Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

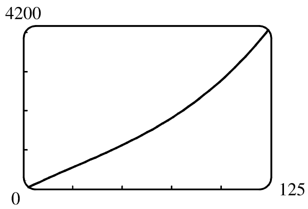
73. $L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1 \Rightarrow L'(t) = 0.04323t^2 - 0.8354t + 2.703$. Use the quadratic formula

to solve $L'(t) = 0$. $t = \dfrac{0.8354 \pm \sqrt{(0.8354)^2 - 4(0.04323)(2.703)}}{2(0.04323)} \approx 4.1$ or 15.2. For $0 \le t \le 12$, we have

$L(0) = 1060.1$, $L(4.1) \approx 1065.2$, and $L(12) \approx 1057.3$. Thus, the water level was highest during 2012 about 4.1 months after January 1.

74. (a) The equation of the graph in the figure is

$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872.$



(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \quad \Rightarrow$

$a'(t) = 0.00876t - 0.23106.$

$a'(t) = 0 \quad \Rightarrow \quad t_1 = \frac{0.23106}{0.00876} \approx 26.4$. $a(0) \approx 24.98$, $a(t_1) \approx 21.93$,

and $a(125) \approx 64.54$.

The maximum acceleration is about 64.5 ft/s^2 and the minimum acceleration is about 21.93 ft/s^2.

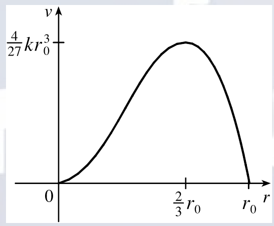
75. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \quad \Rightarrow \quad v'(r) = 2kr_0r - 3kr^2$. $\quad v'(r) = 0 \quad \Rightarrow \quad kr(2r_0 - 3r) = 0 \quad \Rightarrow$

$r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$,

and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



76. $g(x) = 2 + (x - 5)^3 \quad \Rightarrow \quad g'(x) = 3(x - 5)^2 \quad \Rightarrow \quad g'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.

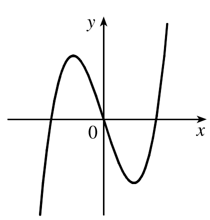
77. $f(x) = x^{101} + x^{51} + x + 1 \quad \Rightarrow \quad f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

78. Suppose that f has a minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a maximum value at c.

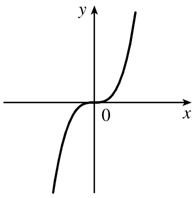
79. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

80. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

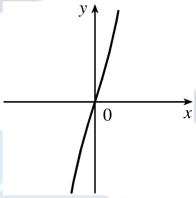
Case (i) [2 critical numbers]: $f(x) = x^3 - 3x \quad \Rightarrow$
$f'(x) = 3x^2 - 3$, so $x = -1, 1$
are critical numbers.



Case (ii) [1 critical number]: $f(x) = x^3 \Rightarrow$
$f'(x) = 3x^2$, so $x = 0$
is the only critical number.



Case (iii) [no critical number]: $f(x) = x^3 + 3x \Rightarrow$
$f'(x) = 3x^2 + 3$,
so there is no critical number.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = k\sin\beta \approx \frac{4}{3}\sin\beta \Leftrightarrow \beta \approx \arcsin\left(\frac{3}{4}\sin\alpha\right)$. We substitute this into $D(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\arcsin\left(\frac{3}{4}\sin\alpha\right)$, and then differentiate to find the minimum:

$D'(\alpha) = 2 - 4\left[1 - \left(\frac{3}{4}\sin\alpha\right)^2\right]^{-1/2}\left(\frac{3}{4}\cos\alpha\right) = 2 - \dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}}$. This is 0 when $\dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}} = 2 \Leftrightarrow$

$\frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\sin^2\alpha \Leftrightarrow \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\left(1 - \cos^2\alpha\right) \Leftrightarrow \frac{27}{16}\cos^2\alpha = \frac{7}{16} \Leftrightarrow \cos\alpha = \sqrt{\frac{7}{27}} \Leftrightarrow$

$\alpha = \arccos\sqrt{\frac{7}{27}} \approx 59.4^\circ$, and so the local minimum is $D(59.4^\circ) \approx 2.4$ radians $\approx 138^\circ$.

To see that this is an absolute minimum, we check the endpoints, which in this case are $\alpha = 0$ and $\alpha = \frac{\pi}{2}$: $D(0) = \pi$ radians $= 180^\circ$, and $D\left(\frac{\pi}{2}\right) \approx 166^\circ$.

Another method: We first calculate $\dfrac{d\beta}{d\alpha}$: $\sin\alpha = \frac{4}{3}\sin\beta \Leftrightarrow \cos\alpha = \frac{4}{3}\cos\beta\,\dfrac{d\beta}{d\alpha} \Leftrightarrow \dfrac{d\beta}{d\alpha} = \dfrac{3}{4}\dfrac{\cos\alpha}{\cos\beta}$, so since $D'(\alpha) = 2 - 4\,\dfrac{d\beta}{d\alpha} = 0 \Leftrightarrow \dfrac{d\beta}{d\alpha} = \dfrac{1}{2}$, the minimum occurs when $3\cos\alpha = 2\cos\beta$. Now we square both sides and substitute $\sin\alpha = \frac{4}{3}\sin\beta$, leading to the same result.

2. If we repeat Problem 1 with k in place of $\frac{4}{3}$, we get $D(\alpha) = \pi + 2\alpha - 4\arcsin\left(\dfrac{1}{k}\sin\alpha\right) \Rightarrow$

$D'(\alpha) = 2 - \dfrac{4\cos\alpha}{k\sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2}}$, which is 0 when $\dfrac{2\cos\alpha}{k} = \sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2} \Leftrightarrow \left(\dfrac{2\cos\alpha}{k}\right)^2 = 1 - \left(\dfrac{\sin\alpha}{k}\right)^2 \Leftrightarrow$

$4\cos^2\alpha = k^2 - \sin^2\alpha \Leftrightarrow 3\cos^2\alpha = k^2 - 1 \Leftrightarrow \alpha = \arccos\sqrt{\dfrac{k^2-1}{3}}$. So for $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx 1.038$ radians, and so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6°$.

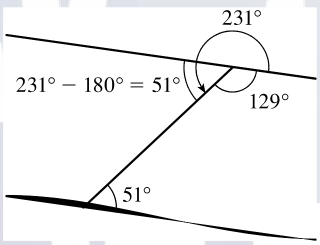
Another method: As in Problem 1, we can instead find $D'(\alpha)$ in terms of $\dfrac{d\beta}{d\alpha}$, and then substitute $\dfrac{d\beta}{d\alpha} = \dfrac{\cos\alpha}{k\cos\beta}$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is $D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. We substitute $\beta = \arcsin\left(\dfrac{\sin\alpha}{k}\right)$ and differentiate, to get

$$D'(\alpha) = 2 - \frac{6\cos\alpha}{k\sqrt{1 - \left(\frac{\sin\alpha}{k}\right)^2}}, \text{ which is 0 when } \frac{3\cos\alpha}{k} = \sqrt{1 - \left(\frac{\sin\alpha}{k}\right)^2} \quad\Leftrightarrow\quad 9\cos^2\alpha = k^2 - \sin^2\alpha \quad\Leftrightarrow$$

$8\cos^2\alpha = k^2 - 1 \quad\Leftrightarrow\quad \cos\alpha = \sqrt{\frac{1}{8}(k^2 - 1)}$. If $k = \frac{4}{3}$, then the minimum occurs at

$\alpha_1 = \arccos\sqrt{\dfrac{(4/3)^2 - 1}{8}} \approx 1.254$ radians. Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231°$, which is equivalent to a *clockwise* rotation of $360° - 231° = 129°$ (see the figure). So the rainbow angle for the secondary rainbow is about $180° - 129° = 51°$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow. For $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx \arccos\sqrt{\dfrac{1.3318^2 - 1}{8}} \approx 1.255$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx \arccos\sqrt{\dfrac{1.3435^2 - 1}{8}} \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6°$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7°, whereas in the secondary rainbow it is about 3°.

4.2 The Mean Value Theorem

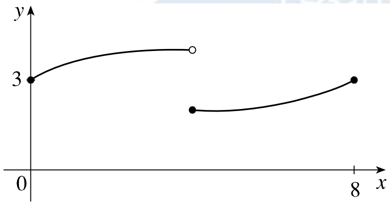
1. **(1)** f is continuous on the closed interval $[0, 8]$.

   (2) f is differentiable on the open interval $(0, 8)$.

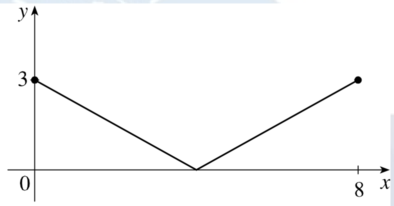
   (3) $f(0) = 3$ and $f(8) = 3$

   Thus, f statisfies the hypotheses of Rolle's Theorem. The numbers $c = 1$ and $c = 5$ satisfy the conclusion of Rolle's Theorem since $f'(1) = f'(5) = 0$.

2. The possible graphs fall into two general categories: **(1)** Not continuous and therefore not differentiable, **(2)** Continuous, but not differentiable.



Not continuous



Not differentiable

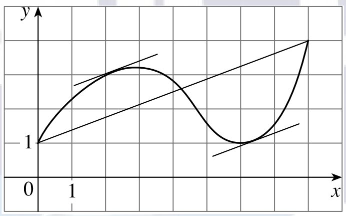
   In either case, there is no number c such that $f'(c) = 0$.

3. (a) **(1)** g is continuous on the closed interval $[0, 8]$.

   (2) g is differentiable on the open interval $(0, 8)$.

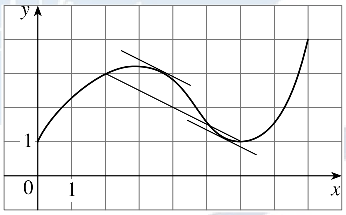
   (b) $g'(c) = \dfrac{g(8) - g(0)}{8 - 0} = \dfrac{4 - 1}{8} = \dfrac{3}{8}$.

   It appears that $g'(c) = \frac{3}{8}$ when $c \approx 2.2$ and 6.4.

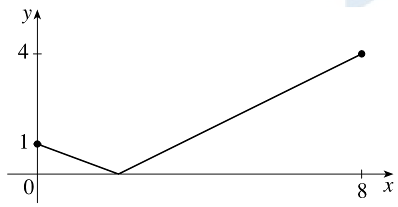


   (c) $g'(c) = \dfrac{g(6) - g(2)}{6 - 2} = \dfrac{1 - 3}{4} = -\dfrac{1}{2}$.

   It appears that $g'(c) = -\frac{1}{2}$ when $c \approx 3.7$ and 5.5.



4.2.3:
(1) change all "f" to "g"
(2) parts b and c: changed values of "c"
(3) replaced two corrected arts

4. 

   The function shown in the figure is continuous on $[0, 8]$ [but not differentiable on $(0, 8)$] with $f(0) = 1$ and $f(8) = 4$. The line passing through the two points has slope $\frac{3}{8}$. There is no number c in $(0, 8)$ such that $f'(c) = \frac{3}{8}$.

5. $f(x) = 2x^2 - 4x + 5$, $[-1, 3]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-1, 3]$ and differentiable on $(-1, 3)$. Since $f(-1) = 11$ and $f(3) = 11$, f satisfies all the hypotheses of Rolle's

Theorem. $f'(c) = 4c - 4$ and $f'(c) = 0 \iff 4c - 4 = 0 \iff c = 1$. $c = 1$ is in the interval $(-1, 3)$, so 1 satisfies the conclusion of Rolle's Theorem.

4.2.6: revise text on last line

6. $f(x) = x^3 - 2x^2 - 4x + 2$, $[-2, 2]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Since $f(-2) = -6$ and $f(2) = -6$, f satisfies all the hypotheses of Rolle's Theorem. $f'(c) = 3c^2 - 4c - 4$ and $f'(c) = 0 \iff (3c + 2)(c - 2) = 0 \iff c = -\frac{2}{3}$ or 2. $c = -\frac{2}{3}$ is in the open interval $(-2, 2)$ (but 2 *isn't*), so only $-\frac{2}{3}$ satisfies the conclusion of Rolle's Theorem.

7. $f(x) = \sin(x/2)$, $[\pi/2, 3\pi/2]$. f, being the composite of the sine function and the polynomial $x/2$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$. Also, $f\left(\frac{\pi}{2}\right) = \frac{1}{2}\sqrt{2} = f\left(\frac{3\pi}{2}\right)$. $f'(c) = 0 \iff \frac{1}{2}\cos(c/2) = 0 \iff \cos(c/2) = 0 \iff c/2 = \frac{\pi}{2} + n\pi \iff c = \pi + 2n\pi$, n an integer. Only $c = \pi$ is in $(\pi/2, 3\pi/2)$, so π satisfies the conclusion of Rolle's Theorem.

8. $f(x) = x + 1/x$, $\left[\frac{1}{2}, 2\right]$. $f'(x) = 1 - 1/x^2 = \dfrac{x^2 - 1}{x^2}$. f is a rational function that is continuous on its domain, $(-\infty, 0) \cup (0, \infty)$, so it is continuous on $\left[\frac{1}{2}, 2\right]$. f' has the same domain and is differentiable on $\left(\frac{1}{2}, 2\right)$. Also, $f\left(\frac{1}{2}\right) = \frac{5}{2} = f(2)$. $f'(c) = 0 \iff \dfrac{c^2 - 1}{c^2} = 0 \iff c^2 - 1 = 0 \iff c = \pm 1$. Only 1 is in $\left(\frac{1}{2}, 2\right)$, so 1 satisfies the conclusion of Rolle's Theorem.

9. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

10. $f(x) = \tan x$. $f(0) = \tan 0 = 0 = \tan \pi = f(\pi)$. $f'(x) = \sec^2 x \geq 1$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'\left(\frac{\pi}{2}\right)$ does not exist, and so f is not differentiable on $(0, \pi)$. (Also, $f(x)$ is not continuous on $[0, \pi]$.)

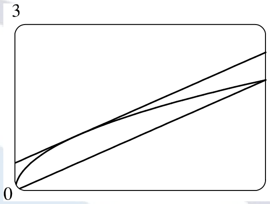
11. $f(x) = 2x^2 - 3x + 1$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff 4c - 3 = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{3 - 1}{2} = 1 \iff 4c = 4 \iff c = 1$, which is in $(0, 2)$.

12. $f(x) = x^3 - 3x + 2$, $[-2, 2]$. f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff 3c^2 - 3 = \dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 0}{4} = 1 \iff 3c^2 = 4 \iff c^2 = \dfrac{4}{3} \iff c = \pm\dfrac{2}{\sqrt{3}}$, which are both in $(-2, 2)$.

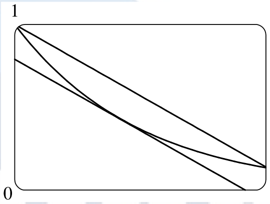
13. $f(x) = \ln x$, $[1, 4]$. f is continuous and differentiable on $(0, \infty)$, so f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff \dfrac{1}{c} = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{\ln 4 - 0}{3} = \dfrac{\ln 4}{3} \iff c = \dfrac{3}{\ln 4} \approx 2.16$, which is in $(1, 4)$.

14. $f(x) = \frac{1}{x}$, $[1, 3]$. f is continuous and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continous on $[1, 3]$ and differentiable on $(1, 3)$. $f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow -\frac{1}{c^2} = \frac{f(3) - f(1)}{3 - 1} = \frac{\frac{1}{3} - 1}{2} = -\frac{1}{3} \Leftrightarrow c^2 = 3 \Leftrightarrow c = \pm\sqrt{3}$, but only $\sqrt{3}$ is in $(1, 3)$.

15. $f(x) = \sqrt{x}$, $[0, 4]$. $f'(c) = \frac{f(4) - f(0)}{4 - 0} \Leftrightarrow \frac{1}{2\sqrt{c}} = \frac{2 - 0}{4} \Leftrightarrow \frac{1}{2\sqrt{c}} = \frac{1}{2} \Leftrightarrow \sqrt{c} = 1 \Leftrightarrow c = 1$. The secant line and the tangent line are parallel.



16. $f(x) = e^{-x}$, $[0, 2]$. $f'(c) = \frac{f(2) - f(0)}{2 - 0} \Leftrightarrow -e^{-c} = \frac{e^{-2} - 1}{2} \Leftrightarrow e^{-c} = \frac{1 - e^{-2}}{2} \Leftrightarrow -c = \ln\frac{1 - e^{-2}}{2} \Leftrightarrow c = -\ln\frac{1 - e^{-2}}{2} \approx 0.8386$. The secant line and the tangent line are parallel.



17. $f(x) = (x - 3)^{-2} \Rightarrow f'(x) = -2\,(x - 3)^{-3}$. $f(4) - f(1) = f'(c)(4 - 1) \Rightarrow \frac{1}{1^2} - \frac{1}{(-2)^2} = \frac{-2}{(c - 3)^3} \cdot 3 \Rightarrow \frac{3}{4} = \frac{-6}{(c - 3)^3} \Rightarrow (c - 3)^3 = -8 \Rightarrow c - 3 = -2 \Rightarrow c = 1$, which is not in the open interval $(1, 4)$. This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$.

18. $f(x) = 2 - |2x - 1| = \begin{cases} 2 - (2x - 1) & \text{if } 2x - 1 \geq 0 \\ 2 - [-(2x - 1)] & \text{if } 2x - 1 < 0 \end{cases} = \begin{cases} 3 - 2x & \text{if } x \geq \frac{1}{2} \\ 1 + 2x & \text{if } x < \frac{1}{2} \end{cases} \Rightarrow f'(x) = \begin{cases} -2 & \text{if } x > \frac{1}{2} \\ 2 & \text{if } x < \frac{1}{2} \end{cases}$

$f(3) - f(0) = f'(c)(3 - 0) \Rightarrow -3 - 1 = f'(c) \cdot 3 \Rightarrow f'(c) = -\frac{4}{3}$ [not ± 2]. This does not contradict the Mean Value Theorem since f is not differentiable at $x = \frac{1}{2}$.

19. Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$. Since f is the sum of the polynomial $2x$ and the trignometric function $\cos x$, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-\pi, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one root.

20. Let $f(x) = x^3 + e^x$. Then $f(-1) = -1 + 1/e < 0$ and $f(0) = 1 > 0$. Since f is the sum of a polynomial and the natural exponential function, f is continous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-1, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b

with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 3r^2 + e^r > 0$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one root.

21. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.

22. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

23. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

    (b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

24. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$ there is a number c such that $a < c < b$ and $f'(c) = 0$.

    (b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to $f(x)$ on $[a, b]$ and $[b, c]$ there are numbers $a < d < b$ and $b < e < c$ with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to $f'(x)$ on $[d, e]$ there is a number g with $d < g < e$ such that $f''(g) = 0$.

    (c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

25. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

26. If $3 \leq f'(x) \leq 5$ for all x, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8-2)$ for some c in $[2, 8]$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \quad \Rightarrow \quad 18 \leq f(8) - f(2) \leq 30$.

27. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

28. Let $h = f - g$. Note that since $f(a) = g(a)$, $h(a) = f(a) - g(a) = 0$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b - a)$. Since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.

29. Consider the function $f(x) = \sin x$, which is continuous and differentiable on $\mathbb{R}$. Let a be a number such that $0 < a < 2\pi$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. By the Mean Value Theorem, there is a number c in $(0, a)$ such that $f(a) - f(0) = f'(c)(a - 0)$; that is, $\sin a - 0 = (\cos c)(a)$. Now $\cos c < 1$ for $0 < c < 2\pi$, so $\sin a < 1 \cdot a = a$. We took a to be an arbitrary number in $(0, 2\pi)$, so $\sin x < x$ for all x satisfying $0 < x < 2\pi$.

30. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\dfrac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\dfrac{f(b) + f(b)}{2b} = f'(c) \quad \Rightarrow \quad \dfrac{f(b)}{b} = f'(c)$.

31. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a). By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \leq |\cos c|\,|b - a| \leq |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \leq |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.

32. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.

33. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

34. Let $f(x) = 2\sin^{-1} x - \cos^{-1}(1 - 2x^2)$. Then $f'(x) = \dfrac{2}{\sqrt{1 - x^2}} - \dfrac{4x}{\sqrt{1 - (1 - 2x^2)^2}} = \dfrac{2}{\sqrt{1 - x^2}} - \dfrac{4x}{2x\sqrt{1 - x^2}} = 0$ [since $x \geq 0$]. Thus, $f'(x) = 0$ for all $x \in (0, 1)$. Thus, $f(x) = C$ on $(0, 1)$. To find C, let $x = 0.5$. Thus, $2\sin^{-1}(0.5) - \cos^{-1}(0.5) = 2\left(\frac{\pi}{6}\right) - \frac{\pi}{3} = 0 = C$. We conclude that $f(x) = 0$ for x in $(0, 1)$. By continuity of f, $f(x) = 0$ on $[0, 1]$. Therefore, we see that $f(x) = 2\,\sin^{-1} x - \cos^{-1}(1 - 2x^2) = 0 \quad \Rightarrow \quad 2\,\sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

35. Let $f(x) = \arcsin\left(\dfrac{x-1}{x+1}\right) - 2\arctan\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus,

$$f'(x) = \frac{1}{\sqrt{1 - \left(\dfrac{x-1}{x+1}\right)^2}} \frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x+1)} - \frac{1}{\sqrt{x}\,(x+1)} = 0.$$

Then $f(x) = C$ on $(0, \infty)$ by Theorem 5. By continuity of f, $f(x) = C$ on $[0, \infty)$. To find C, we let $x = 0 \Rightarrow$ $\arcsin(-1) - 2\arctan(0) + \frac{\pi}{2} = C \Rightarrow -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \Rightarrow$ $\arcsin\left(\dfrac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \frac{\pi}{2}$.

36. Let $v(t)$ be the velocity of the car t hours after 2:00 PM. Then $\dfrac{v(1/6) - v(0)}{1/6 - 0} = \dfrac{50-30}{1/6} = 120$. By the Mean Value Theorem, there is a number c such that $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2:00 PM is exactly $120\ \text{mi/h}^2$.

37. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c, with $0 < c < b$, such that $f'(c) = \dfrac{f(b) - f(0)}{b - 0}$. But $f(b) = f(0) = 0$, so $f'(c) = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$. So at time c, both runners have the same speed $g'(c) = h'(c)$.

38. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But then $f'(c) = \dfrac{b-a}{b-a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

4.3 How Derivatives Affect the Shape of a Graph

1. (a) f is increasing on $(1, 3)$ and $(4, 6)$.
   (b) f is decreasing on $(0, 1)$ and $(3, 4)$.
   (c) f is concave upward on $(0, 2)$.
   (d) f is concave downward on $(2, 4)$ and $(4, 6)$.
   (e) The point of inflection is $(2, 3)$.

2. (a) f is increasing on $(0, 1)$ and $(3, 7)$.
   (b) f is decreasing on $(1, 3)$.
   (c) f is concave upward on $(2, 4)$ and $(5, 7)$.
   (d) f is concave downward on $(0, 2)$ and $(4, 5)$.
   (e) The points of inflection are $(2, 2)$, $(4, 3)$, and $(5, 4)$.

3. (a) Use the Increasing/Decreasing (I/D) Test.
   (b) Use the Concavity Test.
   (c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

4. (a) See the First Derivative Test.
   (b) See the Second Derivative Test and the note that precedes Example 7.

5. (a) Since $f'(x) > 0$ on $(1, 5)$, f is increasing on this interval. Since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$. Since $f'(x) = 0$ at $x = 5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 5$.

6. (a) $f'(x) > 0$ and f is increasing on $(0, 1)$ and $(5, 7)$. $f'(x) < 0$ and f is decreasing on $(1, 5)$ and $(7, 8)$.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 7$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 7$. Since $f'(x) = 0$ at $x = 5$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 5$.

7. (a) There is an IP at $x = 3$ because the graph of f changes from CD to CU there. There is an IP at $x = 5$ because the graph of f changes from CU to CD there.

(b) There is an IP at $x = 2$ and at $x = 6$ because $f'(x)$ has a maximum value there, and so $f''(x)$ changes from positive to negative there. There is an IP at $x = 4$ because $f'(x)$ has a minimum value there and so $f''(x)$ changes from negative to positive there.

(c) There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

8. (a) f is increasing when f' is positive. This happens on the intervals $(0, 4)$ and $(6, 8)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$ and $x = 8$). Similarly, f has a local minimum where f' changes from negative to positive (at $x = 6$).

(c) f is concave upward where f' is increasing (hence f'' is positive). This happens on $(0, 1)$, $(2, 3)$, and $(5, 7)$. Similarly, f is concave downward where f' is decreasing, that is, on $(1, 2)$, $(3, 5)$, and $(7, 9)$.

(d) f has an inflection point where the concavity changes. This happens at $x = 1$, 2, 3, 5, and 7.

9. (a) $f(x) = x^3 - 3x^2 - 9x + 4 \Rightarrow f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3)$.

Interval	$x + 1$	$x - 3$	$f'(x)$	f
$x < -1$	$-$	$-$	$+$	increasing on $(-\infty, -1)$
$-1 < x < 3$	$+$	$-$	$-$	decreasing on $(-1, 3)$
$x > 3$	$+$	$+$	$+$	increasing on $(3, \infty)$

(b) f changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = 3$. Thus, $f(-1) = 9$ is a local maximum value and $f(3) = -23$ is a local minimum vlaue.

(c) $f''(x) = 6x - 6 = 6(x - 1)$. $f''(x) > 0 \Leftrightarrow x > 1$ and $f''(x) < 0 \Leftrightarrow x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, -7)$.

10. (a) $f(x) = 2x^3 - 9x^2 + 12x - 3 \quad \Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$.

Interval	$x-1$	$x-2$	$f'(x)$	f
$x < 1$	$-$	$-$	$+$	increasing on $(-\infty, 1)$
$1 < x < 2$	$+$	$-$	$-$	decreasing on $(1, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = 1$ and from decreasing to increasing at $x = 2$. Thus, $f(1) = 2$ is a local maximum value and $f(2) = 1$ is a local minimum value.

(c) $f''(x) = 12x - 18 = 12\left(x - \frac{3}{2}\right)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x > \frac{3}{2}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < \frac{3}{2}$. Thus, f is concave upward on $\left(\frac{3}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{3}{2}\right)$. There is an inflection point at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

11. (a) $f(x) = x^4 - 2x^2 + 3 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x+1)(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

12. (a) $f(x) = \dfrac{x}{x^2 + 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1 - x^2}{(x^2+1)^2} = -\dfrac{(x+1)(x-1)}{(x^2+1)^2}$. Thus, $f'(x) > 0$ if $(x+1)(x-1) < 0 \quad \Leftrightarrow \quad -1 < x < 1$, and $f'(x) < 0$ if $x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) f changes from decreasing to increasing at $x = -1$ and from increasing to decreasing at $x = 1$. Thus, $f(-1) = -\frac{1}{2}$ is a local minimum value and $f(1) = \frac{1}{2}$ is a local maximum value.

(c) $f''(x) = \dfrac{(x^2+1)^2(-2x) - (1-x^2)[2(x^2+1)(2x)]}{[(x^2+1)^2]^2} = \dfrac{(x^2+1)(-2x)[(x^2+1) + 2(1-x^2)]}{(x^2+1)^4} = \dfrac{2x(x^2-3)}{(x^2+1)^3}$.

$f''(x) > 0 \quad \Leftrightarrow \quad -\sqrt{3} < x < 0$ or $x > \sqrt{3}$, and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\sqrt{3}$ or $0 < x < \sqrt{3}$. Thus, f is concave upward on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$ and concave downward on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$. There are inflection points at $\left(-\sqrt{3}, -\sqrt{3}/4\right)$, $(0, 0)$, and $\left(\sqrt{3}, \sqrt{3}/4\right)$.

13. (a) $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$. $f'(x) = \cos x - \sin x = 0 \quad \Rightarrow \quad \cos x = \sin x \quad \Rightarrow \quad 1 = \dfrac{\sin x}{\cos x} \quad \Rightarrow$ $\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad \cos x - \sin x > 0 \quad \Leftrightarrow \quad \cos x > \sin x \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{4}$ or $\frac{5\pi}{4} < x < 2\pi$ and $f'(x) < 0 \quad \Leftrightarrow \quad \cos x < \sin x \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$. So f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{\pi}{4}$ and from decreasing to increasing at $x = \frac{5\pi}{4}$. Thus, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum value and $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum value.

(c) $f''(x) = -\sin x - \cos x = 0 \quad \Rightarrow \quad -\sin x = \cos x \quad \Rightarrow \quad \tan x = -1 \quad \Rightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Divide the interval $(0, 2\pi)$ into subintervals with these numbers as endpoints and complete a second derivative chart.

Interval	$f''(x) = -\sin x - \cos x$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$f''\left(\frac{\pi}{2}\right) = -1 < 0$	downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$f''(\pi) = 1 > 0$	upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$f''\left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3} < 0$	downward

There are inflection points at $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

14. (a) $f(x) = \cos^2 x - 2\sin x,\ \ 0 \le x \le 2\pi$. $f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x\,(1 + \sin x)$. Note that $1 + \sin x \ge 0$ [since $\sin x \ge -1$], with equality $\Leftrightarrow \quad \sin x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{2}$ [since $0 \le x \le 2\pi$] $\Rightarrow$ $\cos x = 0$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad \cos x < 0 \quad \Leftrightarrow \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$ and $f'(x) < 0 \quad \Leftrightarrow \quad \cos x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) f changes from decreasing to increasing at $x = \frac{\pi}{2}$ and from increasing to decreasing at $x = \frac{3\pi}{2}$. Thus, $f\left(\frac{\pi}{2}\right) = -2$ is a local minimum value and $f\left(\frac{3\pi}{2}\right) = 2$ is a local maximum value.

(c) $f''(x) = 2\sin x\,(1 + \sin x) - 2\cos^2 x = 2\sin x + 2\sin^2 x - 2(1 - \sin^2 x)$

$= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1)$

so $f''(x) > 0 \quad \Leftrightarrow \quad \sin x > \frac{1}{2} \quad \Leftrightarrow \quad \frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0 \quad \Leftrightarrow \quad \sin x < \frac{1}{2}$ and $\sin x \ne -1 \quad \Leftrightarrow$ $0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and concave downward on $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

15. (a) $f(x) = e^{2x} + e^{-x} \quad \Rightarrow \quad f'(x) = 2e^{2x} - e^{-x}$. $f'(x) > 0 \quad \Leftrightarrow \quad 2e^{2x} > e^{-x} \quad \Leftrightarrow \quad e^{3x} > \frac{1}{2} \quad \Leftrightarrow \quad 3x > \ln\frac{1}{2} \quad \Leftrightarrow$ $x > \frac{1}{3}(\ln 1 - \ln 2) \quad \Leftrightarrow \quad x > -\frac{1}{3}\ln 2$ [≈ -0.23] and $f'(x) < 0$ if $x < -\frac{1}{3}\ln 2$. So f is increasing on $\left(-\frac{1}{3}\ln 2, \infty\right)$ and f is decreasing on $\left(-\infty, -\frac{1}{3}\ln 2\right)$.

(b) f changes from decreasing to increasing at $x = -\frac{1}{3}\ln 2$. Thus,

$f\left(-\frac{1}{3}\ln 2\right) = f\left(\ln\sqrt[3]{1/2}\right) = e^{2\ln\sqrt[3]{1/2}} + e^{-\ln\sqrt[3]{1/2}} = e^{\ln\sqrt[3]{1/4}} + e^{\ln\sqrt[3]{2}} = \sqrt[3]{1/4} + \sqrt[3]{2} = 2^{-2/3} + 2^{1/3}$ [≈ 1.89] is a local minimum value.

(c) $f''(x) = 4e^{2x} + e^{-x} > 0$ [the sum of two positive terms]. Thus, f is concave upward on $(-\infty, \infty)$ and there is no point of inflection.

16. (a) $f(x) = x^2 \ln x \quad \Rightarrow \quad f'(x) = x^2(1/x) + (\ln x)(2x) = x + 2x \ln x = x(1 + 2\ln x)$. The domain of f is $(0, \infty)$, so the sign of f' is determined solely by the factor $1 + 2\ln x$. $f'(x) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$ [≈ 0.61] and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < e^{-1/2}$. So f is increasing on $(e^{-1/2}, \infty)$ and f is decreasing on $(0, e^{-1/2})$.

(b) f changes from decreasing to increasing at $x = e^{-1/2}$. Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$ [≈ -0.18] is a local minimum value.

(c) $f'(x) = x(1 + 2\ln x) \quad \Rightarrow \quad f''(x) = x(2/x) + (1 + 2\ln x) \cdot 1 = 2 + 1 + 2\ln x = 3 + 2\ln x$. $f''(x) > 0 \quad \Leftrightarrow$ $3 + 2\ln x > 0 \quad \Leftrightarrow \quad \ln x > -3/2 \quad \Leftrightarrow \quad x > e^{-3/2}$ [≈ 0.22]. Thus, f is concave upward on $(e^{-3/2}, \infty)$ and f is concave downward on $(0, e^{-3/2})$. $f(e^{-3/2}) = (e^{-3/2})^2 \ln e^{-3/2} = e^{-3}(-3/2) = -3/(2e^3)$ [≈ -0.07]. There is a point of inflection at $\left(e^{-3/2}, f(e^{-3/2})\right) = \left(e^{-3/2}, -3/(2e^3)\right)$.

17. (a) $f(x) = x^2 - x - \ln x \quad \Rightarrow \quad f'(x) = 2x - 1 - \dfrac{1}{x} = \dfrac{2x^2 - x - 1}{x} = \dfrac{(2x+1)(x-1)}{x}$. Thus, $f'(x) > 0$ if $x > 1$ [note that $x > 0$] and $f'(x) < 0$ if $0 < x < 1$. So f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.

(b) f changes from decreasing to increasing at $x = 1$. Thus, $f(1) = 0$ is a local minimum value.

(c) $f''(x) = 2 + 1/x^2 > 0$ for all x, so f is concave upward on $(0, \infty)$. There is no inflection point.

18. (a) $f(x) = x^4 e^{-x} \quad \Rightarrow \quad f'(x) = x^4(-e^{-x}) + e^{-x}(4x^3) = x^3 e^{-x}(-x + 4)$. Thus, $f'(x) > 0$ if $0 < x < 4$ and $f'(x) < 0$ if $x < 0$ or $x > 4$. So f is increasing on $(0, 4)$ and decreasing on $(-\infty, 0)$ and $(4, \infty)$.

(b) f changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 4$. Thus, $f(0) = 0$ is a local minimum value and $f(4) = 256/e^4$ is a local maximum value.

(c) $f'(x) = e^{-x}(-x^4 + 4x^3) \quad \Rightarrow$

$$\begin{aligned} f''(x) &= e^{-x}(-4x^3 + 12x^2) + (-x^4 + 4x^3)(-e^{-x}) = e^{-x}[(-4x^3 + 12x^2) - (-x^4 + 4x^3)] \\ &= e^{-x}(x^4 - 8x^3 + 12x^2) = x^2 e^{-x}(x^2 - 8x + 12) = x^2 e^{-x}(x - 2)(x - 6) \end{aligned}$$

$f''(x) > 0 \quad \Leftrightarrow \quad x < 2$ [excluding 0] or $x > 6$ and $f''(x) < 0 \quad \Leftrightarrow \quad 2 < x < 6$. Thus, f is concave upward on $(-\infty, 2)$ and $(6, \infty)$ and f is concave downward on $(2, 6)$. There are inflection points at $(2, 16e^{-2})$ and $(6, 1296e^{-6})$.

19. $f(x) = 1 + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 6x - 6x^2 = 6x(1 - x)$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad 0 < x < 1$ and $f'(x) < 0 \quad \Rightarrow \quad x < 0$ or $x > 1$. Since f' changes from negative to positive at $x = 0$, $f(0) = 1$ is a local minimum value; and since f' changes from positive to negative at $x = 1$, $f(1) = 2$ is a local maximum value.

Second Derivative Test: $f''(x) = 6 - 12x$. $f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 1$. $f''(0) = 6 > 0 \quad \Rightarrow \quad f(0) = 1$ is a local minimum value. $f''(1) = -6 < 0 \quad \Rightarrow \quad f(1) = 2$ is a local maximum value.

Preference: For this function, the two tests are equally easy.

20. $f(x) = \dfrac{x^2}{x-1} \quad \Rightarrow \quad f'(x) = \dfrac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad \Rightarrow \quad 0 < x < 1$ or $1 < x < 2$. Since f' changes from positive to negative at $x = 0$, $f(0) = 0$ is a local maximum value; and since f' changes from negative to positive at $x = 2$, $f(2) = 4$ is a local minimum value.

Second Derivative Test:

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{[(x-1)^2]^2} = \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} = \frac{2}{(x-1)^3}.$$

$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 2$. $f''(0) = -2 < 0 \quad \Rightarrow \quad f(0) = 0$ is a local maximum value. $f''(2) = 2 > 0 \quad \Rightarrow \quad f(2) = 4$ is a local minimum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

21. $f(x) = \sqrt{x} - \sqrt[4]{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{4}x^{-3/4} = \dfrac{1}{4}x^{-3/4}(2x^{1/4} - 1) = \dfrac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}}$

First Derivative Test: $2\sqrt[4]{x} - 1 > 0 \quad \Rightarrow \quad x > \frac{1}{16}$, so $f'(x) > 0 \quad \Rightarrow \quad x > \frac{1}{16}$ and $f'(x) < 0 \quad \Rightarrow \quad 0 < x < \frac{1}{16}$. Since f' changes from negative to positive at $x = \frac{1}{16}$, $f(\frac{1}{16}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ is a local minimum value.

Second Derivative Test: $f''(x) = -\dfrac{1}{4}x^{-3/2} + \dfrac{3}{16}x^{-7/4} = -\dfrac{1}{4\sqrt{x^3}} + \dfrac{3}{16\sqrt[4]{x^7}}$.

$f'(x) = 0 \quad \Leftrightarrow \quad x = \frac{1}{16}$. $f''(\frac{1}{16}) = -16 + 24 = 8 > 0 \quad \Rightarrow \quad f(\frac{1}{16}) = -\frac{1}{4}$ is a local minimum value.

Preference: The First Derivative Test may be slightly easier to apply in this case.

22. (a) $f(x) = x^4(x-1)^3 \quad \Rightarrow \quad f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$

The critical numbers are 0, 1, and $\frac{4}{7}$.

(b) $$\begin{aligned} f''(x) &= 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7 \\ &= x^2(x-1)[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)] \end{aligned}$$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

$f''(\frac{4}{7}) = (\frac{4}{7})^2(\frac{4}{7} - 1)[0 + 0 + 7(\frac{4}{7})(\frac{4}{7} - 1)] = (\frac{4}{7})^2(-\frac{3}{7})(4)(-\frac{3}{7}) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

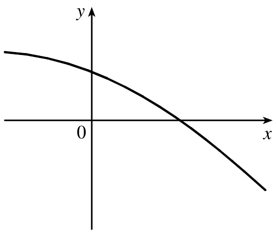
(c) f' is positive on $(-\infty, 0)$, negative on $(0, \frac{4}{7})$, positive on $(\frac{4}{7}, 1)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

23. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

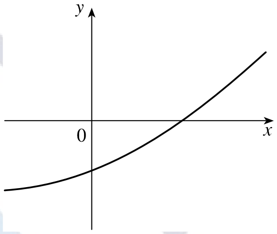
24. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave downward (since the second derivative is always negative).



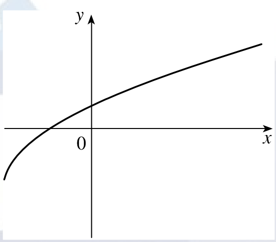
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave upward (since the second derivative is always positive).



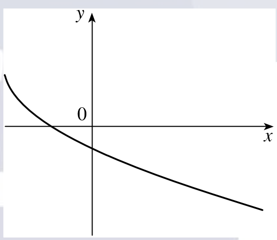
25. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave downward (since the second derivative is always negative).



(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave upward (since the second derivative is always positive).



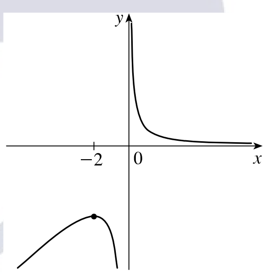
26. Vertical asymptote $x = 0$

$f'(x) > 0$ if $x < -2 \quad \Rightarrow \quad f$ is increasing on $(-\infty, -2)$.

$f'(x) < 0$ if $x > -2$ $(x \neq 0) \quad \Rightarrow \quad f$ is decreasing on $(-2, 0)$ and $(0, \infty)$.

$f''(x) < 0$ if $x < 0 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 0)$.

$f''(x) > 0$ if $x > 0 \quad \Rightarrow \quad f$ is concave upward on $(0, \infty)$.



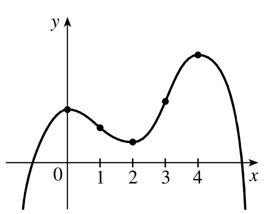
27. $f'(0) = f'(2) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0, 2, 4$.

$f'(x) > 0$ if $x < 0$ or $2 < x < 4 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 0)$ and $(2, 4)$.

$f'(x) < 0$ if $0 < x < 2$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(0, 2)$ and $(4, \infty)$.

$f''(x) > 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave upward on $(1, 3)$.

$f''(x) < 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.



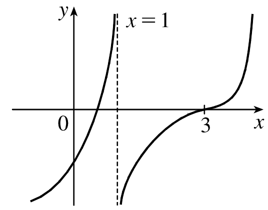
28. $f'(x) > 0$ for all $x \neq 1 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 1)$ and $(1, \infty)$.

Vertical asymptote $x = 1$

$f''(x) > 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 1)$ and $(3, \infty)$.

$f''(x) < 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave downward on $(1, 3)$.

There is an inflection point at $x = 3$.

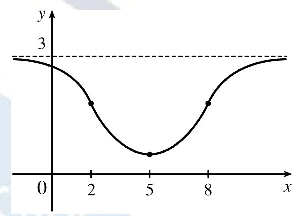


29. $f'(5) = 0 \quad \Rightarrow \quad$ horizontal tangent at $x = 5$.

$f'(x) < 0$ when $x < 5 \quad \Rightarrow \quad f$ is decreasing on $(-\infty, 5)$.

$f'(x) > 0$ when $x > 5 \quad \Rightarrow \quad f$ is increasing on $(5, \infty)$.

$f''(2) = 0, \quad f''(8) = 0, \quad f''(x) < 0$ when $x < 2$ or $x > 8$,



$f''(x) > 0$ for $2 < x < 8 \quad \Rightarrow \quad f$ is concave upward on $(2, 8)$ and concave downward on $(-\infty, 2)$ and $(8, \infty)$.

There are inflection points at $x = 2$ and $x = 8$.

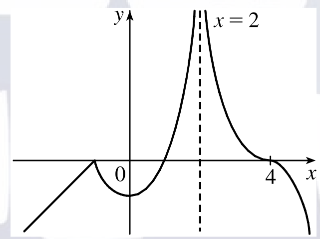
$\lim\limits_{x \to \infty} f(x) = 3, \quad \lim\limits_{x \to -\infty} f(x) = 3 \quad \Rightarrow \quad y = 3$ is a horizontal asymptote.

30. $f'(0) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0$ and 4.

$f'(x) = 1$ if $x < -1 \quad \Rightarrow \quad f$ is a line with slope 1 on $(-\infty, -1)$.

$f'(x) > 0$ if $0 < x < 2 \quad \Rightarrow \quad f$ is increasing on $(0, 2)$.

$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(-1, 0)$, $(2, 4)$, and $(4, \infty)$.



$\lim\limits_{x \to 2^-} f'(x) = \infty \quad \Rightarrow \quad f'$ increases without bound as $x \to 2^-$.

$\lim\limits_{x \to 2^+} f'(x) = -\infty \quad \Rightarrow \quad f'$ decreases without bound as $x \to 2^+$.

$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4 \quad \Rightarrow \quad f$ is concave upward on $(-1, 2)$ and $(2, 4)$.

$f''(x) < 0$ if $x > 4 \quad \Rightarrow \quad f$ is concave downward on $(4, \infty)$.

31. $f'(x) > 0$ if $x \neq 2 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 2)$ and $(2, \infty)$.

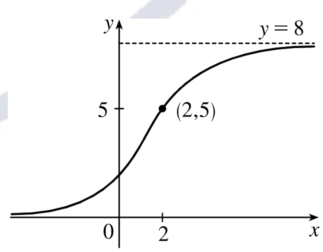
$f''(x) > 0$ if $x < 2 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 2)$.

$f''(x) < 0$ if $x > 2 \quad \Rightarrow \quad f$ is concave downward on $(2, \infty)$.

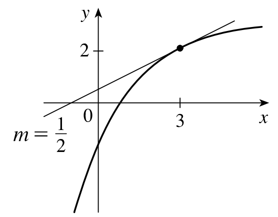
f has inflection point $(2, 5) \quad \Rightarrow \quad f$ changes concavity at the point $(2, 5)$.

$\lim\limits_{x \to \infty} f(x) = 8 \quad \Rightarrow \quad f$ has a horizontal asymptote of $y = 8$ as $x \to \infty$.

$\lim\limits_{x \to -\infty} f(x) = 0 \quad \Rightarrow \quad f$ has a horizontal asymptote of $y = 0$ as $x \to -\infty$.



32. (a) $f(3) = 2 \Rightarrow$ the point $(3, 2)$ is on the graph of f. $f'(3) = \frac{1}{2} \Rightarrow$ the slope of the tangent line at $(3, 2)$ is $\frac{1}{2}$. $f'(x) > 0$ for all $x \Rightarrow f$ is increasing on $\mathbb{R}$. $f''(x) < 0$ for all $x \Rightarrow f$ is concave downward on $\mathbb{R}$. A possible graph for f is shown.

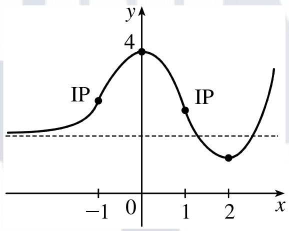


(b) The tangent line at $(3, 2)$ has equation $y - 2 = \frac{1}{2}(x - 3)$, or $y = \frac{1}{2}x + \frac{1}{2}$, and x-intercept -1. Since f is concave downward on $\mathbb{R}$, f is below the x-axis at $x = -1$, and hence changes sign at least once. Since f is increasing on $\mathbb{R}$, it changes sign at most once. Thus, it changes sign exactly once and there is one solution of the equation $f(x) = 0$.

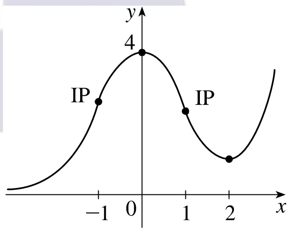
(c) $f'' < 0 \Rightarrow f'$ is decreasing. Since $f'(3) = \frac{1}{2}$, $f'(2)$ must be greater than $\frac{1}{2}$, so no, it is not possible that $f'(2) = \frac{1}{3}$.

33. (a) Intuitively, since f is continuous, increasing, and concave upward for $x > 2$, it cannot have an absolute maximum. For a proof, we appeal to the MVT. Let $x = d > 2$. Then by the MVT, $f(d) - f(2) = f'(c)(d - 2)$ for some c such that $2 < c < d$. So $f(d) = f(2) + f'(c)(d - 2)$ where $f(2)$ is positive since $f(x) > 0$ for all x and $f'(c)$ is positive since $f'(x) > 0$ for $x > 2$. Thus, as $d \to \infty$, $f(d) \to \infty$, and no absolute maximum exists.

(b) Yes, the local minimum at $x = 2$ can be an absolute minimum.



(c) Here $f(x) \to 0$ as $x \to -\infty$, but f does not achieve an absolute minimum.



34. (a) $\dfrac{dy}{dx} > 0$ (f is increasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point B.

(b) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} < 0$ (f is concave downward) at point E.

(c) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point A.

Note: At C, $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$. At D, $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} \leq 0$.

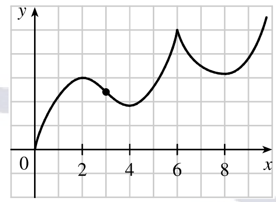
35. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



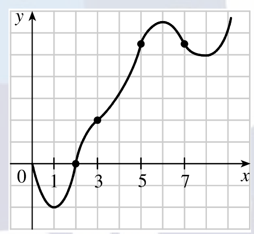
36. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)

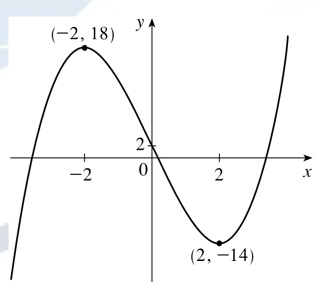


37. (a) $f(x) = x^3 - 12x + 2 \quad \Rightarrow \quad f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$. $f'(x) > 0 \quad \Leftrightarrow \quad x < -2$ or $x > 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad -2 < x < 2$. So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and f is decreasing on $(-2, 2)$.

(b) f changes from increasing to decreasing at $x = -2$, so $f(-2) = 18$ is a local maximum value. f changes from decreasing to increasing at $x = 2$, so $f(2) = -14$ is a local minimum value.

(c) $f''(x) = 6x$. $f''(x) = 0 \quad \Leftrightarrow \quad x = 0$. $f''(x) > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$. So f is concave upward on $(0, \infty)$ and f is concave downward on $(-\infty, 0)$. There is an inflection point at $(0, 2)$.

(d)

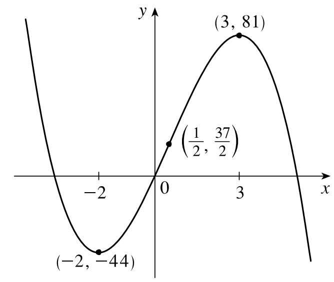


38. (a) $f(x) = 36x + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x + 2)(x - 3)$. $f'(x) > 0 \quad \Leftrightarrow$ $-2 < x < 3$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $x > 3$. So f is increasing on $(-2, 3)$ and f is decreasing on $(-\infty, -2)$ and $(3, \infty)$.

(b) f changes from increasing to decreasing at $x = 3$, so $f(3) = 81$ is a local maximum value. f changes from decreasing to increasing at $x = -2$, so $f(-2) = -44$ is a local minimum value.

(c) $f''(x) = 6 - 12x$. $f''(x) = 0 \Leftrightarrow x = \frac{1}{2}$. $f''(x) > 0$ on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x) < 0$ on $\left(\frac{1}{2}, \infty\right)$. So f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, \frac{37}{2}\right)$.

(d)

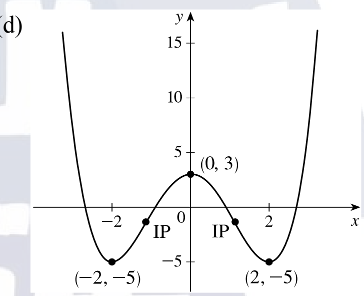


39. (a) $f(x) = \frac{1}{2}x^4 - 4x^2 + 3 \Rightarrow f'(x) = 2x^3 - 8x = 2x(x^2 - 4) = 2x(x+2)(x-2)$. $f'(x) > 0 \Leftrightarrow -2 < x < 0$ or $x > 2$, and $f'(x) < 0 \Leftrightarrow x < -2$ or $0 < x < 2$. So f is increasing on $(-2, 0)$ and $(2, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$, so $f(0) = 3$ is a local maximum value.
f changes from decreasing to increasing at $x = \pm 2$, so $f(\pm 2) = -5$ is a local minimum value.

(c) $f''(x) = 6x^2 - 8 = 6\left(x^2 - \frac{4}{3}\right) = 6\left(x + \frac{2}{\sqrt{3}}\right)\left(x - \frac{2}{\sqrt{3}}\right)$.
$f''(x) = 0 \Leftrightarrow x = \pm\frac{2}{\sqrt{3}}$. $f''(x) > 0$ on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and $f''(x) < 0$ on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. So f is CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$, and f is CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. There are inflection points at $\left(\pm\frac{2}{\sqrt{3}}, -\frac{13}{9}\right)$.

(d)

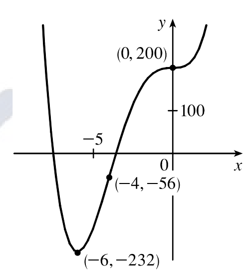


40. (a) $g(x) = 200 + 8x^3 + x^4 \Rightarrow g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$.
$g'(x) > 0 \Leftrightarrow x > -6$ $[x \neq 0]$ and $g'(x) < 0 \Leftrightarrow x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$.
$g''(x) > 0 \Leftrightarrow x < -4$ or $x > 0$ and $g''(x) < 0 \Leftrightarrow -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. There are inflection points at $(-4, -56)$ and $(0, 200)$.

(d)

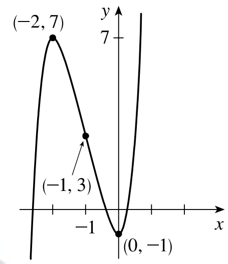


41. (a) $h(x) = (x+1)^5 - 5x - 2 \Rightarrow h'(x) = 5(x+1)^4 - 5$. $h'(x) = 0 \Leftrightarrow 5(x+1)^4 = 5 \Leftrightarrow (x+1)^4 = 1 \Rightarrow (x+1)^2 = 1 \Rightarrow x + 1 = 1$ or $x + 1 = -1 \Rightarrow x = 0$ or $x = -2$. $h'(x) > 0 \Leftrightarrow x < -2$ or $x > 0$ and $h'(x) < 0 \Leftrightarrow -2 < x < 0$. So h is increasing on $(-\infty, -2)$ and $(0, \infty)$ and h is decreasing on $(-2, 0)$.

(b) $h(-2) = 7$ is a local maximum value and $h(0) = -1$ is a local minimum value.

(c) $h''(x) = 20(x+1)^3 = 0 \iff x = -1$. $h''(x) > 0 \iff x > -1$ and $h''(x) < 0 \iff x < -1$, so h is CU on $(-1, \infty)$ and h is CD on $(-\infty, -1)$. There is a point of inflection at $(-1, h(-1)) = (-1, 3)$.

(d)

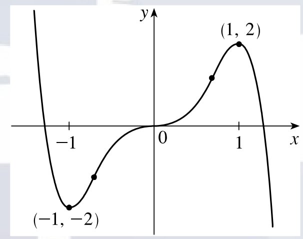


42. (a) $h(x) = 5x^3 - 3x^5 \Rightarrow h'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1+x)(1-x)$. $h'(x) > 0 \iff -1 < x < 0$ and $0 < x < 1$ [note that $h'(0) = 0$] and $h'(x) < 0 \iff x < -1$ or $x > 1$. So h is increasing on $(-1, 1)$ and h is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) h changes from decreasing to increasing at $x = -1$, so $h(-1) = -2$ is a local minimum value. h changes from increasing to decreasing at $x = 1$, so $h(1) = 2$ is a local maximum value.

(c) $h''(x) = 30x - 60x^3 = 30x(1 - 2x^2)$. $h''(x) = 0 \iff x = 0$ or $1 - 2x^2 = 0 \iff x = 0$ or $x = \pm 1/\sqrt{2}$. $h''(x) > 0$ on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and $h''(x) < 0$ on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. So h is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and h is CD on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. There are inflection points at $\left(-1/\sqrt{2}, -7/\left(4\sqrt{2}\right)\right)$, $(0, 0)$, and $\left(1/\sqrt{2}, 7/\left(4\sqrt{2}\right)\right)$.

(d)



43. (a) $F(x) = x\sqrt{6 - x} \Rightarrow$

$$F'(x) = x \cdot \tfrac{1}{2}(6-x)^{-1/2}(-1) + (6-x)^{1/2}(1) = \tfrac{1}{2}(6-x)^{-1/2}[-x + 2(6-x)] = \frac{-3x + 12}{2\sqrt{6-x}}.$$

$F'(x) > 0 \iff -3x + 12 > 0 \iff x < 4$ and $F'(x) < 0 \iff 4 < x < 6$. So F is increasing on $(-\infty, 4)$ and F is decreasing on $(4, 6)$.

(b) F changes from increasing to decreasing at $x = 4$, so $F(4) = 4\sqrt{2}$ is a local maximum value. There is no local minimum value.

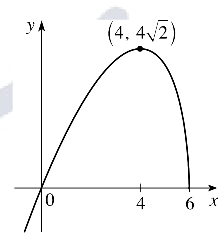
(c) $F'(x) = -\frac{3}{2}(x-4)(6-x)^{-1/2} \Rightarrow$

$$F''(x) = -\tfrac{3}{2}\left[(x-4)\left(-\tfrac{1}{2}(6-x)^{-3/2}(-1)\right) + (6-x)^{-1/2}(1)\right]$$

$$= -\frac{3}{2} \cdot \frac{1}{2}(6-x)^{-3/2}[(x-4) + 2(6-x)] = \frac{3(x-8)}{4(6-x)^{3/2}}$$

$F''(x) < 0$ on $(-\infty, 6)$, so F is CD on $(-\infty, 6)$. There is no inflection point.

(d)



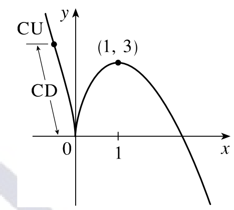
44. (a) $G(x) = 5x^{2/3} - 2x^{5/3} \Rightarrow G'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}x^{-1/3}(1 - x) = \dfrac{10(1-x)}{3x^{1/3}}$.

$G'(x) > 0 \iff 0 < x < 1$ and $G'(x) < 0 \iff x < 0$ or $x > 1$. So G is increasing on $(0, 1)$ and G is decreasing on $(-\infty, 0)$ and $(1, \infty)$.

(b) G changes from decreasing to increasing at $x = 0$, so $G(0) = 0$ is a local minimum value. G changes from increasing to decreasing at $x = 1$, so $G(1) = 3$ is a local maximum value. Note that the First Derivative Test applies at $x = 0$ even though G' is not defined at $x = 0$, since G is continuous at 0.

(c) $G''(x) = -\frac{10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3} = -\frac{10}{9}x^{-4/3}(1 + 2x)$. $G''(x) > 0 \quad \Leftrightarrow \quad x < -\frac{1}{2}$ and $G''(x) < 0 \quad \Leftrightarrow \quad -\frac{1}{2} < x < 0$ or $x > 0$. So G is CU on $\left(-\infty, -\frac{1}{2}\right)$ and G is CD on $\left(-\frac{1}{2}, 0\right)$ and $(0, \infty)$. The only change in concavity occurs at $x = -\frac{1}{2}$, so there is an inflection point at $\left(-\frac{1}{2}, 6/\sqrt[3]{4}\right)$.

(d)



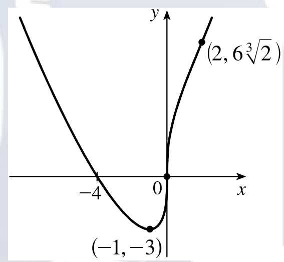
45. (a) $C(x) = x^{1/3}(x + 4) = x^{4/3} + 4x^{1/3} \quad \Rightarrow \quad C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x + 1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x - 2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)



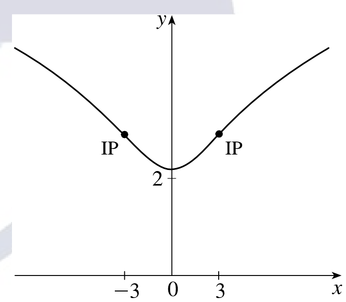
46. (a) $f(x) = \ln(x^2 + 9) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2+9} \cdot 2x = \dfrac{2x}{x^2+9}$. $f'(x) > 0 \quad \Leftrightarrow \quad 2x > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$, so $f(0) = \ln 9$ is a local minimum value. There is no local maximum value.

(c) $f''(x) = \dfrac{(x^2+9) \cdot 2 - 2x(2x)}{(x^2+9)^2} = \dfrac{18 - 2x^2}{(x^2+9)^2} = \dfrac{-2(x+3)(x-3)}{(x^2+9)^2}$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \pm 3$. $f''(x) > 0$ on $(-3, 3)$ and $f''(x) < 0$ on $(-\infty, -3)$ and $(3, \infty)$. So f is CU on $(-3, 3)$, and f is CD on $(-\infty, -3)$ and $(3, \infty)$. There are inflection points at $(\pm 3, \ln 18)$.

(d)



47. (a) $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi \quad \Rightarrow \quad f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta) = -2\sin\theta\,(1 + \cos\theta)$.

$f'(\theta) = 0 \quad \Leftrightarrow \quad \theta = 0, \pi$, and 2π. $f'(\theta) > 0 \quad \Leftrightarrow \quad \pi < \theta < 2\pi$ and $f'(\theta) < 0 \quad \Leftrightarrow \quad 0 < \theta < \pi$. So f is increasing on $(\pi, 2\pi)$ and f is decreasing on $(0, \pi)$.

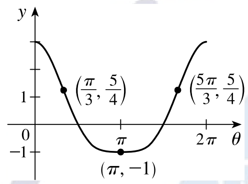
(b) $f(\pi) = -1$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta\,(1+\cos\theta) \quad\Rightarrow$

$$f''(\theta) = -2\sin\theta\,(-\sin\theta) + (1+\cos\theta)(-2\cos\theta) = 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta$$
$$= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta = -4\cos^2\theta - 2\cos\theta + 2$$
$$= -2(2\cos^2\theta + \cos\theta - 1) = -2(2\cos\theta - 1)(\cos\theta + 1)$$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \neq \pi$], $f''(\theta) > 0 \quad\Rightarrow\quad 2\cos\theta - 1 < 0 \quad\Rightarrow\quad \cos\theta < \frac{1}{2} \quad\Rightarrow\quad \frac{\pi}{3} < \theta < \frac{5\pi}{3}$ and $f''(\theta) < 0 \quad\Rightarrow\quad \cos\theta > \frac{1}{2} \quad\Rightarrow\quad 0 < \theta < \frac{\pi}{3}$ or $\frac{5\pi}{3} < \theta < 2\pi$. So f is CU on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and f is CD on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are points of inflection at $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5\pi}{3}, \frac{5}{4}\right)$.

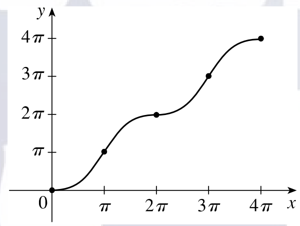
(d)



48. (a) $S(x) = x - \sin x,\ 0 \le x \le 4\pi \quad\Rightarrow\quad S'(x) = 1 - \cos x$. $S'(x) = 0 \quad\Leftrightarrow\quad \cos x = 1 \quad\Leftrightarrow\quad x = 0, 2\pi$, and 4π. $S'(x) > 0 \quad\Leftrightarrow\quad \cos x < 1$, which is true for all x except integer multiples of 2π, so S is increasing on $(0, 4\pi)$ since $S'(2\pi) = 0$.

(b) There is no local maximum or minimum.

(c)



(d) $S''(x) = \sin x$. $S''(x) > 0$ if $0 < x < \pi$ or $2\pi < x < 3\pi$, and $S''(x) < 0$ if $\pi < x < 2\pi$ or $3\pi < x < 4\pi$. So S is CU on $(0, \pi)$ and $(2\pi, 3\pi)$, and S is CD on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$. There are inflection points at (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

49. $f(x) = 1 + \dfrac{1}{x} - \dfrac{1}{x^2}$ has domain $(-\infty, 0) \cup (0, \infty)$.

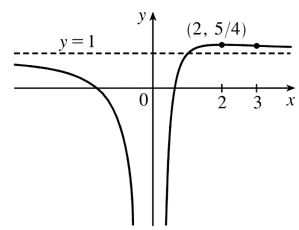
(a) $\displaystyle\lim_{x\to\pm\infty}\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) = 1$, so $y = 1$ is a HA. $\displaystyle\lim_{x\to 0^+}\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{x^2 + x - 1}{x^2}\right) = -\infty$ since $(x^2 + x - 1) \to -1$ and $x^2 \to 0$ as $x \to 0^+$ [a similar argument can be made for $x \to 0^-$], so $x = 0$ is a VA.

(b) $f'(x) = -\dfrac{1}{x^2} + \dfrac{2}{x^3} = -\dfrac{1}{x^3}(x - 2)$. $f'(x) = 0 \quad\Leftrightarrow\quad x = 2$. $f'(x) > 0 \quad\Leftrightarrow\quad 0 < x < 2$ and $f'(x) < 0 \quad\Leftrightarrow\quad x < 0$ or $x > 2$. So f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

(c) f changes from increasing to decreasing at $x = 2$, so $f(2) = \frac{5}{4}$ is a local maximum value. There is no local minimum value.

(d) $f''(x) = \dfrac{2}{x^3} - \dfrac{6}{x^4} = \dfrac{2}{x^4}(x - 3)$. $f''(x) = 0 \quad\Leftrightarrow\quad x = 3$. $f''(x) > 0 \quad\Leftrightarrow\quad x > 3$ and $f''(x) < 0 \quad\Leftrightarrow\quad x < 0$ or $0 < x < 3$. So f is CU on $(3, \infty)$ and f is CD on $(-\infty, 0)$ and $(0, 3)$. There is an inflection point at $\left(3, \frac{11}{9}\right)$.

(e)



50. $f(x) = \dfrac{x^2-4}{x^2+4}$ has domain $\mathbb{R}$.

(a) $\displaystyle\lim_{x\to\pm\infty} \frac{x^2-4}{x^2+4} = \lim_{x\to\pm\infty} \frac{1-4/x^2}{1+4/x^2} = \frac{1}{1} = 1$, so $y = 1$ is a HA. There is no vertical asymptote.

(b) $f'(x) = \dfrac{(x^2+4)(2x)-(x^2-4)(2x)}{(x^2+4)^2} = \dfrac{2x[(x^2+4)-(x^2-4)]}{(x^2+4)^2} = \dfrac{16x}{(x^2+4)^2}$. $f'(x) > 0 \quad\Leftrightarrow\quad x > 0$ and

$f'(x) < 0 \quad\Leftrightarrow\quad x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(c) f changes from decreasing to increasing at $x = 0$, so $f(0) = -1$ is a local minimum value.

(d) $f''(x) = \dfrac{(x^2+4)^2(16) - 16x \cdot 2(x^2+4)(2x)}{[(x^2+4)^2]^2} = \dfrac{16(x^2+4)[(x^2+4)-4x^2]}{(x^2+4)^4} = \dfrac{16(4-3x^2)}{(x^2+4)^3}$.

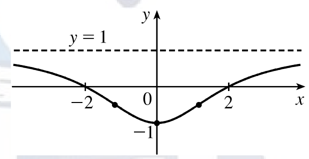
$f''(x) = 0 \quad\Leftrightarrow\quad x = \pm 2/\sqrt{3}$. $f''(x) > 0 \quad\Leftrightarrow\quad -2/\sqrt{3} < x < 2/\sqrt{3}$

and $f''(x) < 0 \quad\Leftrightarrow\quad x < -2/\sqrt{3}$ or $x > 2/\sqrt{3}$. So f is CU on

$\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$ and f is CD on $\left(-\infty, -2/\sqrt{3}\right)$ and $\left(2/\sqrt{3}, \infty\right)$.

There are inflection points at $\left(\pm 2/\sqrt{3}, -\frac{1}{2}\right)$.

(e)



51. (a) $\displaystyle\lim_{x\to-\infty} \left(\sqrt{x^2+1} - x\right) = \infty$ and

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0, \text{ so } y = 0 \text{ is a HA.}$$

(b) $f(x) = \sqrt{x^2+1} - x \quad\Rightarrow\quad f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1$. Since $\dfrac{x}{\sqrt{x^2+1}} < 1$ for all x, $f'(x) < 0$, so f is decreasing on $\mathbb{R}$.

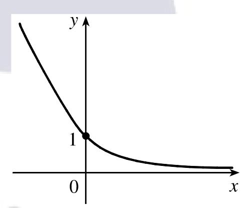
(c) No minimum or maximum

(d) $f''(x) = \dfrac{(x^2+1)^{1/2}(1) - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$

$$= \frac{(x^2+1)^{1/2} - \dfrac{x^2}{(x^2+1)^{1/2}}}{x^2+1} = \frac{(x^2+1)-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0,$$

so f is CU on $\mathbb{R}$. No IP

(e)



52. $f(x) = \dfrac{e^x}{1-e^x}$ has domain $\{x \mid 1 - e^x \neq 0\} = \{x \mid e^x \neq 1\} = \{x \mid x \neq 0\}$.

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{1-e^x} = \lim_{x\to\infty} \frac{e^x/e^x}{(1-e^x)/e^x} = \lim_{x\to\infty} \frac{1}{1/e^x - 1} = \frac{1}{0-1} = -1$, so $y = -1$ is a HA.

$\displaystyle\lim_{x\to-\infty} \frac{e^x}{1-e^x} = \frac{0}{1-0} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty$ and $\displaystyle\lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty$, so $x = 0$ is a VA.

(b) $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x)+e^x]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$. $f'(x) > 0$ for $x \neq 0$, so f is increasing on

$(-\infty, 0)$ and $(0, \infty)$.

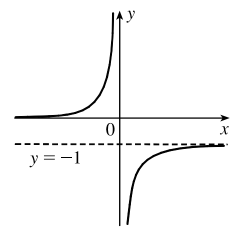
(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(1-e^x)^2 e^x - e^x \cdot 2(1-e^x)(-e^x)}{[(1-e^x)^2]^2}$

$= \dfrac{(1-e^x)e^x[(1-e^x)+2e^x]}{(1-e^x)^4} = \dfrac{e^x(e^x+1)}{(1-e^x)^3}$

$f''(x) > 0 \;\Leftrightarrow\; (1-e^x)^3 > 0 \;\Leftrightarrow\; e^x < 1 \;\Leftrightarrow\; x < 0$ and $f''(x) < 0 \;\Leftrightarrow\; x > 0$. So f is CU on $(-\infty, 0)$ and f is CD on $(0, \infty)$. There is no inflection point.

(e)



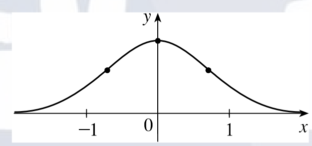
53. (a) $\lim\limits_{x\to\pm\infty} e^{-x^2} = \lim\limits_{x\to\pm\infty} \dfrac{1}{e^{x^2}} = 0$, so $y = 0$ is a HA. There is no VA.

(b) $f(x) = e^{-x^2} \;\Rightarrow\; f'(x) = e^{-x^2}(-2x)$. $f'(x) = 0 \;\Leftrightarrow\; x = 0$. $f'(x) > 0 \;\Leftrightarrow\; x < 0$ and $f'(x) < 0 \;\Leftrightarrow\; x > 0$. So f is increasing on $(-\infty, 0)$ and f is decreasing on $(0, \infty)$.

(c) f changes from increasing to decreasing at $x = 0$, so $f(0) = 1$ is a local maximum value. There is no local minimum value.

(d) $f''(x) = e^{-x^2}(-2) + (-2x)e^{-x^2}(-2x) = -2e^{-x^2}(1 - 2x^2)$.
$f''(x) = 0 \;\Leftrightarrow\; x^2 = \frac{1}{2} \;\Leftrightarrow\; x = \pm 1/\sqrt{2}$. $f''(x) > 0 \;\Leftrightarrow\; x < -1/\sqrt{2}$ or $x > 1/\sqrt{2}$ and $f''(x) < 0 \;\Leftrightarrow\; -1/\sqrt{2} < x < 1/\sqrt{2}$. So f is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, \infty\right)$, and f is CD on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$.
There are inflection points at $\left(\pm 1/\sqrt{2}, e^{-1/2}\right)$.

(e)



54. $f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$ has domain $(0, \infty)$.

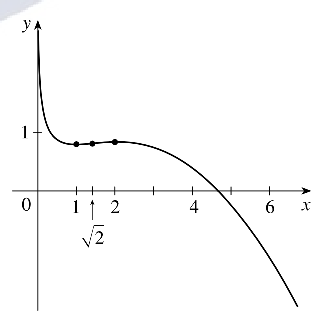
(a) $\lim\limits_{x\to 0^+} \left(x - \frac{1}{6}x^2 - \frac{2}{3}\ln x\right) = \infty$ since $\ln x \to -\infty$ as $x \to 0^+$, so $x = 0$ is a VA. There is no HA.

(b) $f'(x) = 1 - \dfrac{1}{3}x - \dfrac{2}{3x} = \dfrac{3x - x^2 - 2}{3x} = \dfrac{-(x^2 - 3x + 2)}{3x} = \dfrac{-(x-1)(x-2)}{3x}$. $f'(x) > 0 \;\Leftrightarrow\; (x-1)(x-2) < 0 \;\Leftrightarrow\; 1 < x < 2$ and $f'(x) < 0 \;\Leftrightarrow\; 0 < x < 1$ or $x > 2$. So f is increasing on $(1, 2)$ and f is decreasing on $(0, 1)$ and $(2, \infty)$.

(c) f changes from decreasing to increasing at $x = 1$, so $f(1) = \frac{5}{6}$ is a local minimum value. f changes from increasing to decreasing at $x = 2$, so $f(2) = \frac{4}{3} - \frac{2}{3}\ln 2 \approx 0.87$ is a local maximum value.

(d) $f''(x) = -\dfrac{1}{3} + \dfrac{2}{3x^2} = \dfrac{2 - x^2}{3x^2}$. $f''(x) > 0 \;\Leftrightarrow\; 0 < x < \sqrt{2}$ and $f''(x) < 0 \;\Leftrightarrow\; x > \sqrt{2}$. So f is CU on $\left(0, \sqrt{2}\right)$ and f is CD on $\left(\sqrt{2}, \infty\right)$. There is an inflection point at $\left(\sqrt{2}, \sqrt{2} - \frac{1}{3} - \frac{1}{3}\ln 2\right)$.

(e)



55. $f(x) = \ln(1 - \ln x)$ is defined when $x > 0$ (so that $\ln x$ is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined]. The second condition is equivalent to $1 > \ln x \quad \Leftrightarrow \quad x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+$, $\ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-$, $\ln x \to 1^-$, so $1 - \ln x \to 0^+$ and $f(x) \to -\infty$. Thus, $x = 0$ and $x = e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x) = \dfrac{1}{1 - \ln x}\left(-\dfrac{1}{x}\right) = -\dfrac{1}{x(1 - \ln x)} < 0$ on $(0, e)$. Thus, f is decreasing on its domain, $(0, e)$.

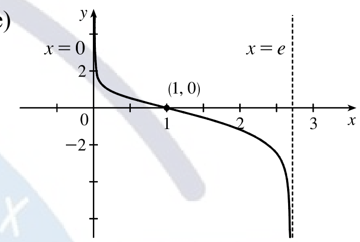
(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $f''(x) = -\dfrac{-[x(1 - \ln x)]'}{[x(1 - \ln x)]^2} = \dfrac{x(-1/x) + (1 - \ln x)}{x^2(1 - \ln x)^2}$

$= -\dfrac{\ln x}{x^2(1 - \ln x)^2}$

so $f''(x) > 0 \quad \Leftrightarrow \quad \ln x < 0 \quad \Leftrightarrow \quad 0 < x < 1$. Thus, f is CU on $(0, 1)$ and CD on $(1, e)$. There is an inflection point at $(1, 0)$.

(e)



56. (a) $\lim\limits_{x \to \infty} \arctan x = \frac{\pi}{2}$, so $\lim\limits_{x \to \infty} e^{\arctan x} = e^{\pi/2}$ [≈ 4.81], so $y = e^{\pi/2}$ is a HA.

$\lim\limits_{x \to -\infty} e^{\arctan x} = e^{-\pi/2}$ [≈ 0.21], so $y = e^{-\pi/2}$ is a HA. No VA.

(b) $f(x) = e^{\arctan x} \quad \Rightarrow \quad f'(x) = e^{\arctan x} \cdot \dfrac{1}{1 + x^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

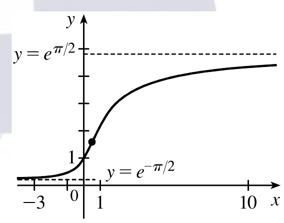
(c) There is no local maximum or minimum.

(d) $f''(x) = e^{\arctan x}\left[\dfrac{-2x}{(1 + x^2)^2}\right] + \dfrac{1}{1 + x^2} \cdot e^{\arctan x} \cdot \dfrac{1}{1 + x^2}$

$= \dfrac{e^{\arctan x}}{(1 + x^2)^2}(-2x + 1)$

$f''(x) > 0 \quad \Leftrightarrow \quad -2x + 1 > 0 \quad \Leftrightarrow \quad x < \frac{1}{2}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x > \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, e^{\arctan(1/2)}\right) \approx \left(\frac{1}{2}, 1.59\right)$.

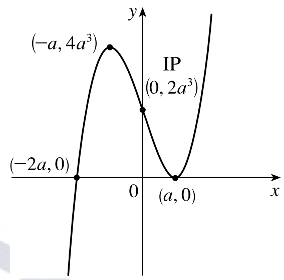
(e)



57. The nonnegative factors $(x + 1)^2$ and $(x - 6)^4$ do not affect the sign of $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$. So $f'(x) > 0 \quad \Rightarrow \quad (x - 3)^5 > 0 \quad \Rightarrow \quad x - 3 > 0 \quad \Rightarrow \quad x > 3$. Thus, f is increasing on the interval $(3, \infty)$.

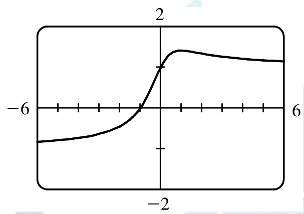
58. $y = f(x) = x^3 - 3a^2x + 2a^3$, $a > 0$. The y-intercept is $f(0) = 2a^3$. $\quad y' = 3x^2 - 3a^2 = 3(x^2 - a^2) = 3(x + a)(x - a)$. The critical numbers are $-a$ and a. $\quad f' < 0$ on $(-a, a)$, so f is decreasing on $(-a, a)$ and f is increasing on $(-\infty, -a)$ and (a, ∞). $f(-a) = 4a^3$ is a local maximum value and $f(a) = 0$ is a local minimum value. Since $f(a) = 0$, a is an x-intercept, and $x - a$ is a factor of f. Synthetically dividing $y = x^3 - 3a^2x + 2a^3$ by $x - a$ gives us the following result: $y = x^3 - 3a^2x + 2a^3 = (x - a)(x^2 + ax - 2a^2) = (x - a)(x - a)(x + 2a) = (x - a)^2(x + 2a)$, which tells us

that the only x-intercepts are $-2a$ and a. $y' = 3x^2 - 3a^2 \Rightarrow y'' = 6x$, so $y'' > 0$ on $(0, \infty)$ and $y'' < 0$ on $(-\infty, 0)$. This tells us that f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. There is an inflection point at $(0, 2a^3)$. The graph illustrates these features.



What the curves in the family have in common is that they are all CD on $(-\infty, 0)$, CU on $(0, \infty)$, and have the same basic shape. But as a increases, the four key points shown in the figure move further away from the origin.

59. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

$f(x) = \dfrac{x+1}{\sqrt{x^2+1}} \Rightarrow f'(x) = \dfrac{1-x}{(x^2+1)^{3/2}}$.

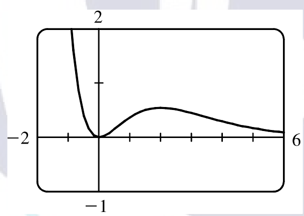
$f'(x) = 0 \Leftrightarrow x = 1$. $f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$f''(x) = \dfrac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \Leftrightarrow x = \dfrac{3 \pm \sqrt{17}}{4}$. $x = \dfrac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f'.

The maximum value of f' occurs at $x = \frac{3-\sqrt{17}}{4} \approx -0.28$.

60. (a)



Tracing the graph gives us estimates of $f(0) = 0$ for a local minimum value and $f(2) = 0.54$ for a local maximum value.

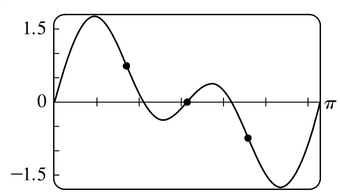
$f(x) = x^2 e^{-x} \Rightarrow f'(x) = xe^{-x}(2-x)$. $f'(x) = 0 \Leftrightarrow x = 0$ or 2.

$f(0) = 0$ and $f(2) = 4e^{-2}$ are the exact values.

(b) From the graph in part (a), f increases most rapidly around $x = \frac{3}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'. $f''(x) = e^{-x}(x^2 - 4x + 2) = 0 \Rightarrow x = 2 \pm \sqrt{2}$. $x = 2 + \sqrt{2}$ corresponds to the *minimum* value of f'. The maximum value of f' is at $\left(2-\sqrt{2}, (2-\sqrt{2})^2 e^{-2+\sqrt{2}}\right) \approx (0.59, 0.19)$.

61. $f(x) = \sin 2x + \sin 4x \Rightarrow f'(x) = 2\cos 2x + 4\cos 4x \Rightarrow f''(x) = -4\sin 2x - 16\sin 4x$

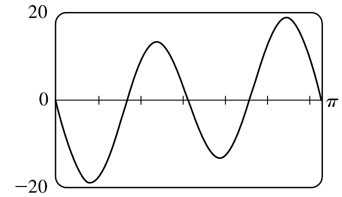
(a) From the graph of f, it seems that f is CD on $(0, 0.8)$, CU on $(0.8, 1.6)$, CD on $(1.6, 2.3)$, and CU on $(2.3, \pi)$. The inflection points appear to be at $(0.8, 0.7)$, $(1.6, 0)$, and $(2.3, -0.7)$.



4.3.61(a): transpose all instances of "CD" and "CU" to match as shown here.

4.3.61(b): transpose all instances of "CD" and "CU" to match as shown here.

(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.85)$, CU on $(0.85, 1.57)$, CD on $(1.57, 2.29)$, and CU on $(2.29, \pi)$. Refined estimates of the inflection points are $(0.85, 0.74)$, $(1.57, 0)$, and $(2.29, -0.74)$.

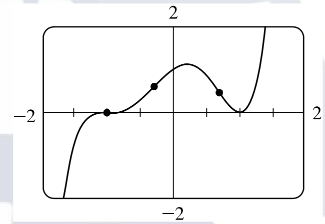


62. $f(x) = (x-1)^2(x+1)^3 \quad \Rightarrow$

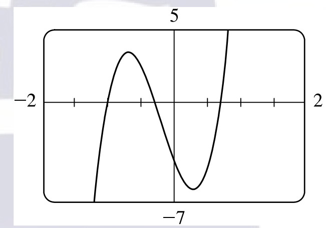
$$f'(x) = (x-1)^2 3(x+1)^2 + (x+1)^3 2(x-1)$$
$$= (x-1)(x+1)^2\,[3(x-1) + 2(x+1)] = (x-1)(x+1)^2(5x-1) \quad \Rightarrow$$

$$f''(x) = (1)(x+1)^2(5x-1) + (x-1)(2)(x+1)(5x-1) + (x-1)(x+1)^2(5)$$
$$= (x+1)[(x+1)(5x-1) + 2(x-1)(5x-1) + 5(x-1)(x+1)]$$
$$= (x+1)[5x^2 + 4x - 1 + 10x^2 - 12x + 2 + 5x^2 - 5]$$
$$= (x+1)(20x^2 - 8x - 4) = 4(x+1)(5x^2 - 2x - 1)$$

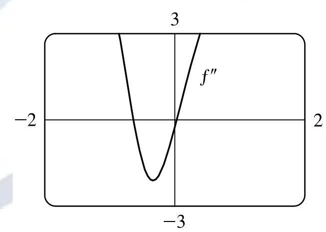
(a) From the graph of f, it seems that f is CD on $(-\infty, -1)$, CU on $(-1, -0.3)$, CD on $(-0.3, 0.7)$, and CU on $(0.7, \infty)$. The inflection points appear to be at $(-1, 0)$, $(-0.3, 0.6)$, and $(0.7, 0.5)$.



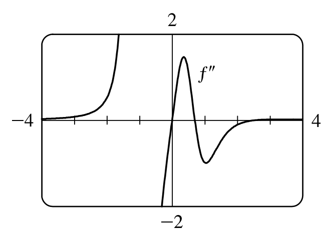
(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-1, 0)$, CU on $(-1, -0.29)$, CD on $(-0.29, 0.69)$, and CU on $(0.69, \infty)$. Refined estimates of the inflection points are $(-1, 0)$, $(-0.29, 0.60)$, and $(0.69, 0.46)$.



63. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-2..2);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-2,2}]`. We see that $f'' > 0$ for $x < -0.6$ and $x > 0.0$ $[\approx 0.03]$ and $f'' < 0$ for $-0.6 < x < 0.0$. So f is CU on $(-\infty, -0.6)$ and $(0.0, \infty)$ and CD on $(-0.6, 0.0)$.



64. $f(x) = \dfrac{x^2 \tan^{-1} x}{1 + x^3}$. It appears that f'' is positive (and thus f is concave upward) on $(-\infty, -1)$, $(0, 0.7)$, and $(2.5, \infty)$; and f'' is negative (and thus f is concave downward) on $(-1, 0)$ and $(0.7, 2.5)$.



65. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t = 8$ hours.

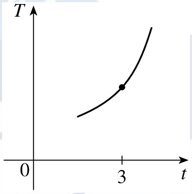
(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

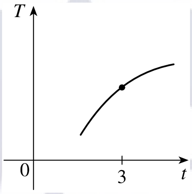
66. If $S(t)$ is the average SAT score as a function of time t, then $S'(t) < 0$ (since the SAT scores are declining) and $S''(t) > 0$ (since the rate of decrease of the scores is increasing—becoming less negative).

67. If $D(t)$ is the size of the national deficit as a function of time t, then at the time of the speech $D'(t) > 0$ (since the deficit is increasing), and $D''(t) < 0$ (since the rate of increase of the deficit is decreasing).

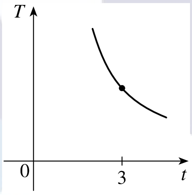
68. (a) I'm very unhappy. It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, and $f''(3) = 4$ indicates that the rate of increase is increasing. (The temperature is rapidly getting warmer.)



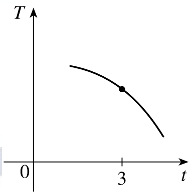
(b) I'm still unhappy, but not as unhappy as in part (a). It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, but $f''(3) = -4$ indicates that the rate of increase is decreasing. (The temperature is slowly getting warmer.)



(c) I'm somewhat happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, but $f''(3) = 4$ indicates that the rate of change is increasing. (The rate of change is negative but it's becoming less negative. The temperature is slowly getting cooler.)

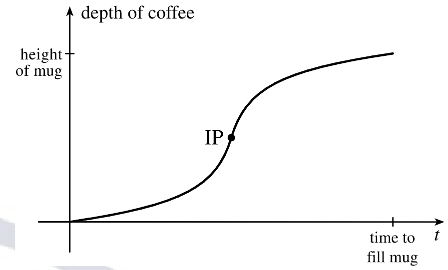


(d) I'm very happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, and $f''(3) = -4$ indicates that the rate of change is decreasing, that is, becoming more negative. (The temperature is rapidly getting cooler.)

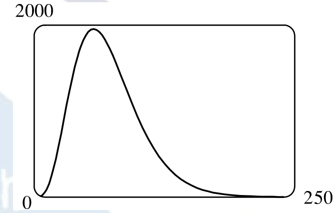


69. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

70. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



71. $S(t) = At^p e^{-kt}$ with $A = 0.01$, $p = 4$, and $k = 0.07$. We will find the zeros of f'' for $f(t) = t^p e^{-kt}$.



$$f'(t) = t^p(-ke^{-kt}) + e^{-kt}(pt^{p-1}) = e^{-kt}(-kt^p + pt^{p-1})$$

$$\begin{aligned} f''(t) &= e^{-kt}(-kpt^{p-1} + p(p-1)t^{p-2}) + (-kt^p + pt^{p-1})(-ke^{-kt}) \\ &= t^{p-2}e^{-kt}[-kpt + p(p-1) + k^2t^2 - kpt] \\ &= t^{p-2}e^{-kt}(k^2t^2 - 2kpt + p^2 - p) \end{aligned}$$

Using the given values of p and k gives us $f''(t) = t^2 e^{-0.07t}(0.0049t^2 - 0.56t + 12)$. So $S''(t) = 0.01f''(t)$ and its zeros are $t = 0$ and the solutions of $0.0049t^2 - 0.56t + 12 = 0$, which are $t_1 = \frac{200}{7} \approx 28.57$ and $t_2 = \frac{600}{7} \approx 85.71$.

At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest and at t_2 minutes, the rate of decrease is the greatest.

72. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$.

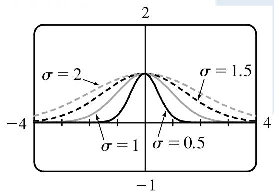
$f'(x) = 0 \Leftrightarrow x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find $f''(x) = -\dfrac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \dfrac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2)$.

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$.

So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $(\pm\sigma, e^{-1/2})$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

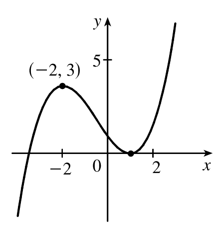
(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

73. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$.

We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)$.



74. $f(x) = axe^{bx^2} \Rightarrow f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}(2bx^2 + 1)$. For $f(2) = 1$ to be a maximum value, we must have $f'(2) = 0$. $f(2) = 1 \Rightarrow 1 = 2ae^{4b}$ and $f'(2) = 0 \Rightarrow 0 = (8b + 1)ae^{4b}$. So $8b + 1 = 0$ $[a \neq 0]$ $\Rightarrow$ $b = -\frac{1}{8}$ and now $1 = 2ae^{-1/2} \Rightarrow a = \sqrt{e}/2$.

75. (a) $f(x) = x^3 + ax^2 + bx \Rightarrow f'(x) = 3x^2 + 2ax + b$. f has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = 1/\sqrt{3}$, so $f'(\frac{1}{\sqrt{3}}) = 0 \Rightarrow 1 + \frac{2}{\sqrt{3}}a + b = 0$ **(1)** and $f(\frac{1}{\sqrt{3}}) = -\frac{2}{9}\sqrt{3} \Rightarrow \frac{1}{9}\sqrt{3} + \frac{1}{3}a + \frac{1}{3}\sqrt{3}b = -\frac{2}{9}\sqrt{3}$ **(2)**.

Rewrite the system of equations as

$$\begin{aligned} \tfrac{2}{3}\sqrt{3}a + b &= -1 \quad \textbf{(3)} \\ \tfrac{1}{3}a + \tfrac{1}{3}\sqrt{3}b &= -\tfrac{1}{3}\sqrt{3} \quad \textbf{(4)} \end{aligned}$$

and then multiplying **(4)** by $-2\sqrt{3}$ gives us the system

$$\begin{aligned} \tfrac{2}{3}\sqrt{3}a + b &= -1 \\ -\tfrac{2}{3}\sqrt{3}a - 2b &= 2 \end{aligned}$$

Adding the equations gives us $-b = 1 \Rightarrow b = -1$. Substituting -1 for b into **(3)** gives us $\frac{2}{3}\sqrt{3}a - 1 = -1 \Rightarrow \frac{2}{3}\sqrt{3}a = 0 \Rightarrow a = 0$. Thus, $f(x) = x^3 - x$.

(b) To find the smallest slope, we want to find the minimum of the slope function, f', so we'll find the critical numbers of f'. $f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1 \Rightarrow f''(x) = 6x$. $f''(x) = 0 \Leftrightarrow x = 0$. At $x = 0$, $y = 0$, $f'(x) = -1$, and f'' changes from negative to positive. Thus, we have a minimum for f' and $y - 0 = -1(x - 0)$, or $y = -x$, is the tangent line that has the smallest slope.

76. The original equation can be written as $(x^2 + b)y + ax = 0$. Call this (**1**). Since $(2, 2.5)$ is on this curve, we have $(4 + b)\left(\frac{5}{2}\right) + 2a = 0$, or $20 + 5b + 4a = 0$. Let's rewrite that as $4a + 5b = -20$ and call it (**A**). Differentiating (**1**) gives (after regrouping) $(x^2 + b)y' = -(2xy + a)$. Call this (**2**). Differentiating again gives $(x^2 + b)y'' + (2x)y' = -2xy' - 2y$, or $(x^2 + b)y'' + 4xy' + 2y = 0$. Call this (**3**). At $(2, 2.5)$, equations (**2**) and (**3**) say that $(4 + b)y' = -(10 + a)$ and $(4 + b)y'' + 8y' + 5 = 0$. If $(2, 2.5)$ is an inflection point, then $y'' = 0$ there, so the second condition becomes $8y' + 5 = 0$, or $y' = -\frac{5}{8}$. Substituting in the first condition, we get $-(4 + b)\frac{5}{8} = -(10 + a)$, or $20 + 5b = 80 + 8a$, which simplifies to $-8a + 5b = 60$. Call this (**B**). Subtracting (**B**) from (**A**) yields $12a = -80$, so $a = -\frac{20}{3}$. Substituting that value in (**A**) gives $-\frac{80}{3} + 5b = -20 = -\frac{60}{3}$, so $5b = \frac{20}{3}$ and $b = \frac{4}{3}$. Thus far we've shown that IF the curve has an inflection point at $(2, 2.5)$, then $a = -\frac{20}{3}$ and $b = \frac{4}{3}$.

To prove the converse, suppose that $a = -\frac{20}{3}$ and $b = \frac{4}{3}$. Then by (**1**), (**2**), and (**3**), our curve satisfies

$$\left(x^2 + \tfrac{4}{3}\right)y = \tfrac{20}{3}x \quad \textbf{(4)}$$

$$\left(x^2 + \tfrac{4}{3}\right)y' = -2xy + \tfrac{20}{3} \quad \textbf{(5)}$$

and

$$\left(x^2 + \tfrac{4}{3}\right)y'' + 4xy' + 2y = 0. \quad \textbf{(6)}$$

Multiply (**6**) by $\left(x^2 + \frac{4}{3}\right)$ and substitute from (**4**) and (**5**) to obtain $\left(x^2 + \frac{4}{3}\right)^2 y'' + 4x\left(-2xy + \frac{20}{3}\right) + 2\left(\frac{20}{3}x\right) = 0$, or

$\left(x^2+\frac{4}{3}\right)^2 y''-8x^2y+40x=0$. Now multiply by (x^2+b) again and substitute from the first equation to obtain $\left(x^2+\frac{4}{3}\right)^3 y''-8x^2\left(\frac{20}{3}x\right)+40x\left(x^2+\frac{4}{3}\right)=0$, or $\left(x^2+\frac{4}{3}\right)^3 y''-\frac{40}{3}(x^3-4x)=0$. The coefficient of y'' is positive, so the sign of y'' is the same as the sign of $\frac{40}{3}(x^3-4x)$, which is a positive multiple of $x(x+2)(x-2)$. It is clear from this expression that y'' changes sign at $x=0$, $x=-2$, and $x=2$, so the curve changes its direction of concavity at those values of x. By (**4**), the corresponding y-values are 0, -2.5, and 2.5, respectively. Thus when $a=-\frac{20}{3}$ and $b=\frac{4}{3}$, the curve has inflection points, not only at $(2,2.5)$, but also at $(0,0)$ and $(-2,-2.5)$.

77. $y=\dfrac{1+x}{1+x^2} \quad\Rightarrow\quad y'=\dfrac{(1+x^2)(1)-(1+x)(2x)}{(1+x^2)^2}=\dfrac{1-2x-x^2}{(1+x^2)^2} \quad\Rightarrow$

$$y''=\frac{(1+x^2)^2(-2-2x)-(1-2x-x^2)\cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2}=\frac{2(1+x^2)[(1+x^2)(-1-x)-(1-2x-x^2)(2x)]}{(1+x^2)^4}$$

$$=\frac{2(-1-x-x^2-x^3-2x+4x^2+2x^3)}{(1+x^2)^3}=\frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3}=\frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3}$$

So $y''=0 \quad\Rightarrow\quad x=1,-2\pm\sqrt{3}$. Let $a=-2-\sqrt{3}$, $b=-2+\sqrt{3}$, and $c=1$. We can show that $f(a)=\frac{1}{4}\left(1-\sqrt{3}\right)$, $f(b)=\frac{1}{4}\left(1+\sqrt{3}\right)$, and $f(c)=1$. To show that these three points of inflection lie on one straight line, we'll show that the slopes m_{ac} and m_{bc} are equal.

$$m_{ac}=\frac{f(c)-f(a)}{c-a}=\frac{1-\frac{1}{4}\left(1-\sqrt{3}\right)}{1-\left(-2-\sqrt{3}\right)}=\frac{\frac{3}{4}+\frac{1}{4}\sqrt{3}}{3+\sqrt{3}}=\frac{1}{4}$$

$$m_{bc}=\frac{f(c)-f(b)}{c-b}=\frac{1-\frac{1}{4}\left(1+\sqrt{3}\right)}{1-\left(-2+\sqrt{3}\right)}=\frac{\frac{3}{4}-\frac{1}{4}\sqrt{3}}{3-\sqrt{3}}=\frac{1}{4}$$

78. $y=f(x)=e^{-x}\sin x \quad\Rightarrow\quad y'=e^{-x}\cos x+\sin x(-e^{-x})=e^{-x}(\cos x-\sin x) \quad\Rightarrow$

$y''=e^{-x}(-\sin x-\cos x)+(\cos x-\sin x)(-e^{-x})=e^{-x}(-\sin x-\cos x-\cos x+\sin x)=e^{-x}(-2\cos x)$.

So $y''=0 \quad\Rightarrow\quad \cos x=0 \quad\Rightarrow\quad x=\frac{\pi}{2}+n\pi$. At these values of x, f has points of inflection and since $\sin\left(\frac{\pi}{2}+n\pi\right)=\pm 1$, we get $y=\pm e^{-x}$, so f intersects the other curves at its inflection points.

Let $g(x)=e^{-x}$ and $h(x)=-e^{-x}$, so that $g'(x)=-e^{-x}$ and $h'(x)=e^{-x}$. Now $f'\left(\frac{\pi}{2}+n\pi\right)=e^{-(\pi/2+n\pi)}\left[\cos\left(\frac{\pi}{2}+n\pi\right)-\sin\left(\frac{\pi}{2}+n\pi\right)\right]=-e^{-(\pi/2+n\pi)}\sin\left(\frac{\pi}{2}+n\pi\right)$. If n is odd, then $f'\left(\frac{\pi}{2}+n\pi\right)=e^{-(\pi/2+n\pi)}=h'\left(\frac{\pi}{2}+n\pi\right)$. If n is even, then $f'\left(\frac{\pi}{2}+n\pi\right)=-e^{-(\pi/2+n\pi)}=g'\left(\frac{\pi}{2}+n\pi\right)$.

Thus, at $x=\frac{\pi}{2}+n\pi$, f has the same slope as either g or h, and hence, g and h touch f at its inflection points.

79. $y=x\sin x \quad\Rightarrow\quad y'=x\cos x+\sin x \quad\Rightarrow\quad y''=-x\sin x+2\cos x.\quad y''=0 \quad\Rightarrow\quad 2\cos x=x\sin x$ [which is y] $\quad\Rightarrow\quad (2\cos x)^2=(x\sin x)^2 \quad\Rightarrow\quad 4\cos^2 x=x^2\sin^2 x \quad\Rightarrow\quad 4\cos^2 x=x^2(1-\cos^2 x) \quad\Rightarrow\quad 4\cos^2 x+x^2\cos^2 x=x^2 \quad\Rightarrow\quad \cos^2 x(4+x^2)=x^2 \quad\Rightarrow\quad 4\cos^2 x(x^2+4)=4x^2 \quad\Rightarrow\quad y^2(x^2+4)=4x^2$ since $y=2\cos x$ when $y''=0$.

80. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I, then $f''>0$ on I. If f and g are CU on I, then $f''>0$ and $g''>0$ on I, so $(f+g)''=f''+g''>0$ on $I \quad\Rightarrow$ $f+g$ is CU on I.

(b) Since f is positive and CU on I, $f > 0$ and $f'' > 0$ on I. So $g(x) = [f(x)]^2 \Rightarrow g' = 2ff' \Rightarrow$ $g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \Rightarrow g$ is CU on I.

81. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \Rightarrow (fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow$ fg is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

82. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$. $h(x) = f(g(x)) \Rightarrow$ $h'(x) = f'(g(x))\, g'(x) \Rightarrow h''(x) = f''(g(x))\, g'(x)\, g'(x) + f'(g(x))\, g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))\, g''(x) > 0$ if $f' > 0$. So h is CU if f is increasing.

83. $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x$ for $0 < x < \frac{\pi}{2}$.

84. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \Rightarrow e^x - 1 - x \geq 0 \Rightarrow e^x \geq 1 + x$.

(b) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by part (a). Thus, $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \Rightarrow e^x \geq 1 + x + \frac{1}{2}x^2$.

(c) By part (a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} \geq 0$ by assumption. Hence, $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ for every positive integer n, by mathematical induction.

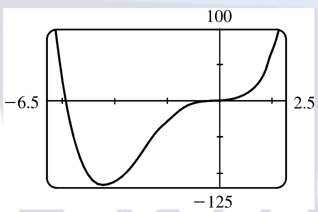
85. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c \quad \Rightarrow \quad f''(x) = 6ax + 2b$.

So f is CU when $6ax + 2b > 0 \quad \Leftrightarrow \quad x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us

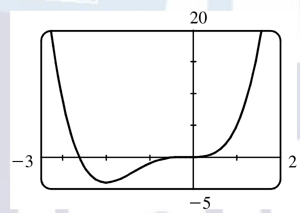
$$f(x) = a[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3]$$

Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

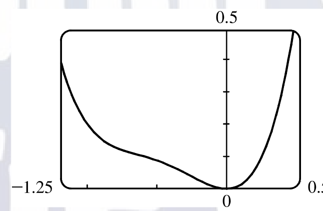
86. $P(x) = x^4 + cx^3 + x^2 \quad \Rightarrow \quad P'(x) = 4x^3 + 3cx^2 + 2x \quad \Rightarrow \quad P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \quad \Leftrightarrow \quad 36c^2 - 96 > 0 \quad \Leftrightarrow \quad |c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \quad \Leftrightarrow \quad c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \quad \Leftrightarrow \quad |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



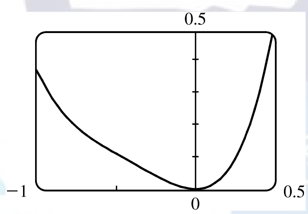
$c = 6$



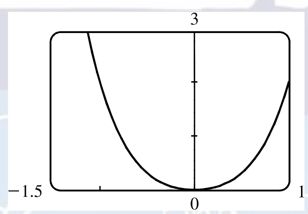
$c = 3$



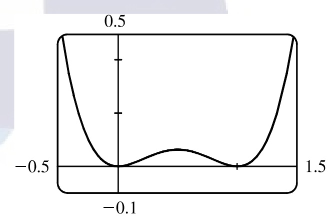
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

87. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

88. $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3 \quad \Rightarrow \quad f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

89. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\,|x| = x\sqrt{x^2} \quad\Rightarrow\quad g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2\,|x| \quad\Rightarrow$ $g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

90. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

91. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

92. $f(x) = cx + \dfrac{1}{x^2 + 3} \quad\Rightarrow\quad f'(x) = c - \dfrac{2x}{(x^2 + 3)^2}$. $f'(x) > 0 \quad\Leftrightarrow\quad c > \dfrac{2x}{(x^2 + 3)^2}$ [call this $g(x)$].

Now f' is positive (and hence f increasing) if $c > g$, so we'll find the maximum value of g.

$$g'(x) = \frac{(x^2 + 3)^2 \cdot 2 - 2x \cdot 2(x^2 + 3) \cdot 2x}{[(x^2 + 3)^2]^2} = \frac{2(x^2 + 3)[(x^2 + 3) - 4x^2]}{(x^2 + 3)^4} = \frac{2(3 - 3x^2)}{(x^2 + 3)^3} = \frac{6(1 + x)(1 - x)}{(x^2 + 3)^3}.$$

$g'(x) = 0 \quad\Leftrightarrow\quad x = \pm 1.$ $g'(x) > 0$ on $(0, 1)$ and $g'(x) < 0$ on $(1, \infty)$, so g is increasing on $(0, 1)$ and decreasing on $(1, \infty)$, and hence g has a maximum value on $(0, \infty)$ of $g(1) = \frac{2}{16} = \frac{1}{8}$. Also since $g(x) \le 0$ if $x \le 0$, the maximum value of g on $(-\infty, \infty)$ is $\frac{1}{8}$. Thus, when $c > \frac{1}{8}$, f is increasing. When $c = \frac{1}{8}$, $f'(x) > 0$ on $(-\infty, 1)$ and $(1, \infty)$, and hence f is increasing on these intervals. Since f is continuous, we may conclude that f is also increasing on $(-\infty, \infty)$ if $c = \frac{1}{8}$. Therefore, f is increasing on $(-\infty, \infty)$ if $c \ge \frac{1}{8}$.

93. (a) $f(x) = x^4 \sin \frac{1}{x} \quad \Rightarrow \quad f'(x) = x^4 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) + \sin \frac{1}{x} (4x^3) = 4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x}$.

$g(x) = x^4 \left(2 + \sin \frac{1}{x}\right) = 2x^4 + f(x) \quad \Rightarrow \quad g'(x) = 8x^3 + f'(x)$.

$h(x) = x^4 \left(-2 + \sin \frac{1}{x}\right) = -2x^4 + f(x) \quad \Rightarrow \quad h'(x) = -8x^3 + f'(x)$.

It is given that $f(0) = 0$, so $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^4 \sin \frac{1}{x} - 0}{x} = \lim_{x\to 0} x^3 \sin \frac{1}{x}$. Since $-\left|x^3\right| \le x^3 \sin \frac{1}{x} \le \left|x^3\right|$ and $\lim_{x\to 0} \left|x^3\right| = 0$, we see that $f'(0) = 0$ by the Squeeze Theorem. Also, $g'(0) = 8(0)^3 + f'(0) = 0$ and $h'(0) = -8(0)^3 + f'(0) = 0$, so 0 is a critical number of f, g, and h.

For $x_{2n} = \frac{1}{2n\pi}$ [n a nonzero integer], $\sin \frac{1}{x_{2n}} = \sin 2n\pi = 0$ and $\cos \frac{1}{x_{2n}} = \cos 2n\pi = 1$, so $f'(x_{2n}) = -x_{2n}^2 < 0$.

For $x_{2n+1} = \frac{1}{(2n+1)\pi}$, $\sin \frac{1}{x_{2n+1}} = \sin(2n+1)\pi = 0$ and $\cos \frac{1}{x_{2n+1}} = \cos(2n+1)\pi = -1$, so $f'(x_{2n+1}) = x_{2n+1}^2 > 0$. Thus, f' changes sign infinitely often on both sides of 0.

Next, $g'(x_{2n}) = 8x_{2n}^3 + f'(x_{2n}) = 8x_{2n}^3 - x_{2n}^2 = x_{2n}^2(8x_{2n} - 1) < 0$ for $x_{2n} < \frac{1}{8}$, but $g'(x_{2n+1}) = 8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(8x_{2n+1} + 1) > 0$ for $x_{2n+1} > -\frac{1}{8}$, so g' changes sign infinitely often on both sides of 0.

Last, $h'(x_{2n}) = -8x_{2n}^3 + f'(x_{2n}) = -8x_{2n}^3 - x_{2n}^2 = -x_{2n}^2(8x_{2n} + 1) < 0$ for $x_{2n} > -\frac{1}{8}$ and $h'(x_{2n+1}) = -8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(-8x_{2n+1} + 1) > 0$ for $x_{2n+1} < \frac{1}{8}$, so h' changes sign infinitely often on both sides of 0.

(b) $f(0) = 0$ and since $\sin \frac{1}{x}$ and hence $x^4 \sin \frac{1}{x}$ is both positive and negative inifinitely often on both sides of 0, and arbitrarily close to 0, f has neither a local maximum nor a local minimum at 0.

Since $2 + \sin \frac{1}{x} \ge 1$, $g(x) = x^4 \left(2 + \sin \frac{1}{x}\right) > 0$ for $x \ne 0$, so $g(0) = 0$ is a local minimum.

Since $-2 + \sin \frac{1}{x} \le -1$, $h(x) = x^4 \left(-2 + \sin \frac{1}{x}\right) < 0$ for $x \ne 0$, so $h(0) = 0$ is a local maximum.

4.4 Indeterminate Forms and l'Hospital's Rule

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$

1. (a) $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x\to a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim_{x\to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim_{x\to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x\to a} [h(x)p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a} [p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x\to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x\to a} [p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.8}{\frac{4}{5}} = \frac{9}{4}$$

6. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.5}{-1} = -\frac{3}{2}$$

7. f and $g = e^x - 1$ are differentiable and $g' = e^x \neq 0$ on an open interval that contains 0. $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 0} g(x) = 0$, so we have the indeterminate form $\frac{0}{0}$ and can apply l'Hospital's Rule.

$$\lim_{x\to 0} \frac{f(x)}{e^x - 1} \overset{\text{H}}{=} \lim_{x\to 0} \frac{f'(x)}{e^x} = \frac{1}{1} = 1$$

Note that $\lim\limits_{x\to 0} f'(x) = 1$ since the graph of f has the same slope as the line $y = x$ at $x = 0$.

8. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9} = \lim\limits_{x\to 3} \dfrac{x-3}{(x+3)(x-3)} = \lim\limits_{x\to 3} \dfrac{1}{x+3} = \dfrac{1}{3+3} = \dfrac{1}{6}$

Note: Alternatively, we could apply l'Hospital's Rule.

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim\limits_{x\to 4} \dfrac{(x-4)(x+2)}{x-4} = \lim\limits_{x\to 4}(x+2) = 4+2 = 6$

Note: Alternatively, we could apply l'Hospital's Rule.

10. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2} \overset{\text{H}}{=} \lim\limits_{x\to -2} \dfrac{3x^2}{1} = 3(-2)^2 = 12$

Note: Alternatively, we could factor and simplify.

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1} \dfrac{x^3-2x^2+1}{x^3-1} \overset{\text{H}}{=} \lim\limits_{x\to 1} \dfrac{3x^2-4x}{3x^2} = -\dfrac{1}{3}$

Note: Alternatively, we could factor and simplify.

12. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1/2} \dfrac{6x^2+5x-4}{4x^2+16x-9} \overset{\text{H}}{=} \lim\limits_{x\to 1/2} \dfrac{12x+5}{8x+16} = \dfrac{6+5}{4+16} = \dfrac{11}{20}$

Note: Alternatively, we could factor and simplify.

13. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to(\pi/2)^+} \dfrac{\cos x}{1-\sin x} \overset{\text{H}}{=} \lim\limits_{x\to(\pi/2)^+} \dfrac{-\sin x}{-\cos x} = \lim\limits_{x\to(\pi/2)^+} \tan x = -\infty$.

14. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\tan 3x}{\sin 2x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{3\sec^2 3x}{2\cos 2x} = \dfrac{3(1)^2}{2(1)} = \dfrac{3}{2}$

15. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0} \dfrac{e^{2t}-1}{\sin t} \overset{\text{H}}{=} \lim\limits_{t\to 0} \dfrac{2e^{2t}}{\cos t} = \dfrac{2(1)}{1} = 2$

16. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{2x}{\sin x} = \lim\limits_{x\to 0} \dfrac{2}{(\sin x)/x} = \dfrac{2}{1} = 2$

17. This limit has the form $\frac{0}{0}$. $\lim\limits_{\theta\to\pi/2} \dfrac{1-\sin\theta}{1+\cos 2\theta} \overset{\text{H}}{=} \lim\limits_{\theta\to\pi/2} \dfrac{-\cos\theta}{-2\sin 2\theta} \overset{\text{H}}{=} \lim\limits_{\theta\to\pi/2} \dfrac{\sin\theta}{-4\cos 2\theta} = \dfrac{1}{4}$

18. The limit can be evaluated by substituting π for θ. $\lim\limits_{\theta\to\pi} \dfrac{1+\cos\theta}{1-\cos\theta} = \dfrac{1+(-1)}{1-(-1)} = \dfrac{0}{2} = 0$

19. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{\frac{1}{2}x^{-1/2}} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

20. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty} \dfrac{x+x^2}{1-2x^2} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1+2x}{-4x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2}{-4} = -\dfrac{1}{2}$.3

A better method is to divide the numerator and the denominator by x^2: $\lim\limits_{x\to\infty} \dfrac{x+x^2}{1-2x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+1}{\frac{1}{x^2}-2} = \dfrac{0+1}{0-2} = -\dfrac{1}{2}$.

21. $\lim\limits_{x\to 0^+} [(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x \to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

22. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty} \dfrac{\ln\sqrt{x}}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}\ln x}{x^2} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{2x}}{2x} = \lim\limits_{x\to\infty} \dfrac{1}{4x^2} = 0$

23. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 1} \dfrac{t^8-1}{t^5-1} \overset{\text{H}}{=} \lim\limits_{t\to 1} \dfrac{8t^7}{5t^4} = \dfrac{8}{5}\lim\limits_{t\to 1} t^3 = \dfrac{8}{5}(1) = \dfrac{8}{5}$

24. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0} \dfrac{8^t-5^t}{t} \overset{\text{H}}{=} \lim\limits_{t\to 0} \dfrac{8^t\ln 8 - 5^t\ln 5}{1} = \ln 8 - \ln 5 = \ln\dfrac{8}{5}$

25. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+2x)^{-1/2}\cdot 2 - \frac{1}{2}(1-4x)^{-1/2}(-4)}{1}$$
$$= \lim_{x\to 0}\left(\frac{1}{\sqrt{1+2x}} + \frac{2}{\sqrt{1-4x}}\right) = \frac{1}{\sqrt{1}} + \frac{2}{\sqrt{1}} = 3$$

26. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{3u^2} \overset{\text{H}}{=} \tfrac{1}{30}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\text{H}}{=} \tfrac{1}{600}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{1} = \tfrac{1}{6000}\lim_{u\to\infty} e^{u/10} = \infty$$

27. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{e^x-1}{2x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

28. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\sinh x - x}{x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\cosh x - 1}{3x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\sinh x}{6x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\cosh x}{6} = \dfrac{1}{6}$

29. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\tanh x}{\tan x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\operatorname{sech}^2 x}{\sec^2 x} = \dfrac{\operatorname{sech}^2 0}{\sec^2 0} = \dfrac{1}{1} = 1$

30. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{x-\sin x}{x-\tan x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{1-\cos x}{1-\sec^2 x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-(-\sin x)}{-2\sec x\,(\sec x\,\tan x)} = -\frac{1}{2}\lim_{x\to 0} \frac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x}$$
$$= -\tfrac{1}{2}\lim_{x\to 0} \cos^3 x = -\tfrac{1}{2}(1)^3 = -\tfrac{1}{2}$$

Another method is to write the limit as $\lim\limits_{x\to 0} \dfrac{1-\dfrac{\sin x}{x}}{1-\dfrac{\tan x}{x}}$.

31. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{1/\sqrt{1-x^2}}{1} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{1-x^2}} = \dfrac{1}{1} = 1$

32. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2(\ln x)(1/x)}{1} = 2\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} 2\lim\limits_{x\to\infty} \dfrac{1/x}{1} = 2(0) = 0$

33. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{x3^x}{3^x-1} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{x3^x\ln 3 + 3^x}{3^x\ln 3} = \lim\limits_{x\to 0} \dfrac{3^x(x\ln 3+1)}{3^x\ln 3} = \lim\limits_{x\to 0} \dfrac{x\ln 3+1}{\ln 3} = \dfrac{1}{\ln 3}$

34. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\tfrac{1}{2}(n^2-m^2)$$

35. This limit can be evaluated by substituting 0 for x. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{\cos x+e^x-1}=\frac{\ln 1}{1+1-1}=\frac{0}{1}=0$

36. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x\sin(x-1)}{2x^2-x-1}\overset{\text{H}}{=}\lim_{x\to 1}\frac{x\cos(x-1)+\sin(x-1)}{4x-1}=\frac{\cos 0}{4-1}=\frac{1}{3}$

37. This limit has the form $\frac{0}{\infty}$, so l'Hospital's Rule doesn't apply. As $x\to 0^+$, $\arctan(2x)\to 0$ and $\ln x\to -\infty$, so

$$\lim_{x\to 0^+}\frac{\arctan(2x)}{\ln x}=0.$$

38. $\displaystyle\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}$. From Example 9, $\displaystyle\lim_{x\to 0^+}x^x=1$, so $\displaystyle\lim_{x\to 0^+}(x^x-1)=0$. As $x\to 0^+$, $\ln x\to -\infty$, so

$\ln x+x-1\to -\infty$ as $x\to 0^+$. Thus, $\displaystyle\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}=0$.

39. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1}$ [for $b\neq 0$] $\displaystyle\overset{\text{H}}{=}\lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}}=\frac{a(1)}{b(1)}=\frac{a}{b}$

40. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{1+1}{1}=2$

41. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\sin x+x}{4x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\cos x+1}{12x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{24x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{24}=\frac{1}{24}$

42. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{x\to a^+}\frac{\cos x\ln(x-a)}{\ln(e^x-e^a)}=\lim_{x\to a^+}\cos x\lim_{x\to a^+}\frac{\ln(x-a)}{\ln(e^x-e^a)}\overset{\text{H}}{=}\cos a\lim_{x\to a^+}\frac{\dfrac{1}{x-a}}{\dfrac{1}{e^x-e^a}\cdot e^x}$$

$$=\cos a\lim_{x\to a^+}\frac{1}{e^x}\cdot\lim_{x\to a^+}\frac{e^x-e^a}{x-a}\overset{\text{H}}{=}\cos a\cdot\frac{1}{e^a}\lim_{x\to a^+}\frac{e^x}{1}=\cos a\cdot\frac{1}{e^a}\cdot e^a=\cos a$$

43. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to\infty}x\sin(\pi/x)=\lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2}=\pi\lim_{x\to\infty}\cos(\pi/x)=\pi(1)=\pi$$

44. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\sqrt{x}\,e^{-x/2}=\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}\,e^{x/2}}=0$$

45. This limit has the form $0\cdot\infty$. We'll change it to the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\sin 5x\csc 3x=\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{5\cos 5x}{3\cos 3x}=\frac{5\cdot 1}{3\cdot 1}=\frac{5}{3}$

46. This limit has the form $(-\infty)\cdot 0$.

$$\lim_{x\to-\infty}x\ln\!\left(1-\frac{1}{x}\right)=\lim_{x\to-\infty}\frac{\ln\left(1-\dfrac{1}{x}\right)}{\dfrac{1}{x}}\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{\dfrac{1}{1-1/x}\cdot\dfrac{1}{x^2}}{-\dfrac{1}{x^2}}=\lim_{x\to-\infty}\frac{-1}{1-\dfrac{1}{x}}=\frac{-1}{1}=-1$$

47. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \lim\limits_{x\to\infty} \dfrac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3}{4xe^{x^2}} = 0$

48. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to\infty} x^{3/2}\sin(1/x) = \lim\limits_{x\to\infty} x^{1/2} \cdot \dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+} \dfrac{1}{\sqrt{t}} \dfrac{\sin t}{t}$ [where $t = 1/x$] $= \infty$

since as $t \to 0^+$, $\dfrac{1}{\sqrt{t}} \to \infty$ and $\dfrac{\sin t}{t} \to 1$.

49. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 1^+} \ln x \, \tan(\pi x/2) = \lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

50. This limit has the form $0 \cdot \infty$. $\lim\limits_{x\to(\pi/2)^-} \cos x \sec 5x = \lim\limits_{x\to(\pi/2)^-} \dfrac{\cos x}{\cos 5x} \overset{\text{H}}{=} \lim\limits_{x\to(\pi/2)^-} \dfrac{-\sin x}{-5\sin 5x} = \dfrac{-1}{-5} = \dfrac{1}{5}$

51. This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - (1/x) + \ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to 1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} = \lim_{x\to 1} \frac{x}{1+x} = \frac{1}{1+1} = \frac{1}{2}$$

52. This limit has the form $\infty - \infty$. $\lim\limits_{x\to 0}(\csc x - \cot x) = \lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

53. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^x - 1}{xe^x + e^x - 1} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^x}{xe^x + e^x + e^x} = \frac{1}{0+1+1} = \frac{1}{2}$$

54. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\tan^{-1} x}\right) = \lim_{x\to 0^+} \frac{\tan^{-1} x - x}{x\tan^{-1} x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/(1+x^2) - 1}{x/(1+x^2) + \tan^{-1} x} = \lim_{x\to 0^+} \frac{1 - (1+x^2)}{x + (1+x^2)\tan^{-1} x}$$

$$= \lim_{x\to 0^+} \frac{-x^2}{x + (1+x^2)\tan^{-1} x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-2x}{1 + (1+x^2)(1/(1+x^2)) + (\tan^{-1} x)(2x)}$$

$$= \lim_{x\to 0^+} \frac{-2x}{2 + 2x\tan^{-1} x} = \frac{0}{2+0} = 0$$

55. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \text{ since } \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

56. This limit has the form $\infty - \infty$.

$$\lim_{x\to 1^+}[\ln(x^7 - 1) - \ln(x^5 - 1)] = \lim_{x\to 1^+} \ln \frac{x^7 - 1}{x^5 - 1} = \ln \lim_{x\to 1^+} \frac{x^7 - 1}{x^5 - 1} \overset{\text{H}}{=} \ln \lim_{x\to 1^+} \frac{7x^6}{5x^4} = \ln \frac{7}{5}$$

57. $y = x^{\sqrt{x}} \quad \Rightarrow \quad \ln y = \sqrt{x}\,\ln x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sqrt{x}\,\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = -2\lim_{x\to 0^+} \sqrt{x} = 0 \quad \Rightarrow$$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

58. $y = (\tan 2x)^x \quad \Rightarrow \quad \ln y = x \cdot \ln \tan 2x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \cdot \ln \tan 2x = \lim_{x\to 0^+} \frac{\ln \tan 2x}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+} \frac{-2x^2 \cos 2x}{\sin 2x\, \cos^2 2x}$$

$$= \lim_{x\to 0^+} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0^+} \frac{-x}{\cos 2x} = 1 \cdot 0 = 0 \quad \Rightarrow$$

$$\lim_{x\to 0^+} (\tan 2x)^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

59. $y = (1-2x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\displaystyle\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-2/(1-2x)}{1} = -2 \quad \Rightarrow$

$$\lim_{x\to 0} (1-2x)^{1/x} = \lim_{x\to 0} e^{\ln y} = e^{-2}.$$

60. $y = \left(1 + \dfrac{a}{x}\right)^{bx} \quad \Rightarrow \quad \ln y = bx \ln\left(1 + \dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1+a/x} = ab \quad \Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

61. $y = x^{1/(1-x)} \quad \Rightarrow \quad \ln y = \dfrac{1}{1-x}\ln x$, so $\displaystyle\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{1}{1-x}\ln x = \lim_{x\to 1^+} \frac{\ln x}{1-x} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{-1} = -1 \quad \Rightarrow$

$$\lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} e^{\ln y} = e^{-1} = \frac{1}{e}.$$

62. $y = x^{(\ln 2)/(1+\ln x)} \quad \Rightarrow \quad \ln y = \dfrac{\ln 2}{1+\ln x}\ln x \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1+\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2, \text{ so } \lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$$

63. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x)\ln x \quad \Rightarrow \quad \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

64. $y = x^{e^{-x}} \quad \Rightarrow \quad \ln y = e^{-x}\ln x \quad \Rightarrow \quad \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0 \quad \Rightarrow$

$$\lim_{x\to\infty} x^{e^{-x}} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

65. $y = (4x+1)^{\cot x} \quad \Rightarrow \quad \ln y = \cot x\, \ln(4x+1)$, so $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\dfrac{4}{4x+1}}{\sec^2 x} = 4 \quad \Rightarrow$

$$\lim_{x\to 0^+} (4x+1)^{\cot x} = \lim_{x\to 0^+} e^{\ln y} = e^4.$$

66. $y = (2-x)^{\tan(\pi x/2)} \quad \Rightarrow \quad \ln y = \tan\left(\dfrac{\pi x}{2}\right)\ln(2-x) \quad \Rightarrow$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \left[\tan\left(\frac{\pi x}{2}\right)\ln(2-x)\right] = \lim_{x\to 1} \frac{\ln(2-x)}{\cot\left(\dfrac{\pi x}{2}\right)} \overset{\text{H}}{=} \lim_{x\to 1} \frac{\dfrac{1}{2-x}(-1)}{-\csc^2\left(\dfrac{\pi x}{2}\right)\cdot\dfrac{\pi}{2}} = \frac{2}{\pi}\lim_{x\to 1} \frac{\sin^2\left(\dfrac{\pi x}{2}\right)}{2-x}$$

$$= \frac{2}{\pi}\cdot\frac{1^2}{1} = \frac{2}{\pi} \quad \Rightarrow \quad \lim_{x\to 1} (2-x)^{\tan(\pi x/2)} = \lim_{x\to 1} e^{\ln y} = e^{(2/\pi)}$$

67. $y = (1 + \sin 3x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln(1 + \sin 3x) \quad \Rightarrow$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 3x)}{x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{[1/(1 + \sin 3x)] \cdot 3 \cos 3x}{1} = \lim_{x \to 0^+} \frac{3 \cos 3x}{1 + \sin 3x} = \frac{3 \cdot 1}{1 + 0} = 3 \quad \Rightarrow$$

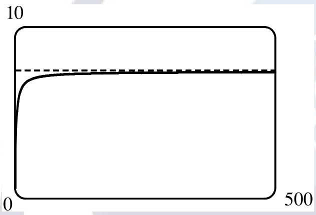
$$\lim_{x \to 0^+} (1 + \sin 3x)^{1/x} = \lim_{x \to 0^+} e^{\ln y} = e^3$$

68. $y = \left(\dfrac{2x - 3}{2x + 5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x + 1) \ln\left(\dfrac{2x - 3}{2x + 5}\right) \quad \Rightarrow$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln(2x - 3) - \ln(2x + 5)}{1/(2x + 1)} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{2/(2x - 3) - 2/(2x + 5)}{-2/(2x + 1)^2} = \lim_{x \to \infty} \frac{-8(2x + 1)^2}{(2x - 3)(2x + 5)}$$

$$= \lim_{x \to \infty} \frac{-8(2 + 1/x)^2}{(2 - 3/x)(2 + 5/x)} = -8 \quad \Rightarrow \quad \lim_{x \to \infty} \left(\frac{2x - 3}{2x + 5}\right)^{2x+1} = e^{-8}$$

69.



From the graph, if $x = 500$, $y \approx 7.36$. The limit has the form 1^∞.

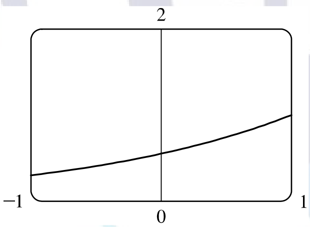
Now $y = \left(1 + \dfrac{2}{x}\right)^x \quad \Rightarrow \quad \ln y = x \ln\left(1 + \dfrac{2}{x}\right) \quad \Rightarrow$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln(1 + 2/x)}{1/x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{\dfrac{1}{1 + 2/x}\left(-\dfrac{2}{x^2}\right)}{-1/x^2}$$

$$= 2 \lim_{x \to \infty} \frac{1}{1 + 2/x} = 2(1) = 2 \quad \Rightarrow$$

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x \to \infty} e^{\ln y} = e^2 \quad [\approx 7.39]$$

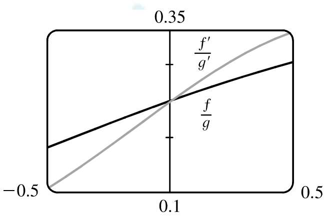
70.



From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \quad [\approx 0.55]$$

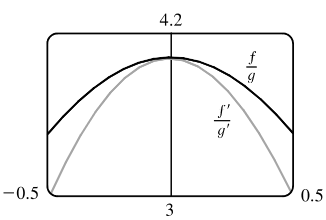
71.



From the graph, it appears that $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.25$.

We calculate $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{e^x}{3x^2 + 4} = \frac{1}{4}$.

72.



From the graph, it appears that $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{2x \sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x \to 0} \frac{2(x \cos x + \sin x)}{\sec x \tan x}$$

$$\overset{\text{H}}{=} \lim_{x \to 0} \frac{2(-x \sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x \tan x)} = \frac{4}{1} = 4$$

73. $\lim_{x\to\infty} \frac{e^x}{x^n} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

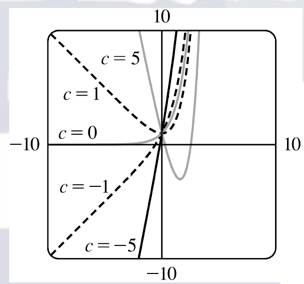
74. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p > 0$.

75. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2/x^2+1/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{1} = 1$

76. $\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{\sec x\,\tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to simplify first:

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{1/\cos x}{\sin x/\cos x} = \lim_{x\to(\pi/2)^-} \frac{1}{\sin x} = \frac{1}{1} = 1$$

77. $f(x) = e^x - cx \Rightarrow f'(x) = e^x - c = 0 \Leftrightarrow e^x = c \Leftrightarrow x = \ln c, c > 0$. $f''(x) = e^x > 0$, so f is CU on $(-\infty,\infty)$. $\lim_{x\to\infty}(e^x - cx) = \lim_{x\to\infty}\left[x\left(\frac{e^x}{x} - c\right)\right] = L_1$. Now $\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$, so $L_1 = \infty$, regardless of the value of c. For $L = \lim_{x\to-\infty}(e^x - cx)$, $e^x \to 0$, so L is determined by $-cx$. If $c > 0$, $-cx \to \infty$, and $L = \infty$. If $c < 0$, $-cx \to -\infty$, and $L = -\infty$. Thus, f has an absolute minimum for $c > 0$. As c increases, the minimum points $(\ln c, c - c\ln c)$, get farther away from the origin.



78. (a) $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}(1-0)$ [because $-ct/m \to -\infty$ as $t\to\infty$]

$= \frac{mg}{c}$, which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim_{c\to 0^+} v = \lim_{c\to 0^+} \frac{mg}{c}(1 - e^{-ct/m}) = mg \lim_{c\to 0^+} \frac{1 - e^{-ct/m}}{c}$ [form is $\frac{0}{0}$]

$\overset{\text{H}}{=} mg \lim_{c\to 0^+} \frac{(-e^{-ct/m})\cdot(-t/m)}{1} = \frac{mgt}{m} \lim_{c\to 0^+} e^{-ct/m} = gt(1) = gt$

The velocity of a falling object in a vacuum is directly proportional to the amount of time it falls.

79. First we will find $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1 + \frac{r}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1 + \frac{r}{n}\right)$, so $\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1 + \frac{r}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1+r/n)}{1/n} \overset{\text{H}}{=} t\lim_{n\to\infty} \frac{(-r/n^2)}{(1+r/n)(-1/n^2)} = t\lim_{n\to\infty} \frac{r}{1+i/n} = tr \Rightarrow$ $\lim_{n\to\infty} y = e^{rt}$. Thus, as $n\to\infty$, $A = A_0\left(1 + \frac{r}{n}\right)^{nt} \to A_0e^{rt}$.

80. (a) $r = 3, \rho = 0.05 \Rightarrow P = \frac{1 - 10^{-\rho r^2}}{\rho r^2 \ln 10} = \frac{1 - 10^{-0.45}}{0.45\ln 10} \approx 0.62$, or about 62%.

(b) $r = 2$, $\rho = 0.05 \quad \Rightarrow \quad P = \dfrac{1 - 10^{-0.2}}{0.2 \ln 10} \approx 0.80$, or about 80%.

Yes, it makes sense. Since measured brightness decreases with light entering farther from the center of the pupil, a smaller pupil radius means that the average brightness measurements are higher than when including light entering at larger radii.

(c) $\displaystyle \lim_{r\to 0^+} P = \lim_{r\to 0^+} \frac{1 - 10^{-\rho r^2}}{\rho r^2 \ln 10} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{-10^{-\rho r^2}(\ln 10)(-2\rho r)}{2\rho r(\ln 10)} = \lim_{r\to 0^+} \frac{1}{10^{\rho r^2}} = 1$, or 100%.

We might expect that 100% of the brightness is sensed at the very center of the pupil, so a limit of 1 would make sense in this context if the radius r could approach 0. This result isn't physically possible because there are limitations on how small the pupil can shrink.

81. (a) $\displaystyle \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{1 + Ae^{-kt}} = \frac{M}{1 + A \cdot 0} = M$

It is to be expected that a population that is growing will eventually reach the maximum population size that can be supported.

(b) $\displaystyle \lim_{M\to\infty} P(t) = \lim_{M\to\infty} \frac{M}{1 + \dfrac{M - P_0}{P_0} e^{-kt}} = \lim_{M\to\infty} \frac{M}{1 + \left(\dfrac{M}{P_0} - 1\right) e^{-kt}} \overset{\text{H}}{=} \lim_{M\to\infty} \frac{1}{\dfrac{1}{P_0} e^{-kt}} = P_0 e^{kt}$

$P_0 e^{kt}$ is an exponential function.

82. (a) $\displaystyle \lim_{R\to r^+} v = \lim_{R\to r^+} \left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -cr^2 \lim_{R\to r^+} \left[\left(\frac{1}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -cr^2 \cdot \frac{1}{r^2} \cdot \ln 1 = -c \cdot 0 = 0$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\displaystyle \lim_{r\to 0^+} v = \lim_{r\to 0^+} \left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -\frac{c}{R^2} \lim_{r\to 0^+} \left[r^2 \ln\left(\frac{r}{R}\right)\right]$ [form is $0 \cdot \infty$]

$$= -\frac{c}{R^2} \lim_{r\to 0^+} \frac{\ln\left(\dfrac{r}{R}\right)}{\dfrac{1}{r^2}} \quad [\text{form is } \infty/\infty] \quad \overset{\text{H}}{=} -\frac{c}{R^2} \lim_{r\to 0^+} \frac{\dfrac{R}{r} \cdot \dfrac{1}{R}}{\dfrac{-2}{r^3}} = -\frac{c}{R^2} \lim_{r\to 0^+} \left(-\frac{r^2}{2}\right) = 0$$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

83. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} \overset{\text{H}}{=} \lim_{x\to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{\frac{1}{2}(2a^3a - a^4)^{-1/2}(2a^3 - 4a^3) - \frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}$$

$$= \frac{(a^4)^{-1/2}(-a^3) - \frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^3(a^4)^{-3/4}} = \frac{-a - \frac{1}{3}a}{-\frac{3}{4}} = \frac{4}{3}\left(\frac{4}{3}a\right) = \frac{16}{9}a$$

84. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ| = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$. So we have $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$. Now by elementary

trigonometry, $B(\theta) = \frac{1}{2}\,|QR|\,|PQ| = \frac{1}{2}(r - |OQ|)\,|PQ| = \frac{1}{2}(r - r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}$$

$$= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}$$

$$= \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta} = \lim_{\theta\to 0^+}\frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos 0} = \frac{1}{3}$$

85. The limit, $L = \lim\limits_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right] = \lim\limits_{x\to\infty}\left[x - x^2\ln\left(\frac{1}{x}+1\right)\right]$. Let $t = 1/x$, so as $x\to\infty$, $t\to 0^+$.

$$L = \lim_{t\to 0^+}\left[\frac{1}{t}-\frac{1}{t^2}\ln(t+1)\right] = \lim_{t\to 0^+}\frac{t-\ln(t+1)}{t^2} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{1-\dfrac{1}{t+1}}{2t} = \lim_{t\to 0^+}\frac{t/(t+1)}{2t} = \lim_{t\to 0^+}\frac{1}{2\,(t+1)} = \frac{1}{2}$$

Note: Starting the solution by factoring out x or x^2 leads to a more complicated solution.

86. $y = [f(x)]^{g(x)} \quad\Rightarrow\quad \ln y = g(x)\ln f(x)$. Since f is a positive function, $\ln f(x)$ is defined. Now $\lim\limits_{x\to a}\ln y = \lim\limits_{x\to a} g(x)\ln f(x) = -\infty$ since $\lim\limits_{x\to a} g(x) = \infty$ and $\lim\limits_{x\to a} f(x) = 0 \quad\Rightarrow\quad \lim\limits_{x\to a}\ln f(x) = -\infty$. Thus, if $t = \ln y$, $\lim\limits_{x\to a} y = \lim\limits_{t\to -\infty} e^t = 0$. Note that the limit, $\lim\limits_{x\to a} g(x)\ln f(x)$, is *not* of the form $\infty\cdot 0$.

87. Since $f(2) = 0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{f'(2+3x)\cdot 3+f'(2+5x)\cdot 5}{1} = f'(2)\cdot 3+f'(2)\cdot 5 = 8f'(2) = 8\cdot 7 = 56$$

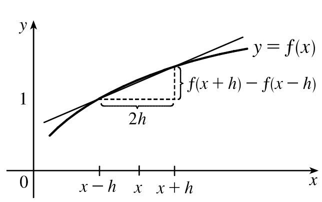
88. $L = \lim\limits_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right) = \lim\limits_{x\to 0}\frac{\sin 2x+ax^3+bx}{x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$. As $x\to 0$, $3x^2\to 0$, and $(2\cos 2x+3ax^2+b)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,

$\lim\limits_{x\to 0}\frac{2\cos 2x+3ax^2-2}{3x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{-4\sin 2x+6ax}{6x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{-8\cos 2x+6a}{6} = \frac{6a-8}{6}$, which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b=-2$ and $a=\frac{4}{3}$.

89. Since $\lim\limits_{h\to 0}[f(x+h)-f(x-h)] = f(x)-f(x) = 0$ (f is differentiable and hence continuous) and $\lim\limits_{h\to 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2} = \frac{f'(x)+f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



90. Since $\lim\limits_{h\to 0}[f(x+h)-2f(x)+f(x-h)] = f(x)-2f(x)+f(x) = 0$ [f is differentiable and hence continuous]

and $\lim\limits_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 89 to $f'(x)$.

91. (a) We show that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}} = \lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty}\frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty}\frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty}\frac{n!}{e^y} = 0 \quad\Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \lim_{x\to 0}x^n\frac{f(x)}{x^{2n}} = \lim_{x\to 0}x^n \lim_{x\to 0}\frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = 0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned}
f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)\,f(x) + p_n(x)\,f'(x)\right] - k_n x^{k_n-1}p_n(x)\,f(x)\right]x^{-2k_n} \\
&= \left[x^{k_n}p_n'(x) + p_n(x)(2/x^3) - k_n x^{k_n-1}p_n(x)\right]f(x)x^{-2k_n} \\
&= \left[x^{k_n+3}p_n'(x) + 2p_n(x) - k_n x^{k_n+2}\,p_n(x)\right]f(x)x^{-(2k_n+3)}
\end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

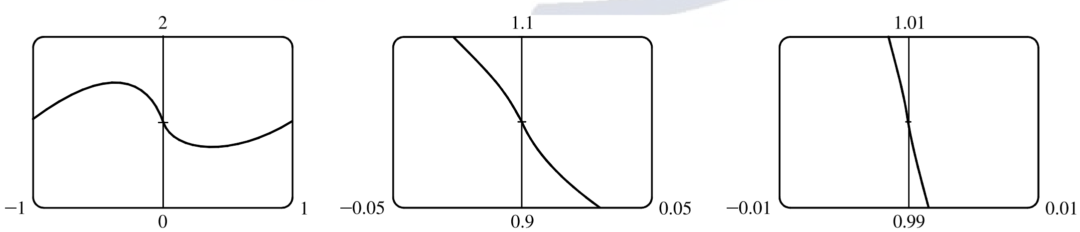
$$\begin{aligned}
f^{(n+1)}(0) &= \lim_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0} = \lim_{x\to 0}\frac{f^{(n)}(x)}{x} = \lim_{x\to 0}\frac{p_n(x)\,f(x)/x^{k_n}}{x} = \lim_{x\to 0}\frac{p_n(x)\,f(x)}{x^{k_n+1}} \\
&= \lim_{x\to 0}p_n(x)\lim_{x\to 0}\frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0
\end{aligned}$$

92. (a) For f to be continuous, we need $\lim\limits_{x\to 0}f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln|x|^x = x\ln|x|$.

So $\lim\limits_{x\to 0}\ln f(x) = \lim\limits_{x\to 0}x\ln|x| = \lim\limits_{x\to 0}\dfrac{\ln|x|}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{1/x}{-1/x^2} = 0$. Therefore, $\lim\limits_{x\to 0}f(x) = \lim\limits_{x\to 0}e^{\ln f(x)} = e^0 = 1$.

So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.

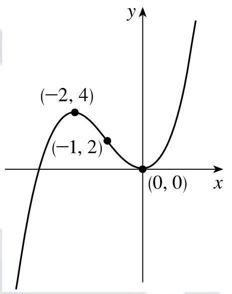


(c) To find f', we use logarithmic differentiation: $\ln f(x) = x \ln |x| \quad \Rightarrow \quad \dfrac{f'(x)}{f(x)} = x\left(\dfrac{1}{x}\right) + \ln |x| \quad \Rightarrow$

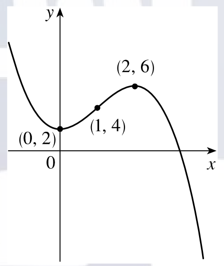
$f'(x) = f(x)(1 + \ln |x|) = |x|^x (1 + \ln |x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln |x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln |x| \to -\infty$ very slowly as $x \to 0$.

4.5 Summary of Curve Sketching

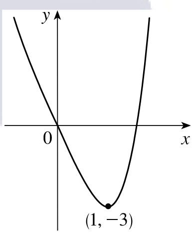
1. $y = f(x) = x^3 + 3x^2 = x^2(x + 3)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** y-intercept $= f(0) = 0$, x-intercepts are 0 and -3 **C.** No symmetry **D.** No asympote **E.** $f'(x) = 3x^2 + 6x = 3x(x + 2) > 0 \quad \Leftrightarrow \quad x < -2$ or $x > 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$. **F.** Local maximum value $f(-2) = 4$, local minimum value $f(0) = 0$ **G.** $f''(x) = 6x + 6 = 6(x + 1) > 0 \quad \Leftrightarrow \quad x > -1$, so f is CU on $(-1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 2)$ **H.**



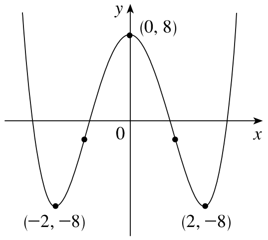
2. $y = f(x) = 2 + 3x^2 - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 6x - 3x^2 = 3x(2 - x) > 0 \quad \Leftrightarrow$ $0 < x < 2$, so f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$. **F.** Local maximum value $f(2) = 6$, local minimum value $f(0) = 2$ **G.** $f''(x) = 6 - 6x = 6(1 - x) > 0 \quad \Leftrightarrow \quad x < 1$, so f is CU on $(-\infty, 1)$ and CD on $(1, \infty)$. IP at $(1, 4)$ **H.**



3. $y = f(x) = x^4 - 4x = x(x^3 - 4)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and $\sqrt[3]{4}$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1) > 0 \quad \Leftrightarrow \quad x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** Local minimum value $f(1) = -3$, no local maximum **G.** $f''(x) = 12x^2 > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP **H.**



4. $y = f(x) = x^4 - 8x^2 + 8$ **A.** $D = \mathbb{R}$ **B.** y-intercept $f(0) = 8$; x-intercepts: $f(x) = 0 \quad \Rightarrow$ [by the quadratic formula] $x = \pm\sqrt{4 \pm 2\sqrt{2}} \approx \pm 2.61, \pm 1.08$ **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis **D.** No asymptote **E.** $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2) > 0 \quad \Leftrightarrow \quad -2 < x < 0$ or $x > 2$, so f is increasing on $(-2, 0)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(0, 2)$. **F.** Local maximum value $f(0) = 8$, local minimum values $f(\pm 2) = -8$ **G.** $f''(x) = 12x^2 - 16 = 4(3x^2 - 4) > 0 \quad \Rightarrow \quad |x| > 2/\sqrt{3}$ [≈ 1.15], so f is CU on $\left(-\infty, -2/\sqrt{3}\right)$ and $\left(2/\sqrt{3}, \infty\right)$, and f is CD on $\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$. IP at $\left(\pm 2/\sqrt{3}, -\frac{8}{9}\right)$ **H.**



5. $y = f(x) = x(x-4)^3$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and 4, y-intercept $f(0) = 0$ **C.** No symmetry

D. No asymptote

E. $f'(x) = x \cdot 3(x-4)^2 + (x-4)^3 \cdot 1 = (x-4)^2[3x + (x-4)]$

$= (x-4)^2(4x-4) = 4(x-1)(x-4)^2 > 0 \quad \Leftrightarrow$

$x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

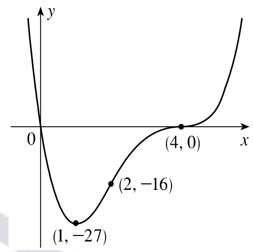
F. Local minimum value $f(1) = -27$, no local maximum value

G. $f''(x) = 4[(x-1) \cdot 2(x-4) + (x-4)^2 \cdot 1] = 4(x-4)[2(x-1) + (x-4)]$

$= 4(x-4)(3x-6) = 12(x-4)(x-2) < 0 \quad \Leftrightarrow$

$2 < x < 4$, so f is CD on $(2, 4)$ and CU on $(-\infty, 2)$ and $(4, \infty)$. IPs at $(2, -16)$ and $(4, 0)$

H.



6. $y = f(x) = x^5 - 5x = x(x^4 - 5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $\pm\sqrt[4]{5}$ and 0, y-intercept $= f(0) = 0$

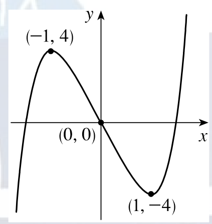
C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2-1)(x^2+1)$

$= 5(x+1)(x-1)(x^2+1) > 0 \quad \Leftrightarrow$

$x < -1$ or $x > 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and f is decreasing on $(-1, 1)$. **F.** Local maximum value $f(-1) = 4$, local minimum value $f(1) = -4$ **G.** $f''(x) = 20x^3 > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. IP at $(0, 0)$

H.



7. $y = f(x) = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x = x\left(\frac{1}{5}x^4 - \frac{8}{3}x^2 + 16\right)$ **A.** $D = \mathbb{R}$ **B.** x-intercept 0, y-intercept $= f(0) = 0$

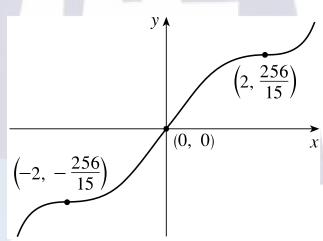
C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x^4 - 8x^2 + 16 = (x^2-4)^2 = (x+2)^2(x-2)^2 > 0$ for all x except ± 2, so f is increasing on $\mathbb{R}$. **F.** There is no local maximum or minimum value.

G. $f''(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x+2)(x-2) > 0 \quad \Leftrightarrow$

$-2 < x < 0$ or $x > 2$, so f is CU on $(-2, 0)$ and $(2, \infty)$, and f is CD on $(-\infty, -2)$ and $(0, 2)$. IP at $\left(-2, -\frac{256}{15}\right)$, $(0, 0)$, and $\left(2, \frac{256}{15}\right)$

H.



8. $y = f(x) = (4 - x^2)^5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 4^5 = 1024$; x-intercepts: ± 2 **C.** $f(-x) = f(x) \quad \Rightarrow$ f is even; the curve is symmetric about the y-axis. **D.** No asymptote **E.** $f'(x) = 5(4-x^2)^4(-2x) = -10x(4-x^2)^4$, so for $x \neq \pm 2$ we have $f'(x) > 0 \quad \Leftrightarrow \quad x < 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x > 0$. Thus, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. **F.** Local maximum value $f(0) = 1024$

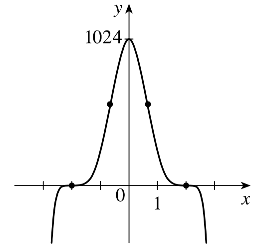
G. $f''(x) = -10x \cdot 4(4-x^2)^3(-2x) + (4-x^2)^4(-10)$

$= -10(4-x^2)^3[-8x^2 + 4 - x^2] = -10(4-x^2)^3(4-9x^2)$

so $f''(x) = 0 \quad \Leftrightarrow \quad x = \pm 2, \pm\frac{2}{3}$. $f''(x) > 0 \quad \Leftrightarrow \quad -2 < x < -\frac{2}{3}$ and $\frac{2}{3} < x < 2$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -2, -\frac{2}{3} < x < \frac{2}{3}$, and $x > 2$, so f is CU on $(-\infty, 2)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$, and $(2, \infty)$, and CD on $\left(-2, -\frac{2}{3}\right)$ and $\left(\frac{2}{3}, 2\right)$.

IP at $(\pm 2, 0)$ and $\left(\pm\frac{2}{3}, \left(\frac{32}{9}\right)^5\right) \approx (\pm 0.67, 568.25)$

H.



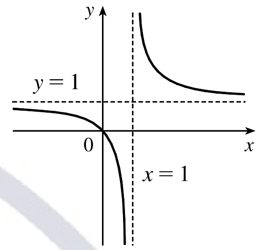
9. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$

C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{x-1} = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to 1^-} \dfrac{x}{x-1} = -\infty$, $\lim\limits_{x \to 1^+} \dfrac{x}{x-1} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1)-x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{2}{(x-1)^3} > 0 \quad \Leftrightarrow \quad x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



10. $y = f(x) = \dfrac{x^2 + 5x}{25 - x^2} = \dfrac{x(x+5)}{(5+x)(5-x)} = \dfrac{x}{5-x}$ for $x \neq -5$. There is a hole in the graph at $\left(-5, -\frac{1}{2}\right)$.

A. $D = \{x \mid x \neq \pm 5\} = (-\infty, -5) \cup (-5, 5) \cup (5, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

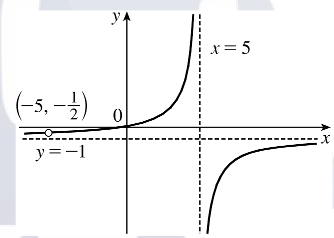
D. $\lim\limits_{x \to \pm\infty} \dfrac{x}{5-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x \to 5^-} \dfrac{x}{5-x} = \infty$, $\lim\limits_{x \to 5^+} \dfrac{x}{5-x} = -\infty$, so $x = 5$ is a VA.

E. $f'(x) = \dfrac{(5-x)(1) - x(-1)}{(5-x)^2} = \dfrac{5}{(5-x)^2} > 0$ for all x in D, so f is increasing on $(-\infty, -5)$, $(-5, 5)$, and $(5, \infty)$. **F.** No extrema

G. $f'(x) = 5(5-x)^{-2} \quad \Rightarrow$

$f''(x) = -10(5-x)^{-3}(-1) = \dfrac{10}{(5-x)^3} > 0 \quad \Leftrightarrow \quad x < 5$, so f is CU on $(-\infty, -5)$ and $(-5, 5)$, and f is CD on $(5, \infty)$. No IP

H.



11. $y = f(x) = \dfrac{x - x^2}{2 - 3x + x^2} = \dfrac{x(1-x)}{(1-x)(2-x)} = \dfrac{x}{2-x}$ for $x \neq 1$. There is a hole in the graph at $(1, 1)$.

A. $D = \{x \mid x \neq 1, 2\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

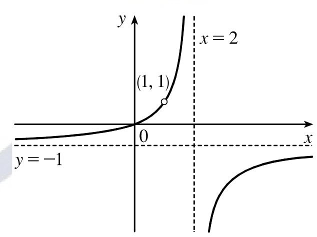
D. $\lim\limits_{x \to \pm\infty} \dfrac{x}{2-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x \to 2^-} \dfrac{x}{2-x} = \infty$, $\lim\limits_{x \to 2^+} \dfrac{x}{2-x} = -\infty$, so $x = 2$ is a VA.

E. $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2} > 0$ $[x \neq 1, 2]$, so f is increasing on $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. **F.** No extrema

G. $f'(x) = 2(2-x)^{-2} \quad \Rightarrow$

$f''(x) = -4(2-x)^{-3}(-1) = \dfrac{4}{(2-x)^3} > 0 \quad \Leftrightarrow \quad x < 2$, so f is CU on $(-\infty, 1)$ and $(1, 2)$, and f is CD on $(2, \infty)$. No IP

H.



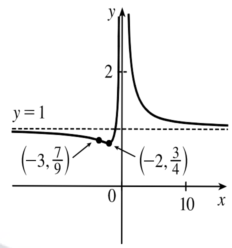
12. $y = f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{x^2 + x + 1}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** y-intercept: none $[x \neq 0]$;

x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x^2 + x + 1 = 0$, there is no real solution, and hence, no x-intercept **C.** No symmetry

D. $\lim\limits_{x \to \pm\infty} \left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to 0} f(x) = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} = \dfrac{-x-2}{x^3}$.

$f'(x) > 0 \quad \Leftrightarrow \quad -2 < x < 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $x > 0$, so f is increasing on $(-2, 0)$ and decreasing on $(-\infty, -2)$ and $(0, \infty)$. **F.** Local minimum value $f(-2) = \frac{3}{4}$; no local maximum **G.** $f''(x) = \dfrac{2}{x^3} + \dfrac{6}{x^4} = \dfrac{2x+6}{x^4}$. $f''(x) < 0 \;\Leftrightarrow\; x < -3$ and $f''(x) > 0 \;\Leftrightarrow\; -3 < x < 0$ and $x > 0$, so f is CD on $(-\infty, -3)$ and CU on $(-3, 0)$ and $(0, \infty)$. IP at $\left(-3, \frac{7}{9}\right)$

H.



13. $y = f(x) = \dfrac{x}{x^2-4} = \dfrac{x}{(x+2)(x-2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x\to 2^+} \dfrac{x}{x^2-4} = \infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to -2^+} f(x) = \infty$, $\lim\limits_{x\to -2^-} f(x) = -\infty$, so $x = \pm 2$ are VAs.

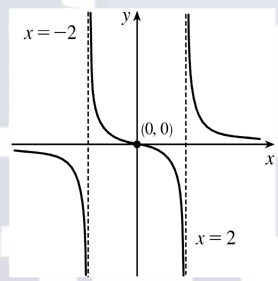
$\lim\limits_{x\to\pm\infty} \dfrac{x}{x^2-4} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2} < 0$ for all x in D, so f is decreasing on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

F. No local extrema

G.
$$f''(x) = -\frac{(x^2-4)^2(2x) - (x^2+4)2(x^2-4)(2x)}{[(x^2-4)^2]^2}$$
$$= -\frac{2x(x^2-4)[(x^2-4) - 2(x^2+4)]}{(x^2-4)^4}$$
$$= -\frac{2x(-x^2-12)}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x+2)^3(x-2)^3}.$$

$f''(x) < 0$ if $x < -2$ or $0 < x < 2$, so f is CD on $(-\infty, -2)$ and $(0, 2)$, and CU on $(-2, 0)$ and $(2, \infty)$. IP at $(0, 0)$

H.



14. $y = f(x) = \dfrac{1}{x^2-4} = \dfrac{1}{(x+2)(x-2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** No x-intercept, y-intercept $= f(0) = -\frac{1}{4}$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

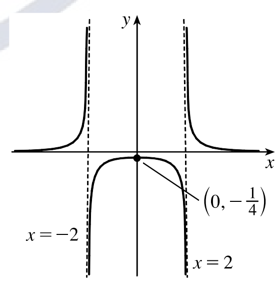
D. $\lim\limits_{x\to 2^+} \dfrac{1}{x^2-4} = \infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to -2^+} f(x) = -\infty$, $\lim\limits_{x\to -2^-} f(x) = \infty$, so $x = \pm 2$ are VAs. $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a HA. **E.** $f'(x) = -\dfrac{2x}{(x^2-4)^2}$ [Reciprocal Rule] > 0 if $x < 0$ and x is in D, so f is increasing on $(-\infty, -2)$ and $(-2, 0)$. f is decreasing on $(0, 2)$ and $(2, \infty)$.

F. Local maximum value $f(0) = -\frac{1}{4}$, no local minimum value

G.
$$f''(x) = \frac{(x^2-4)^2(-2) - (-2x)2(x^2-4)(2x)}{[(x^2-4)^2]^2}$$
$$= \frac{-2(x^2-4)\left[(x^2-4) - 4x^2\right]}{(x^2-4)^4}$$
$$= \frac{-2(-3x^2-4)}{(x^2-4)^3} = \frac{2(3x^2+4)}{(x^2-4)^3}$$

$f''(x) < 0 \;\Leftrightarrow\; -2 < x < 2$, so f is CD on $(-2, 2)$ and CU on $(-\infty, -2)$ and $(2, \infty)$. No IP

H.



15. $y = f(x) = \dfrac{x^2}{x^2+3} = \dfrac{(x^2+3)-3}{x^2+3} = 1 - \dfrac{3}{x^2+3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$;

x-intercepts: $f(x) = 0 \iff x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{x^2+3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x) = -3 \cdot \dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$.

$f'(x) > 0 \iff x > 0$ and $f'(x) < 0 \iff x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

F. Local minimum value $f(0) = 0$, no local maximum.

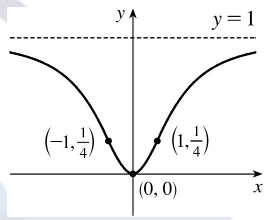
G. $f''(x) = \dfrac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2}$

$= \dfrac{6(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \dfrac{6(3-3x^2)}{(x^2+3)^3} = \dfrac{-18(x+1)(x-1)}{(x^2+3)^3}$

$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$,

so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

H.



16. $y = f(x) = \dfrac{(x-1)^2}{x^2+1} \geq 0$ with equality $\iff x = 1$. **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 1$; x-intercept 1 **C.** No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2-2x+1}{x^2+1} = \lim_{x\to\pm\infty} \frac{1 - 2/x + 1/x^2}{1 + 1/x^2} = 1$, so $y = 1$ is a HA. No VA

E. $f'(x) = \dfrac{(x^2+1)2(x-1) - (x-1)^2(2x)}{(x^2+1)^2} = \dfrac{2(x-1)\left[(x^2+1) - x(x-1)\right]}{(x^2+1)^2} = \dfrac{2(x-1)(x+1)}{(x^2+1)^2} < 0 \iff$

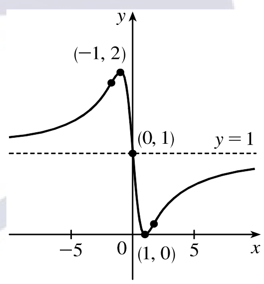
$-1 < x < 1$, so f is decreasing on $(-1, 1)$ and increasing on $(-\infty, -1)$ and $(1, \infty)$ **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = 0$

G. $f''(x) = \dfrac{(x^2+1)^2(4x) - (2x^2-2)2(x^2+1)(2x)}{[(x^2+1)^2]^2} = \dfrac{4x(x^2+1)\left[(x^2+1) - (2x^2-2)\right]}{(x^2+1)^4} = \dfrac{4x(3-x^2)}{(x^2+1)^3}$.

$f''(x) > 0 \iff x < -\sqrt{3}$ or $0 < x < \sqrt{3}$, so f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and f is CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$.

$f\left(\pm\sqrt{3}\right) = \frac{1}{4}\left(\sqrt{3} \mp 1\right)^2 = \frac{1}{4}\left(4 \mp 2\sqrt{3}\right) = 1 \mp \frac{1}{2}\sqrt{3}$ [$\approx 0.13, 1.87$], so there are IPs at $\left(-\sqrt{3}, 1 + \frac{1}{2}\sqrt{3}\right)$, $(0, 1)$, and $\left(\sqrt{3}, 1 - \frac{1}{2}\sqrt{3}\right)$. Note that the graph is symmetric about the point $(0, 1)$.

H.



17. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \iff x = 1$

C. No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 0} \frac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

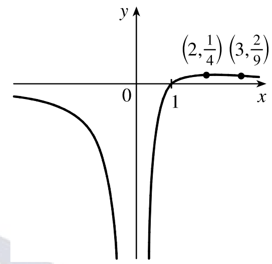
E. $f'(x) = \dfrac{x^2 \cdot 1 - (x-1) \cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \iff 0 < x < 2$ and $f'(x) < 0 \iff$

$x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$. **F.** No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3 \cdot (-1) - [-(x-2)] \cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $\left(3, \frac{2}{9}\right)$

H.



18. $y = f(x) = \dfrac{x}{x^3 - 1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{x^3 - 1} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

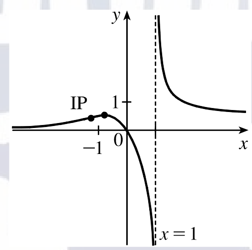
E. $f'(x) = \dfrac{(x^3 - 1)(1) - x(3x^2)}{(x^3 - 1)^2} = \dfrac{-2x^3 - 1}{(x^3 - 1)^2}$. $f'(x) = 0 \Rightarrow x = -\sqrt[3]{1/2}$. $f'(x) > 0 \Leftrightarrow x < -\sqrt[3]{1/2}$ and $f'(x) < 0 \Leftrightarrow -\sqrt[3]{1/2} < x < 1$ and $x > 1$, so f is increasing on $\left(-\infty, -\sqrt[3]{1/2}\right)$ and decreasing on $\left(-\sqrt[3]{1/2}, 1\right)$ and $(1, \infty)$. **F.** Local maximum value $f\left(-\sqrt[3]{1/2}\right) = \frac{2}{3}\sqrt[3]{1/2}$; no local minimum

G. $f''(x) = \dfrac{(x^3 - 1)^2(-6x^2) - (-2x^3 - 1)2(x^3 - 1)(3x^2)}{[(x^3 - 1)^2]^2}$

$= \dfrac{-6x^2(x^3 - 1)[(x^3 - 1) - (2x^3 + 1)]}{(x^3 - 1)^4} = \dfrac{6x^2(x^3 + 2)}{(x^3 - 1)^3}$.

$f''(x) > 0 \Leftrightarrow x < -\sqrt[3]{2}$ and $x > 1$, $f''(x) < 0 \Leftrightarrow -\sqrt[3]{2} < x < 0$ and $0 < x < 1$, so f is CU on $\left(-\infty, -\sqrt[3]{2}\right)$ and $(1, \infty)$ and CD on $\left(-\sqrt[3]{2}, 1\right)$. IP at $\left(-\sqrt[3]{2}, \frac{1}{3}\sqrt[3]{2}\right)$

H.



19. $y = f(x) = \dfrac{x^3}{x^3 + 1} = \dfrac{x^3}{(x + 1)(x^2 - x + 1)}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x^3}{x^3 + 1} = \dfrac{1}{1 + 1/x^3} = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^-} f(x) = \infty$ and $\lim\limits_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA. **E.** $f'(x) = \dfrac{(x^3 + 1)(3x^2) - x^3(3x^2)}{(x^3 + 1)^2} = \dfrac{3x^2}{(x^3 + 1)^2}$. $f'(x) > 0$ for $x \neq -1$ (not in the domain) and $x \neq 0$ ($f' = 0$), so f is increasing on $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$, and furthermore, by Exercise 4.3.91, f is increasing on $(-\infty, -1)$, and $(-1, \infty)$. **F.** No local extrema

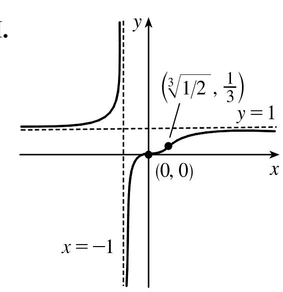
4.5.19: deleted word "for" on line 4

G. $f''(x) = \dfrac{(x^3 + 1)^2(6x) - 3x^2[2(x^3 + 1)(3x^2)]}{[(x^3 + 1)^2]^2}$

$= \dfrac{(x^3 + 1)(6x)[(x^3 + 1) - 3x^3]}{(x^3 + 1)^4} = \dfrac{6x(1 - 2x^3)}{(x^3 + 1)^3}$

$f''(x) > 0 \Leftrightarrow x < -1$ or $0 < x < \sqrt[3]{\frac{1}{2}}$ [≈ 0.79], so f is CU on $(-\infty, -1)$ and $\left(0, \sqrt[3]{\frac{1}{2}}\right)$ and CD on $(-1, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$. There are IPs at $(0, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$.

H.



20. $y = f(x) = \dfrac{x^3}{x-2} = x^2 + 2x + 4 + \dfrac{8}{x-2}$ [by long division] **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x\to 2^-} \dfrac{x^3}{x-2} = -\infty$ and $\lim\limits_{x\to 2^+} \dfrac{x^3}{x-2} = \infty$, so $x = 2$ is a VA. There are no horizontal or slant asymptotes. *Note:* Since $\lim\limits_{x\to\pm\infty} \dfrac{8}{x-2} = 0$, the parabola $y = x^2 + 2x + 4$ is approached asymptotically as $x \to \pm\infty$.

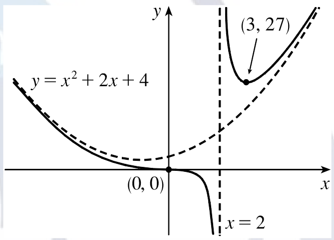
E. $f'(x) = \dfrac{(x-2)(3x^2) - x^3(1)}{(x-2)^2} = \dfrac{x^2[3(x-2) - x]}{(x-2)^2} = \dfrac{x^2(2x-6)}{(x-2)^2} = \dfrac{2x^2(x-3)}{(x-2)^2} > 0 \;\Leftrightarrow\; x > 3$ and $f'(x) < 0 \;\Leftrightarrow\; x < 0$ or $0 < x < 2$ or $2 < x < 3$, so f is increasing on $(3, \infty)$ and f is decreasing on $(-\infty, 2)$ and $(2, 3)$.

F. Local minimum value $f(3) = 27$, no local maximum value **G.** $f'(x) = 2\dfrac{x^3 - 3x^2}{(x-2)^2} \Rightarrow$

$$f''(x) = 2\frac{(x-2)^2(3x^2-6x) - (x^3-3x^2)2(x-2)}{[(x-2)^2]^2}$$
$$= 2\frac{(x-2)x[(x-2)(3x-6) - (x^2-3x)2]}{(x-2)^4}$$
$$= \frac{2x(3x^2 - 12x + 12 - 2x^2 + 6x)}{(x-2)^3}$$
$$= \frac{2x(x^2 - 6x + 12)}{(x-2)^3} > 0 \;\Leftrightarrow$$

$x < 0$ or $x > 2$, so f is CU on $(-\infty, 0)$ and $(2, \infty)$, and f is CD on $(0, 2)$. IP at $(0, 0)$

H.

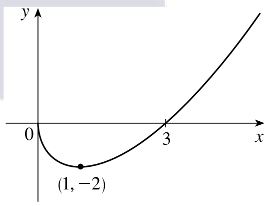


21. $y = f(x) = (x-3)\sqrt{x} = x^{3/2} - 3x^{1/2}$ **A.** $D = [0, \infty)$ **B.** x-intercepts: $0, 3$; y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x-1) = \dfrac{3(x-1)}{2\sqrt{x}} > 0 \;\Leftrightarrow\; x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local minimum value $f(1) = -2$, no local maximum value

G. $f''(x) = \frac{3}{4}x^{-1/2} + \frac{3}{4}x^{-3/2} = \frac{3}{4}x^{-3/2}(x+1) = \dfrac{3(x+1)}{4x^{3/2}} > 0$ for $x > 0$, so f is CU on $(0, \infty)$. No IP

H.



22. $y = f(x) = (x-4)\sqrt[3]{x} = x^{4/3} - 4x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$; x-intercepts: 0 and 4

C. No symmetry **D.** No asymptote

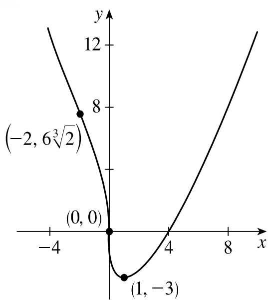
E. $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \dfrac{4(x-1)}{3x^{2/3}}$. $f'(x) > 0 \;\Leftrightarrow$ $x > 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, 1)$.

F. Local minimum value $f(1) = -3$

G. $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x+2) = \dfrac{4(x+2)}{9x^{5/3}}$.

$f''(x) < 0 \;\Leftrightarrow\; -2 < x < 0$, so f is CD on $(-2, 0)$, and f is CU on $(-\infty, -2)$ and $(0, \infty)$. There are IPs at $\left(-2, 6\sqrt[3]{2}\right)$ and $(0, 0)$.

H.



23. $y = f(x) = \sqrt{x^2 + x - 2} = \sqrt{(x+2)(x-1)}$ **A.** $D = \{x \mid (x+2)(x-1) \geq 0\} = (-\infty, -2] \cup [1, \infty)$

B. y-intercept: none; x-intercepts: -2 and 1 **C.** No symmetry **D.** No asymptote

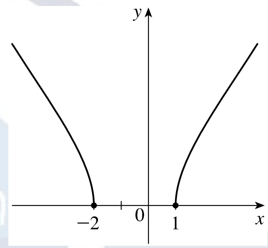
E. $f'(x) = \frac{1}{2}(x^2 + x - 2)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x-2}}$, $f'(x) = 0$ if $x = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the domain.

$f'(x) > 0 \;\Rightarrow\; x > -\frac{1}{2}$ and $f'(x) < 0 \;\Rightarrow\; x < -\frac{1}{2}$, so (considering the domain) f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, -2)$. **F.** No local extrema

G.
$$f''(x) = \frac{2(x^2+x-2)^{1/2}(2) - (2x+1)\cdot 2\cdot \frac{1}{2}(x^2+x-2)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x-2}\right)^2}$$
$$= \frac{(x^2+x-2)^{-1/2}\left[4(x^2+x-2) - (4x^2+4x+1)\right]}{4(x^2+x-2)}$$
$$= \frac{-9}{4(x^2+x-2)^{3/2}} < 0$$

so f is CD on $(-\infty, -2)$ and $(1, \infty)$. No IP

H.



24. $y = f(x) = \sqrt{x^2 + x} - x = \sqrt{x(x+1)} - x$ **A.** $D = (-\infty, -1] \cup [0, \infty)$ **B.** y-intercept: $f(0) = 0$;

x-intercepts: $f(x) = 0 \;\Rightarrow\; \sqrt{x^2+x} = x \;\Rightarrow\; x^2 + x = x^2 \;\Rightarrow\; x = 0$ **C.** No symmetry

D.
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$$
$$= \lim_{x\to\infty}\frac{x/x}{\left(\sqrt{x^2+x}+x\right)/x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1} = \frac{1}{2},$$
so $y = \frac{1}{2}$ is a HA. No VA

E. $f'(x) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) - 1 = \dfrac{2x+1}{2\sqrt{x^2+x}} - 1 > 0 \;\Leftrightarrow\; 2x+1 > 2\sqrt{x^2+x} \;\Leftrightarrow$

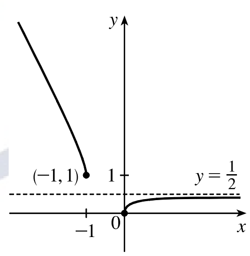
$x + \frac{1}{2} > \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}$. Keep in mind that the domain excludes the interval $(-1, 0)$. When $x + \frac{1}{2}$ is positive (for $x \geq 0$), the last inequality is *true* since the value of the radical is less than $x + \frac{1}{2}$. When $x + \frac{1}{2}$ is negative (for $x \leq -1$), the last inequality is *false* since the value of the radical is positive. So f is increasing on $(0, \infty)$ and decreasing on $(-\infty, -1)$.

F. No local extrema

G.
$$f''(x) = \frac{2(x^2+x)^{1/2}(2) - (2x+1)\cdot 2\cdot\frac{1}{2}(x^2+x)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x}\right)^2}$$
$$= \frac{(x^2+x)^{-1/2}[4(x^2+x) - (2x+1)^2]}{4(x^2+x)} = \frac{-1}{4(x^2+x)^{3/2}}.$$

$f''(x) < 0$ when it is defined, so f is CD on $(-\infty, -1)$ and $(0, \infty)$. No IP

H.



25. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D.
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}}$$

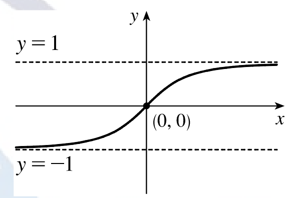
$$= \frac{1}{-\sqrt{1+0}} = -1 \quad \text{so } y = \pm 1 \text{ are HA. No VA}$$

E. $f'(x) = \dfrac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values

G. $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.



26. $y = f(x) = x\sqrt{2-x^2}$ **A.** $D = \left[-\sqrt{2}, \sqrt{2}\right]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm\sqrt{2}$. **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x \cdot \dfrac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \dfrac{-x^2+2-x^2}{\sqrt{2-x^2}} = \dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2} < x < -1$ and $1 < x < \sqrt{2}$, and positive for $-1 < x < 1$, so f is decreasing on $\left(-\sqrt{2}, -1\right)$ and $\left(1, \sqrt{2}\right)$ and increasing on $(-1, 1)$.

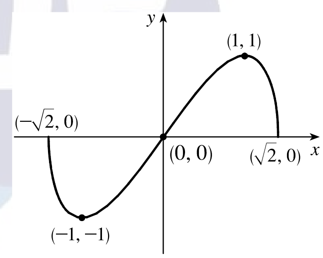
F. Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

G. $f''(x) = \dfrac{\sqrt{2-x^2}(-4x) - (2-2x^2)\dfrac{-x}{\sqrt{2-x^2}}}{[(2-x^2)^{1/2}]^2}$

$$= \frac{(2-x^2)(-4x) + (2-2x^2)x}{(2-x^2)^{3/2}} = \frac{2x^3-6x}{(2-x^2)^{3/2}} = \frac{2x(x^2-3)}{(2-x^2)^{3/2}}$$

Since $x^2 - 3 < 0$ for x in $\left[-\sqrt{2}, \sqrt{2}\right]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2}, 0\right)$ and CD on $\left(0, \sqrt{2}\right)$. The only IP is $(0, 0)$.

H.

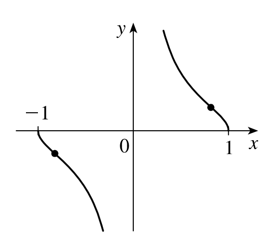


27. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \{x \mid |x| \le 1, x \ne 0\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\displaystyle\lim_{x\to0^+} \frac{\sqrt{1-x^2}}{x} = \infty$, $\displaystyle\lim_{x\to0^-} \frac{\sqrt{1-x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \quad\Leftrightarrow\quad -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

H.



28. $y = f(x) = x/\sqrt{x^2-1}$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} = -1$, so $y = \pm 1$ are HA.

$\lim\limits_{x\to 1^+} f(x) = +\infty$ and $\lim\limits_{x\to -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

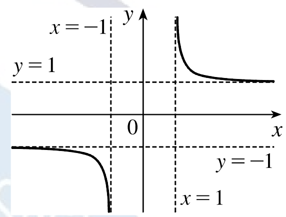
E. $f'(x) = \dfrac{\sqrt{x^2-1} - x \cdot \dfrac{x}{\sqrt{x^2-1}}}{[(x^2-1)^{1/2}]^2} = \dfrac{x^2-1-x^2}{(x^2-1)^{3/2}} = \dfrac{-1}{(x^2-1)^{3/2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No extreme values

G. $f''(x) = (-1)(-\frac{3}{2})(x^2-1)^{-5/2} \cdot 2x = \dfrac{3x}{(x^2-1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) > 0$ on $(1, \infty)$, so f is CD on $(-\infty, -1)$ and CU on $(1, \infty)$. No IP

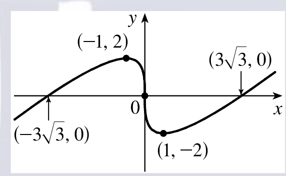
H.



29. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2 - 27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3}-1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is continuous on $(-1, 1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. IP at $(0, 0)$

H.



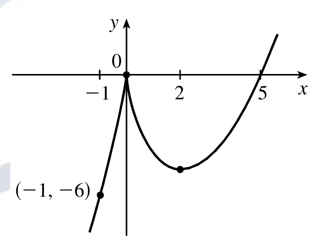
30. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts 0, 5; y-intercept 0 **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty} x^{2/3}(x-5) = \pm\infty$, so there is no asymptote **E.** $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) > 0 \Leftrightarrow x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$. IP at $(-1, -6)$

H.



31. $y = f(x) = \sqrt[3]{x^2-1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -1$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$ **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

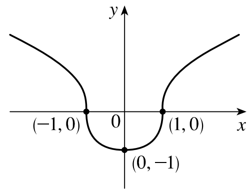
E. $f'(x) = \frac{1}{3}(x^2-1)^{-2/3}(2x) = \dfrac{2x}{3\sqrt[3]{(x^2-1)^2}}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = -1$

G. $f''(x) = \dfrac{2}{3} \cdot \dfrac{(x^2-1)^{2/3}(1) - x \cdot \frac{2}{3}(x^2-1)^{-1/3}(2x)}{[(x^2-1)^{2/3}]^2}$

$= \dfrac{2}{9} \cdot \dfrac{(x^2-1)^{-1/3}[3(x^2-1) - 4x^2]}{(x^2-1)^{4/3}} = -\dfrac{2(x^2+3)}{9(x^2-1)^{5/3}}$

$f''(x) > 0 \quad \Leftrightarrow \quad -1 < x < 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(\pm 1, 0)$

H.



32. $y = f(x) = \sqrt[3]{x^3+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 1 = 0 \quad \Rightarrow \quad x = -1$

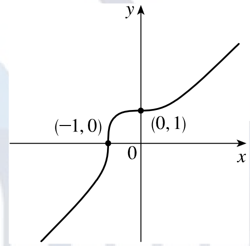
C. No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{3}(x^3+1)^{-2/3}(3x^2) = \dfrac{x^2}{\sqrt[3]{(x^3+1)^2}}$. $f'(x) > 0$ if $x < -1$, $-1 < x < 0$, and $x > 0$, so f is increasing on $\mathbb{R}$. **F.** No local extrema

G. $f''(x) = \dfrac{(x^3+1)^{2/3}(2x) - x^2 \cdot \frac{2}{3}(x^3+1)^{-1/3}(3x^2)}{[(x^3+1)^{2/3}]^2}$

$= \dfrac{x(x^3+1)^{-1/3}[2(x^3+1) - 2x^3]}{(x^3+1)^{4/3}} = \dfrac{2x}{(x^3+1)^{5/3}}$

$f''(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1 < x < 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$ and CD on $(-1, 0)$. IP at $(-1, 0)$ and $(0, 1)$

H.



33. $y = f(x) = \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = n\pi$, n an integer; y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. Also, $f(x + 2\pi) = f(x)$, so f is periodic with period 2π, and we determine **E–G** for $0 \le x \le \pi$. Since f is odd, we can reflect the graph of f on $[0, \pi]$ about the origin to obtain the graph of f on $[-\pi, \pi]$, and then since f has period 2π, we can extend the graph of f for all real numbers.

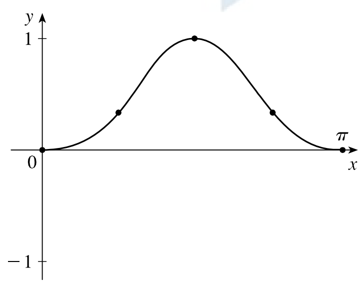
D. No asymptote **E.** $f'(x) = 3\sin^2 x \cos x > 0 \quad \Leftrightarrow \quad \cos x > 0$ and $\sin x \ne 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and f is decreasing on $\left(\frac{\pi}{2}, \pi\right)$. **F.** Local maximum value $f\left(\frac{\pi}{2}\right) = 1$ $\left[\text{local minimum value } f\left(-\frac{\pi}{2}\right) = -1\right]$

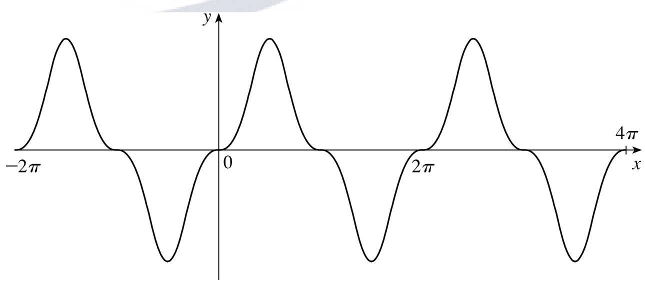
G. $f''(x) = 3\sin^2 x\,(-\sin x) + 3\cos x\,(2\sin x \cos x) = 3\sin x\,(2\cos^2 x - \sin^2 x)$

$= 3\sin x\,[2(1 - \sin^2 x) - \sin^2 x] = 3\sin x(2 - 3\sin^2 x) > 0 \quad \Leftrightarrow$

$\sin x > 0$ and $\sin^2 x < \frac{2}{3} \quad \Leftrightarrow \quad 0 < x < \pi$ and $0 < \sin x < \sqrt{\frac{2}{3}} \quad \Leftrightarrow \quad 0 < x < \sin^{-1}\sqrt{\frac{2}{3}}$ $\left[\text{let } \alpha = \sin^{-1}\sqrt{\frac{2}{3}}\right]$ or $\pi - \alpha < x < \pi$, so f is CU on $(0, \alpha)$ and $(\pi - \alpha, \pi)$, and f is CD on $(\alpha, \pi - \alpha)$. There are inflection points at $x = 0, \pi, \alpha$, and $x = \pi - \alpha$.

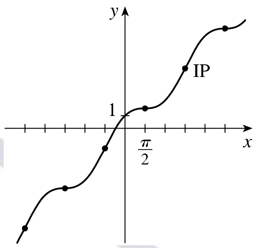
H.





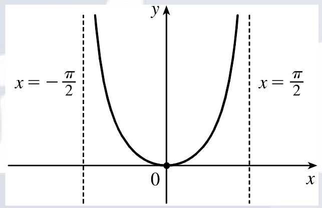
34. $y = f(x) = x + \cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; the x-intercept is about -0.74 and can be found using Newton's method **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 1 - \sin x > 0$ except for $x = \frac{\pi}{2} + 2n\pi$, so f is increasing on $\mathbb{R}$. **F.** No local extrema **G.** $f''(x) = -\cos x$. $f''(x) > 0 \Rightarrow -\cos x > 0 \Rightarrow \cos x < 0 \Rightarrow$ x is in $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and $f''(x) < 0 \Rightarrow$ x is in $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$, so f is CU on $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and CD on $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$. IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right) = \left(\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$ [on the line $y = x$]

H.



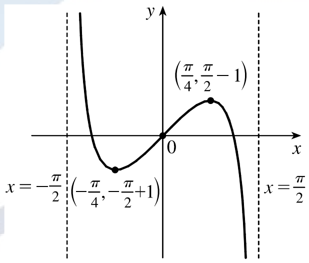
35. $y = f(x) = x \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x \to (\pi/2)^-} x \tan x = \infty$ and $\lim_{x \to -(\pi/2)^+} x \tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = \tan x + x \sec^2 x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$. **F.** Absolute and local minimum value $f(0) = 0$. **G.** $y'' = 2\sec^2 x + 2x \tan x \sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



36. $y = f(x) = 2x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow 2x = \tan x \Leftrightarrow x = 0$ or $x \approx \pm 1.17$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x \to (-\pi/2)^+} (2x - \tan x) = \infty$ and $\lim_{x \to (\pi/2)^-} (2x - \tan x) = -\infty$, so $x = \pm\frac{\pi}{2}$ are VA. No HA. **E.** $f'(x) = 2 - \sec^2 x < 0 \Leftrightarrow |\sec x| > \sqrt{2}$ and $f'(x) > 0 \Leftrightarrow |\sec x| < \sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - 1$, local minimum value $f\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} + 1$ **G.** $f''(x) = -2\sec x \cdot \sec x \tan x = -2\tan x \sec^2 x = -2\tan x(\tan^2 x + 1)$ so $f''(x) > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \Leftrightarrow \tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.

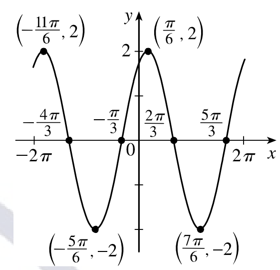


37. $y = f(x) = \sin x + \sqrt{3}\cos x$, $-2\pi \le x \le 2\pi$ **A.** $D = [-2\pi, 2\pi]$ **B.** y-intercept: $f(0) = \sqrt{3}$; x-intercepts: $f(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow \tan x = -\sqrt{3} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$ **C.** f is periodic with period 2π. **D.** No asymptote **E.** $f'(x) = \cos x - \sqrt{3}\sin x$. $f'(x) = 0 \Leftrightarrow \cos x = \sqrt{3}\sin x \Leftrightarrow \tan x = \frac{1}{\sqrt{3}} \Leftrightarrow$ $x = -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}$, or $\frac{7\pi}{6}$. $f'(x) < 0 \Leftrightarrow -\frac{11\pi}{6} < x < -\frac{5\pi}{6}$ or $\frac{\pi}{6} < x < \frac{7\pi}{6}$, so f is decreasing on $\left(-\frac{11\pi}{6}, -\frac{5\pi}{6}\right)$

and $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$, and f is increasing on $\left(-2\pi, -\frac{11\pi}{6}\right)$, $\left(-\frac{5\pi}{6}, \frac{\pi}{6}\right)$, and $\left(\frac{7\pi}{6}, 2\pi\right)$. **F.** Local maximum value $f\left(-\frac{11\pi}{6}\right) = f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \sqrt{3}\left(\frac{1}{2}\sqrt{3}\right) = 2$, local minimum value $f\left(-\frac{5\pi}{6}\right) = f\left(\frac{7\pi}{6}\right) = -\frac{1}{2} + \sqrt{3}\left(-\frac{1}{2}\sqrt{3}\right) = -2$

G. $f''(x) = -\sin x - \sqrt{3}\cos x$. $f''(x) = 0 \iff \sin x = -\sqrt{3}\cos x \iff \tan x = -\dfrac{1}{\sqrt{3}} \iff x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$. $f''(x) > 0 \iff -\frac{4\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{2\pi}{3} < x < \frac{5\pi}{3}$, so f is CU on $\left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$, and f is CD on $\left(-2\pi, -\frac{4\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are IPs at $\left(-\frac{4\pi}{3}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, and $\left(\frac{5\pi}{3}, 0\right)$.

H.



38. $y = f(x) = \csc x - 2\sin x$, $0 < x < \pi$ **A.** $D = (0, \pi)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \iff \csc x = 2\sin x \iff \frac{1}{2} = \sin^2 x \iff \sin x = \pm\frac{1}{2}\sqrt{2} \iff x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ **C.** No symmetry

D. $\lim\limits_{x\to 0^+} f(x) = \infty$ and $\lim\limits_{x\to \pi^-} f(x) = \infty$, so $x = 0$ and $x = \pi$ are VAs.

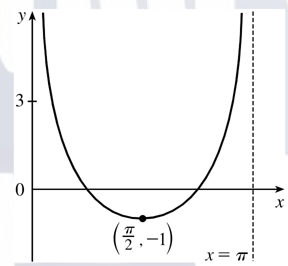
E. $f'(x) = -\csc x\cot x - 2\cos x = -\dfrac{\cos x}{\sin^2 x} - 2\cos x = -\cos x\left(\dfrac{1}{\sin^2 x} + 2\right)$. $f'(x) > 0$ when $-\cos x > 0 \iff \cos x < 0 \iff \frac{\pi}{2} < x < \pi$, so f' is increasing on $\left(\frac{\pi}{2}, \pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{2}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{2}\right) = -1$

G. $f''(x) = (-\csc x)(-\csc^2 x) + (\cot x)(\csc x\ \cot x) + 2\sin x$

$$= \frac{1 + \cos^2 x + 2\sin^4 x}{\sin^3 x}$$

f'' has the same sign as $\sin x$, which is positive on $(0, \pi)$, so f is CU on $(0, \pi)$. No IP

H.



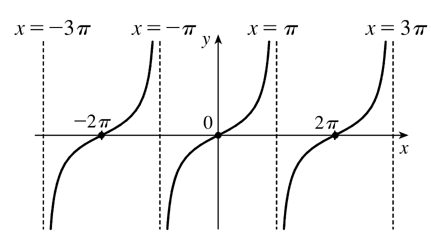
39. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \neq 1}{=} \dfrac{\sin x}{1 + \cos x} \cdot \dfrac{1 - \cos x}{1 - \cos x} = \dfrac{\sin x\,(1 - \cos x)}{\sin^2 x} = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x\right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π; that is, all reals except $(2n + 1)\pi$, where n is an integer. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = 2n\pi$, n an integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x\to(n\pi)^-} f(x) = \infty$ and $\lim\limits_{x\to(n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n. No HA.

E. $f'(x) = \dfrac{(1 + \cos x)\cdot\cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1 + \cos x}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k - 1)\pi, (2k + 1)\pi)$ for each integer k. **F.** No extreme values

G. $f''(x) = \dfrac{\sin x}{(1 + \cos x)^2} > 0 \Rightarrow \sin x > 0 \Rightarrow x \in (2k\pi, (2k + 1)\pi)$ and $f''(x) < 0$ on $((2k - 1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k + 1)\pi)$ and CD on $((2k - 1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.



40. $y = f(x) = \dfrac{\sin x}{2+\cos x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \iff \sin x = 0 \iff x = n\pi$

C. $f(-x) = -f(x)$, so the curve is symmetric about the origin. f is periodic with period 2π, so we determine **E–G** for $0 \le x \le 2\pi$. **D.** No asymptote

E. $f'(x) = \dfrac{(2+\cos x)\cos x - \sin x(-\sin x)}{(2+\cos x)^2} = \dfrac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} = \dfrac{2\cos x + 1}{(2+\cos x)^2}$.

$f'(x) > 0 \iff 2\cos x + 1 > 0 \iff \cos x > -\frac{1}{2} \iff x$ is in $\left(0, \frac{2\pi}{3}\right)$ or $\left(\frac{4\pi}{3}, 2\pi\right)$, so f is increasing on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and f is decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

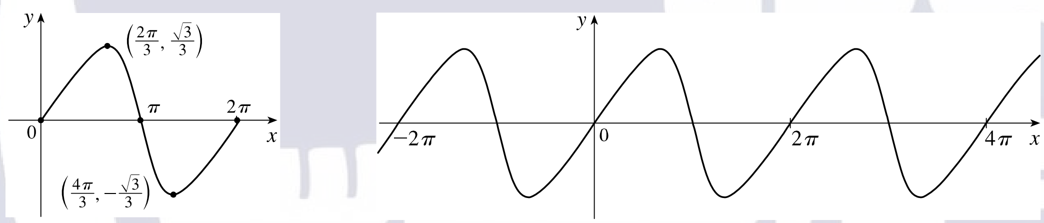
F. Local maximum value $f\left(\frac{2\pi}{3}\right) = \dfrac{\sqrt{3}/2}{2-(1/2)} = \dfrac{\sqrt{3}}{3}$ and local minimum value $f\left(\frac{4\pi}{3}\right) = \dfrac{-\sqrt{3}/2}{2-(1/2)} = -\dfrac{\sqrt{3}}{3}$

G. $f''(x) = \dfrac{(2+\cos x)^2(-2\sin x) - (2\cos x + 1)2(2+\cos x)(-\sin x)}{[(2+\cos x)^2]^2}$

$= \dfrac{-2\sin x\,(2+\cos x)[(2+\cos x) - (2\cos x + 1)]}{(2+\cos x)^4} = \dfrac{-2\sin x\,(1-\cos x)}{(2+\cos x)^3}$

$f''(x) > 0 \iff -2\sin x > 0 \iff \sin x < 0 \iff x$ is in $(\pi, 2\pi)$ [f is CU] and $f''(x) < 0 \iff x$ is in $(0, \pi)$ [f is CD]. The inflection points are $(0,0)$, $(\pi, 0)$, and $(2\pi, 0)$.

H.



41. $y = f(x) = \arctan(e^x)$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = \arctan 1 = \frac{\pi}{4}$. $f(x) > 0$ so there are no x-intercepts.

C. No symmetry **D.** $\lim\limits_{x\to-\infty} \arctan(e^x) = 0$ and $\lim\limits_{x\to\infty} \arctan(e^x) = \frac{\pi}{2}$, so $y = 0$ and $y = \frac{\pi}{2}$ are HAs. No VA

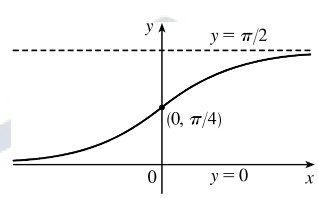
E. $f'(x) = \dfrac{1}{1+(e^x)^2}\dfrac{d}{dx}e^x = \dfrac{e^x}{1+e^{2x}} > 0$, so f is increasing on $(-\infty, \infty)$. **F.** No local extrema

G. $f''(x) = \dfrac{(1+e^{2x})e^x - e^x(2e^{2x})}{(1+e^{2x})^2} = \dfrac{e^x[(1+e^{2x}) - 2e^{2x}]}{(1+e^{2x})^2}$

$= \dfrac{e^x(1-e^{2x})}{(1+e^{2x})^2} > 0 \iff$

$1 - e^{2x} > 0 \iff e^{2x} < 1 \iff 2x < 0 \iff x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{\pi}{4}\right)$

H.



42. $y = f(x) = (1-x)e^x$ **A.** $D = \mathbb{R}$ **B.** x-intercept 1, y-intercept $= f(0) = 1$ **C.** No symmetry

D. $\lim\limits_{x\to-\infty} \dfrac{1-x}{e^{-x}}$ [form $\frac{\infty}{\infty}$] $\overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{-1}{-e^{-x}} = 0$, so $y = 0$ is a HA. No VA

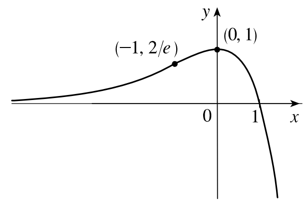
E. $f'(x) = (1-x)e^x + e^x(-1) = e^x[(1-x) + (-1)] = -xe^x > 0 \iff x < 0$, so f is increasing on $(-\infty, 0)$

and decreasing on $(0, \infty)$.

F. Local maximum value $f(0) = 1$, no local minimum value

G. $f''(x) = -xe^x + e^x(-1) = e^x(-x-1) = -(x+1)e^x > 0 \iff$ $x < -1$, so f is CU on $(-\infty, -1)$ and CD on $(-1, \infty)$. IP at $(-1, 2/e)$

H.



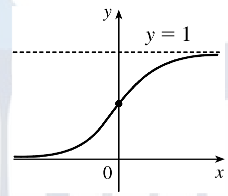
43. $y = 1/(1+e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

D. $\lim\limits_{x\to\infty} 1/(1+e^{-x}) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/(1+e^{-x}) = 0$ since $\lim\limits_{x\to-\infty} e^{-x} = \infty$, so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -(1+e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1+e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values **G.** $f''(x) = \dfrac{(1+e^{-x})^2(-e^{-x}) - e^{-x}(2)(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} = \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$

The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{1}{2}\right)$

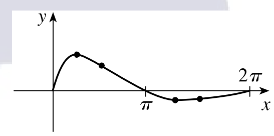
H.



44. $y = f(x) = e^{-x}\sin x,\ 0 \le x \le 2\pi$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \iff \sin x = 0 \iff$ $x = 0, \pi$, and 2π. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^{-x}\cos x + \sin x\,(-e^{-x}) = e^{-x}(\cos x - \sin x)$. $f'(x) = 0 \iff \cos x = \sin x \iff x = \frac{\pi}{4}, \frac{5\pi}{4}$. $f'(x) > 0$ if x is in $\left(0, \frac{\pi}{4}\right)$ or $\left(\frac{5\pi}{4}, 2\pi\right)$ [f is increasing] and $f'(x) < 0$ if x is in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ [f is decreasing]. **F.** Local maximum value $f\left(\frac{\pi}{4}\right)$ and local minimum value $f\left(\frac{5\pi}{4}\right)$

G. $f''(x) = e^{-x}(-\sin x - \cos x) + (\cos x - \sin x)(-e^{-x}) = e^{-x}(-2\cos x)$. $f''(x) > 0 \iff -2\cos x > 0 \iff$ $\cos x < 0 \Rightarrow x$ is in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ [f is CU] and $f''(x) < 0 \iff$ $\cos x > 0 \Rightarrow x$ is in $\left(0, \frac{\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, 2\pi\right)$ [f is CD].
IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right)$

H.

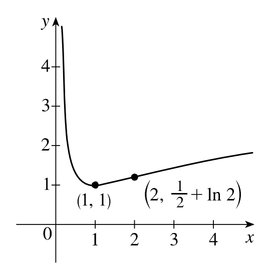


45. $y = f(x) = \dfrac{1}{x} + \ln x$ **A.** $D = (0, \infty)$ [same as $\ln x$] **B.** No y-intercept; no x-intercept $\left[\dfrac{1}{x} \text{ and } \ln x \text{ are both positive on } D\right]$ **C.** No symmetry. **D.** $\lim\limits_{x\to0^+} f(x) = \infty$, so $x = 0$ is a VA.

E. $f'(x) = -\dfrac{1}{x^2} + \dfrac{1}{x} = \dfrac{x-1}{x^2}$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.

F. Local minimum value $f(1) = 1$ **G.** $f''(x) = \dfrac{2}{x^3} - \dfrac{1}{x^2} = \dfrac{2-x}{x^3}$.
$f''(x) > 0$ for $0 < x < 2$, so f is CU on $(0, 2)$, and f is CD on $(2, \infty)$.
IP at $\left(2, \frac{1}{2} + \ln 2\right)$

H.

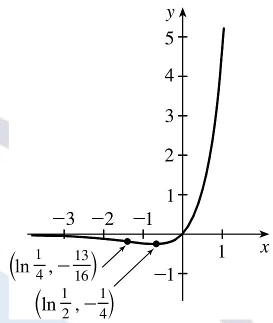


46. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$. **C.** No symmetry **D.** $\lim\limits_{x \to -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA. **E.** $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$.

F. Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{4}$. Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$

H.



47. $y = f(x) = (1 + e^x)^{-2} = \dfrac{1}{(1 + e^x)^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = \frac{1}{4}$. x-intercepts: none [since $f(x) > 0$]

C. No symmetry **D.** $\lim\limits_{x \to \infty} f(x) = 0$ and $\lim\limits_{x \to -\infty} f(x) = 1$, so $y = 0$ and $y = 1$ are HA; no VA

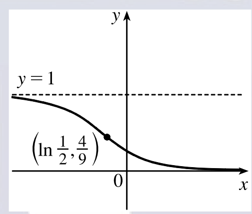
E. $f'(x) = -2(1 + e^x)^{-3}e^x = \dfrac{-2e^x}{(1 + e^x)^3} < 0$, so f is decreasing on $\mathbb{R}$ **F.** No local extrema

G. $f''(x) = (1 + e^x)^{-3}(-2e^x) + (-2e^x)(-3)(1 + e^x)^{-4}e^x$

$$= -2e^x(1 + e^x)^{-4}[(1 + e^x) - 3e^x] = \frac{-2e^x(1 - 2e^x)}{(1 + e^x)^4}.$$

$f''(x) > 0 \Leftrightarrow 1 - 2e^x < 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln\frac{1}{2}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{2}$, so f is CU on $\left(\ln\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \ln\frac{1}{2}\right)$. IP at $\left(\ln\frac{1}{2}, \frac{4}{9}\right)$

H.



48. $y = f(x) = e^x/x^2$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept **C.** No symmetry **D.** $\lim\limits_{x \to -\infty} \dfrac{e^x}{x^2} = 0$, so $y = 0$ is HA. $\lim\limits_{x \to 0} \dfrac{e^x}{x^2} = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{x^2e^x - e^x(2x)}{(x^2)^2} = \dfrac{xe^x(x - 2)}{x^4} = \dfrac{e^x(x - 2)}{x^3} > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$.

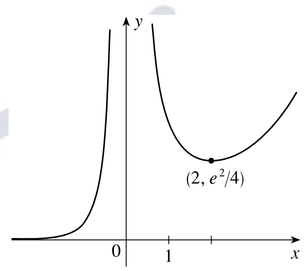
F. Local minimum value $f(2) = e^2/4 \approx 1.85$, no local maximum value

G. $f''(x) = \dfrac{x^3[e^x(1) + (x - 2)e^x] - e^x(x - 2)(3x^2)}{(x^3)^2}$

$$= \frac{x^2e^x[x(x - 1) - 3(x - 2)]}{x^6} = \frac{e^x(x^2 - 4x + 6)}{x^4} > 0$$

for all x in the domain of f; that is, f is CU on $(-\infty, 0)$ and $(0, \infty)$. No IP

H.



49. $y = f(x) = \ln(\sin x)$

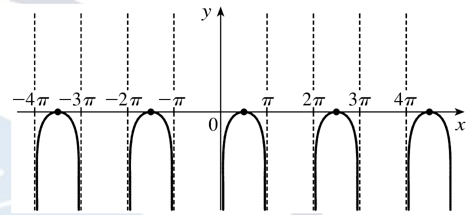
A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup\limits_{n=-\infty}^{\infty} (2n\pi, (2n + 1)\pi) = \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x)=0 \quad\Leftrightarrow\quad \ln(\sin x)=0 \quad\Leftrightarrow\quad \sin x = e^0 = 1 \quad\Leftrightarrow\quad x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim\limits_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim\limits_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n.

F. Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum.

G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.

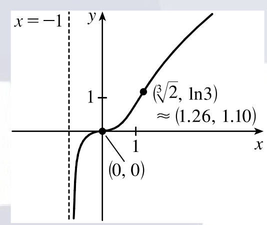


50. $y = f(x) = \ln(1+x^3)$ **A.** $1 + x^3 > 0 \quad\Leftrightarrow\quad x^3 > -1 \quad\Leftrightarrow\quad x > -1$, so $D = (-1, \infty)$. **B.** y-intercept: $f(0) = \ln 1 = 0$; x-intercept: $f(x) = 0 \quad\Leftrightarrow\quad \ln(1+x^3) = 0 \quad\Leftrightarrow\quad 1 + x^3 = e^0 \quad\Leftrightarrow\quad x^3 = 0 \quad\Leftrightarrow\quad x = 0$ **C.** No symmetry. **D.** $\lim\limits_{x\to -1^+} f(x) = -\infty$, so $x = -1$ is a VA **E.** $f'(x) = \dfrac{3x^2}{1+x^3}$. $f'(x) > 0$ on $(-1, 0)$ and $(0, \infty)$ $[f'(x) = 0$ at $x = 0]$, so by Exercise 4.3.91, f is increasing on $(-1, \infty)$. **F.** No extreme values

G. $$f''(x) = \frac{(1+x^3)(6x) - 3x^2(3x^2)}{(1+x^3)^2} = \frac{3x[2(1+x^3) - 3x^3]}{(1+x^3)^2} = \frac{3x(2-x^3)}{(1+x^3)^2}$$

$f''(x) > 0 \quad\Leftrightarrow\quad 0 < x < \sqrt[3]{2}$, so f is CU on $\left(0, \sqrt[3]{2}\right)$ and f is CD on $(-1, 0)$ and $\left(\sqrt[3]{2}, \infty\right)$. IP at $(0, 0)$ and $\left(\sqrt[3]{2}, \ln 3\right)$

H.



51. $y = f(x) = xe^{-1/x}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept **C.** No symmetry

D. $\lim\limits_{x\to 0^-} \dfrac{e^{-1/x}}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to 0^-} \dfrac{e^{-1/x}(1/x^2)}{-1/x^2} = -\lim\limits_{x\to 0^-} e^{-1/x} = -\infty$, so $x = 0$ is a VA. Also, $\lim\limits_{x\to 0^+} xe^{-1/x} = 0$, so the graph approaches the origin as $x \to 0^+$. **E.** $f'(x) = xe^{-1/x}\left(\dfrac{1}{x^2}\right) + e^{-1/x}(1) = e^{-1/x}\left(\dfrac{1}{x} + 1\right) = \dfrac{x+1}{xe^{1/x}} > 0 \quad\Leftrightarrow$ $x < -1$ or $x > 0$, so f is increasing on $(-\infty, -1)$ and $(0, \infty)$, and f is decreasing on $(-1, 0)$.

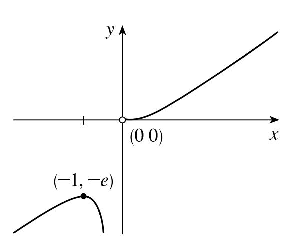
F. Local maximum value $f(-1) = -e$, no local minimum value

G. $f'(x) = e^{-1/x}\left(\dfrac{1}{x} + 1\right) \quad\Rightarrow$

$$f''(x) = e^{-1/x}\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + 1\right)e^{-1/x}\left(\frac{1}{x^2}\right) = \frac{1}{x^2}e^{-1/x}\left[-1 + \left(\frac{1}{x} + 1\right)\right] = \frac{1}{x^3 e^{1/x}} > 0 \quad\Leftrightarrow$$

$x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



52. $y = f(x) = \dfrac{\ln x}{x^2}$ **A.** $D = (0, \infty)$ **B.** y-intercept: none; x-intercept: $f(x) = 0 \iff \ln x = 0 \iff x = 1$

C. No symmetry **D.** $\lim\limits_{x \to 0^+} f(x) = -\infty$, so $x = 0$ is a VA; $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{2x} = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3}$. $f'(x) > 0 \iff 1 - 2\ln x > 0 \iff \ln x < \frac{1}{2} \Rightarrow$ $0 < x < e^{1/2}$ and $f'(x) < 0 \Rightarrow x > e^{1/2}$, so f is increasing on $\left(0, \sqrt{e}\right)$ and decreasing on $\left(\sqrt{e}, \infty\right)$.

F. Local maximum value $f(e^{1/2}) = \dfrac{1/2}{e} = \dfrac{1}{2e}$

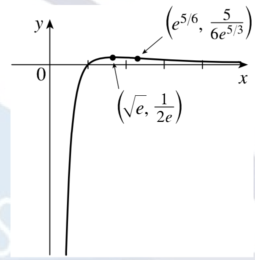
G. $f''(x) = \dfrac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2}$

$= \dfrac{x^2[-2 - 3(1 - 2\ln x)]}{x^6} = \dfrac{-5 + 6\ln x}{x^4}$

$f''(x) > 0 \iff -5 + 6\ln x > 0 \iff \ln x > \frac{5}{6} \Rightarrow x > e^{5/6}$ [f is CU]

and $f''(x) < 0 \iff 0 < x < e^{5/6}$ [f is CD]. IP at $(e^{5/6}, 5/(6e^{5/3}))$

H.



53. $y = f(x) = e^{\arctan x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = e^0 = 1$; no x-intercept since $e^{\arctan x}$ is positive for all x.

C. No symmetry **D.** $\lim\limits_{x \to -\infty} f(x) = e^{-\pi/2}$ [≈ 0.21], so $y = e^{-\pi/2}$ is a HA. $\lim\limits_{x \to \infty} f(x) = e^{\pi/2}$ [≈ 4.81], so $y = e^{\pi/2}$ is a HA. **E.** $f'(x) = e^{\arctan x}\left(\dfrac{1}{1 + x^2}\right)$. $f'(x) > 0$ for all x, so f is increasing on $\mathbb{R}$. **F.** No local extrema

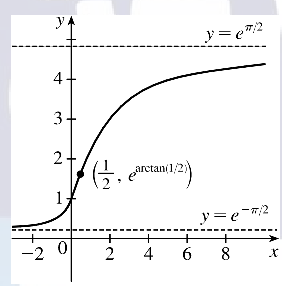
G. $f''(x) = \dfrac{(1 + x^2)e^{\arctan x}\left(\dfrac{1}{1 + x^2}\right) - e^{\arctan x}(2x)}{(1 + x^2)^2}$

$= \dfrac{e^{\arctan x}(1 - 2x)}{(1 + x^2)^2}$

$f''(x) > 0$ for $x < \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$.

IP at $\left(\frac{1}{2}, e^{\arctan 1/2}\right) \approx (0.5, 1.59)$

H.



54. $y = f(x) = \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right)$ **A.** $D = \{x \mid x \neq -1\}$ **B.** x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$

C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right) = \lim\limits_{x \to \pm\infty} \tan^{-1}\left(\dfrac{1 - 1/x}{1 + 1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA.

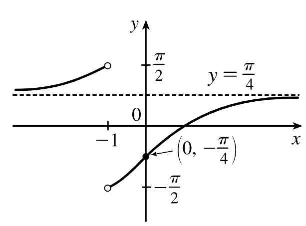
Also $\lim\limits_{x \to -1^+} \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right) = -\dfrac{\pi}{2}$ and $\lim\limits_{x \to -1^-} \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right) = \dfrac{\pi}{2}$.

E. $f'(x) = \dfrac{1}{1 + [(x - 1)/(x + 1)]^2} \dfrac{(x + 1) - (x - 1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2 + (x - 1)^2} = \dfrac{1}{x^2 + 1} > 0$,

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

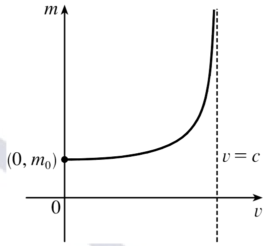
G. $f''(x) = -2x/\left(x^2 + 1\right)^2 > 0 \iff x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

H.



55. $m = f(v) = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. The m-intercept is $f(0) = m_0$. There are no v-intercepts. $\lim\limits_{v \to c^-} f(v) = \infty$, so $v = c$ is a VA.

$f'(v) = -\frac{1}{2} m_0 (1 - v^2/c^2)^{-3/2}(-2v/c^2) = \dfrac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}} = \dfrac{m_0 v}{\dfrac{c^2(c^2 - v^2)^{3/2}}{c^3}} = \dfrac{m_0 cv}{(c^2 - v^2)^{3/2}} > 0$, so f is increasing on $(0, c)$. There are no local extreme values.



$$f''(v) = \frac{(c^2 - v^2)^{3/2}(m_0 c) - m_0 cv \cdot \frac{3}{2}(c^2 - v^2)^{1/2}(-2v)}{[(c^2 - v^2)^{3/2}]^2}$$

$$= \frac{m_0 c(c^2 - v^2)^{1/2}[(c^2 - v^2) + 3v^2]}{(c^2 - v^2)^3} = \frac{m_0 c(c^2 + 2v^2)}{(c^2 - v^2)^{5/2}} > 0,$$

so f is CU on $(0, c)$. There are no inflection points.

56. Let $a = m_0^2 c^4$ and $b = h^2 c^2$, so the equation can be written as $E = f(\lambda) = \sqrt{a + b/\lambda^2} = \sqrt{\dfrac{a\lambda^2 + b}{\lambda^2}} = \dfrac{\sqrt{a\lambda^2 + b}}{\lambda}$.

$\lim\limits_{\lambda \to 0^+} \dfrac{\sqrt{a\lambda^2 + b}}{\lambda} = \infty$, so $\lambda = 0$ is a VA.

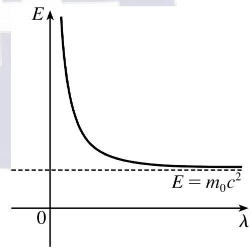
$\lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a\lambda^2 + b}}{\lambda} = \lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a\lambda^2 + b}/\lambda}{\lambda/\lambda} = \lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a + b/\lambda^2}}{1} = \sqrt{a}$, so $E = \sqrt{a} = m_0 c^2$ is a HA.

$$f'(\lambda) = \frac{\lambda \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) - (a\lambda^2 + b)^{1/2}(1)}{\lambda^2} = \frac{(a\lambda^2 + b)^{-1/2}[a\lambda^2 - (a\lambda^2 + b)]}{\lambda^2} = \frac{-b}{\lambda^2\sqrt{a\lambda^2 + b}} < 0,$$

so f is decreasing on $(0, \infty)$. Using the Reciprocal Rule,

$$f''(\lambda) = b \cdot \frac{\lambda^2 \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) + (a\lambda^2 + b)^{1/2}(2\lambda)}{\left(\lambda^2\sqrt{a\lambda^2 + b}\right)^2}$$

$$= \frac{b\lambda(a\lambda^2 + b)^{-1/2}[a\lambda^2 + 2(a\lambda^2 + b)]}{\left(\lambda^2\sqrt{a\lambda^2 + b}\right)^2} = \frac{b(3a\lambda^2 + 2b)}{\lambda^3(a\lambda^2 + b)^{3/2}} > 0,$$



so f is CU on $(0, \infty)$. There are no extrema or inflection points. The graph shows that as λ decreases, the energy increases and as λ increases, the energy decreases. For large wavelengths, the energy is very close to the energy at rest.

57. (a) $p(t) = \dfrac{1}{2} \;\Rightarrow\; \dfrac{1}{2} = \dfrac{1}{1 + ae^{-kt}} \;\Leftrightarrow\; 1 + ae^{-kt} = 2 \;\Leftrightarrow\; ae^{-kt} = 1 \;\Leftrightarrow\; e^{-kt} = \dfrac{1}{a} \;\Leftrightarrow$

$\ln e^{-kt} = \ln a^{-1} \;\Leftrightarrow\; -kt = -\ln a \;\Leftrightarrow\; t = \dfrac{\ln a}{k}$, which is when half the population will have heard the rumor.

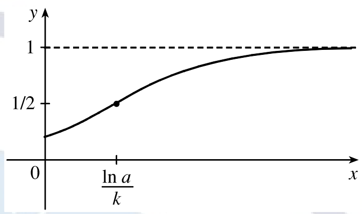
(b) The rate of spread is given by $p'(t) = \dfrac{ake^{-kt}}{(1 + ae^{-kt})^2}$. To find the greatest rate of spread, we'll apply the First Derivative Test to $p'(t)$ [not $p(t)$].

$$[p'(t)]' = p''(t) = \frac{(1 + ae^{-kt})^2(-ak^2e^{-kt}) - ake^{-kt} \cdot 2(1 + ae^{-kt})(-ake^{-kt})}{[(1 + ae^{-kt})^2]^2}$$

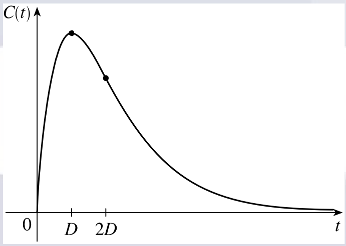
$$= \frac{(1 + ae^{-kt})(-ake^{-kt})[k(1 + ae^{-kt}) - 2ake^{-kt}]}{(1 + ae^{-kt})^4} = \frac{-ake^{-kt}(k)(1 - ae^{-kt})}{(1 + ae^{-kt})^3} = \frac{ak^2e^{-kt}(ae^{-kt} - 1)}{(1 + ae^{-kt})^3}$$

$p''(t) > 0 \quad\Leftrightarrow\quad ae^{-kt} > 1 \quad\Leftrightarrow\quad -kt > \ln a^{-1} \quad\Leftrightarrow\quad t < \dfrac{\ln a}{k}$, so $p'(t)$ is increasing for $t < \dfrac{\ln a}{k}$ and $p'(t)$ is decreasing for $t > \dfrac{\ln a}{k}$. Thus, $p'(t)$, the rate of spread of the rumor, is greatest at the same time, $\dfrac{\ln a}{k}$, as when half the population [by part (a)] has heard it.

(c) $p(0) = \dfrac{1}{1+a}$ and $\lim\limits_{t\to\infty} p(t) = 1$. The graph is shown with $a = 4$ and $k = \frac{1}{2}$.

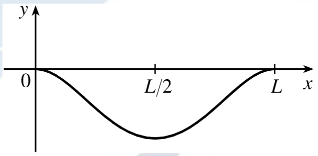


58. $C(t) = K(e^{-at} - e^{-bt})$, where $K > 0$ and $b > a > 0$. $C(0) = K(1-1) = 0$ is the only intercept. $\lim\limits_{t\to\infty} C(t) = 0$, so $C = 0$ is a HA. $C'(t) = K(-ae^{-at} + be^{-bt}) > 0 \quad\Leftrightarrow\quad be^{-bt} > ae^{-at} \quad\Leftrightarrow\quad e^{at}e^{-bt} > \dfrac{a}{b} \quad\Leftrightarrow\quad e^{(a-b)t} > \dfrac{a}{b} \quad\Leftrightarrow$ $(a-b)t > \ln\dfrac{a}{b} \quad\Leftrightarrow\quad t > \dfrac{\ln(a/b)}{a-b}$ or $\dfrac{\ln(b/a)}{b-a}$ [call this value D]. C is increasing for $t < D$ and decreasing for $t > D$, so $C(D)$ is a local maximum [and absolute] value. $C''(t) = K(a^2e^{-at} - b^2e^{-bt}) > 0 \quad\Leftrightarrow\quad a^2e^{-at} > b^2e^{-bt} \quad\Leftrightarrow$ $e^{bt}e^{-at} > \dfrac{b^2}{a^2} \quad\Leftrightarrow\quad e^{(b-a)t} > \left(\dfrac{b}{a}\right)^2 \quad\Leftrightarrow\quad (b-a)t > \ln\left(\dfrac{b}{a}\right)^2 \quad\Leftrightarrow\quad t > \dfrac{2\ln(b/a)}{b-a} = 2D$, so C is CU on $(2D, \infty)$ and CD on $(0, 2D)$. The inflection point is $(2D, C(2D))$. For the graph shown, $K = 1$, $a = 1$, $b = 2$, $D = \ln 2$, $C(D) = \frac{1}{4}$, and $C(2D) = \frac{3}{16}$. The graph tells us that when the drug is injected into the bloodstream, its concentration rises rapidly to a maximum at time D, then falls, reaching its maximum rate of decrease at time $2D$, then continues to decrease more and more slowly, approaching 0 as $t \to \infty$.



59. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2(x^2 - 2Lx + L^2)$

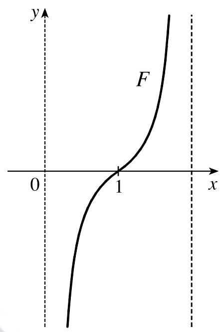
$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$



where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch $f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \quad\Leftrightarrow\quad x(x-L)(2x-L) < 0$ [since $c < 0$] $\quad\Leftrightarrow\quad L/2 < x < L$ and $f'(x) < 0 \quad\Leftrightarrow\quad 0 < x < L/2$. Thus, f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at the point $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$. $f'(x) = 2c[x(x-L)(2x-L)] \quad\Rightarrow$ $f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \quad\Leftrightarrow$ $x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

60. $F(x) = -\dfrac{k}{x^2} + \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$, $x - 2 < 0$, so $F'(x) = \dfrac{2k}{x^3} - \dfrac{2k}{(x-2)^3} > 0$ and F is increasing. $\lim\limits_{x \to 0^+} F(x) = -\infty$ and $\lim\limits_{x \to 2^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



61. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r|l} & x - 1 \\ \hline x+1 & x^2 \qquad + 1 \\ & \underline{x^2 + x} \\ & \quad -x + 1 \\ & \quad \underline{-x - 1} \\ & \qquad\quad 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1 + \frac{1}{x}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$.

So the line $y = x - 1$ is a slant asymptote (SA).

62. $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$. Long division gives us:

$$\begin{array}{r|l} & \qquad\qquad\quad 4x + 2 \\ \hline x^2 - 3x & 4x^3 - 10x^2 - 11x + 1 \\ & \underline{4x^3 - 12x^2} \\ & \qquad\quad 2x^2 - 11x \\ & \qquad\quad \underline{2x^2 - \;\, 6x} \\ & \qquad\qquad\quad -5x + 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x} = 4x + 2 + \dfrac{-5x + 1}{x^2 - 3x}$ and $f(x) - (4x + 2) = \dfrac{-5x + 1}{x^2 - 3x} = \dfrac{-\frac{5}{x} + \frac{1}{x^2}}{1 - \frac{3}{x}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 4x + 2$ is a slant asymptote (SA).

63. $y = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}$. Long division gives us:

$$\begin{array}{r|l} & \qquad\qquad 2x - 3 \\ \hline x^2 - x - 2 & 2x^3 - 5x^2 + 3x \\ & \underline{2x^3 - 2x^2 - 4x} \\ & \quad -3x^2 + 7x \\ & \quad \underline{-3x^2 + 3x + 6} \\ & \qquad\quad 4x - 6 \end{array}$$

Thus, $y = f(x) = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2} = 2x - 3 + \dfrac{4x - 6}{x^2 - x - 2}$ and $f(x) - (2x - 3) = \dfrac{4x - 6}{x^2 - x - 2} = \dfrac{\frac{4}{x} - \frac{6}{x^2}}{1 - \frac{1}{x} - \frac{1}{x^2}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 2x - 3$ is a slant asymptote (SA).

64. $y = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x}$. Long division gives us:

$$\begin{array}{r|l} & -3x + 1 \\ \hline 2x^3 - x & -6x^4 + 2x^3 \qquad\qquad + 3 \\ & -6x^4 \qquad + 3x^2 \\ \hline & \qquad 2x^3 - 3x^2 \\ & \qquad 2x^3 \qquad - x \\ \hline & \qquad\quad -3x^2 + x + 3 \end{array}$$

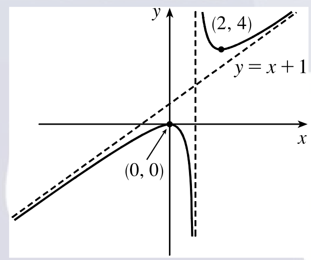
Thus, $y = f(x) = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x} = -3x + 1 + \dfrac{-3x^2 + x + 3}{2x^3 - x}$ and

$f(x) - (-3x + 1) = \dfrac{-3x^2 + x + 3}{2x^3 - x} = \dfrac{-\dfrac{3}{x} + \dfrac{1}{x^2} + \dfrac{3}{x^3}}{2 - \dfrac{1}{x^2}}$ [for $x \neq 0$] $\to \frac{0}{2} = 0$ as $x \to \pm\infty$. So the line $y = -3x + 1$ is a slant asymptote (SA).

65. $y = f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept: $f(x) = 0 \Leftrightarrow x = 0$; y-intercept: $f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (x+1)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{x-1} = 0$, so the line $y = x + 1$ is a SA.

E. $f'(x) = 1 - \dfrac{1}{(x-1)^2} = \dfrac{(x-1)^2 - 1}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 1)$ and $(1, 2)$. **F.** Local maximum value $f(0) = 0$, local minimum value $f(2) = 4$ **G.** $f''(x) = \dfrac{2}{(x-1)^3} > 0$ for $x > 1$, so f is CU on $(1, \infty)$ and f is CD on $(-\infty, 1)$. No IP

H.



66. $y = f(x) = \dfrac{1 + 5x - 2x^2}{x - 2} = -2x + 1 + \dfrac{3}{x-2}$ **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercepts: $f(x) = 0 \Leftrightarrow 1 + 5x - 2x^2 = 0 \Rightarrow x = \dfrac{-5 \pm \sqrt{33}}{-4} \Rightarrow x \approx -0.19, 2.69$; y-intercept: $f(0) = -\frac{1}{2}$ **C.** No symmetry

D. $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$, so $x = 2$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (-2x + 1)] = \lim\limits_{x \to \pm\infty} \dfrac{3}{x-2} = 0$, so $y = -2x + 1$ is a SA.

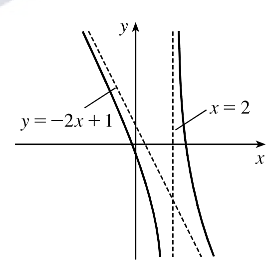
E. $f'(x) = -2 - \dfrac{3}{(x-2)^2} = \dfrac{-2(x^2 - 4x + 4) - 3}{(x-2)^2}$
$= \dfrac{-2x^2 + 8x - 11}{(x-2)^2} < 0$

for $x \neq 2$, so f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. **F.** No local extrema

G. $f''(x) = \dfrac{6}{(x-2)^3} > 0$ for $x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$.

No IP

H.

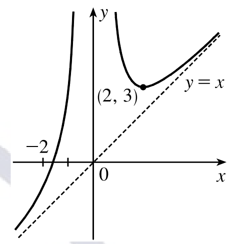


67. $y = f(x) = \dfrac{x^3+4}{x^2} = x + \dfrac{4}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** x-intercept: $f(x) = 0 \Leftrightarrow x = -\sqrt[3]{4}$; no y-intercept

C. No symmetry **D.** $\lim\limits_{x \to 0} f(x) = \infty$, so $x = 0$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - x] = \lim\limits_{x \to \pm\infty} \dfrac{4}{x^2} = 0$, so $y = x$ is a SA.

E. $f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3-8}{x^3} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$. **F.** Local minimum value $f(2) = 3$, no local maximum value **G.** $f''(x) = \dfrac{24}{x^4} > 0$ for $x \neq 0$, so f is CU on $(-\infty, 0)$ and $(0, \infty)$. No IP

H.



68. $y = f(x) = \dfrac{x^3}{(x+1)^2} = x - 2 + \dfrac{3x+2}{(x+1)^2}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept: 0; y-intercept: $f(0) = 0$

C. No symmetry **D.** $\lim\limits_{x \to -1^-} f(x) = -\infty$ and $\lim\limits_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (x-2)] = \lim\limits_{x \to \pm\infty} \dfrac{3x+2}{(x+1)^2} = 0$, so $y = x - 2$ is a SA.

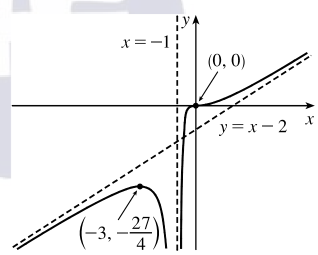
E. $f'(x) = \dfrac{(x+1)^2(3x^2) - x^3 \cdot 2(x+1)}{[(x+1)^2]^2} = \dfrac{x^2(x+1)[3(x+1) - 2x]}{(x+1)^4} = \dfrac{x^2(x+3)}{(x+1)^3} > 0 \Leftrightarrow x < -3$ or $x > -1$ $[x \neq 0]$, so f is increasing on $(-\infty, -3)$ and $(-1, \infty)$, and f is decreasing on $(-3, -1)$.

F. Local maximum value $f(-3) = -\frac{27}{4}$, no local minimum

G.
$$\begin{aligned} f''(x) &= \frac{(x+1)^3(3x^2+6x) - (x^3+3x^2)\cdot 3(x+1)^2}{[(x+1)^3]^2} \\ &= \frac{3x(x+1)^2[(x+1)(x+2) - (x^2+3x)]}{(x+1)^6} \\ &= \frac{3x(x^2+3x+2-x^2-3x)}{(x+1)^4} = \frac{6x}{(x+1)^4} > 0 \quad \Leftrightarrow \end{aligned}$$

$x > 0$, so f is CU on $(0, \infty)$ and f is CD on $(-\infty, -1)$ and $(-1, 0)$. IP at $(0, 0)$

H.



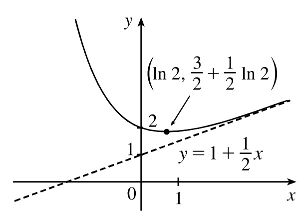
69. $y = f(x) = 1 + \frac{1}{2}x + e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$, no x-intercept [see part **F**] **C.** No symmetry

D. No VA or HA. $\lim\limits_{x \to \infty} \left[f(x) - \left(1 + \frac{1}{2}x\right)\right] = \lim\limits_{x \to \infty} e^{-x} = 0$, so $y = 1 + \frac{1}{2}x$ is a SA. **E.** $f'(x) = \frac{1}{2} - e^{-x} > 0 \Leftrightarrow$ $\frac{1}{2} > e^{-x} \Leftrightarrow -x < \ln\frac{1}{2} \Leftrightarrow x > -\ln 2^{-1} \Leftrightarrow x > \ln 2$, so f is increasing on $(\ln 2, \infty)$ and decreasing on $(-\infty, \ln 2)$. **F.** Local and absolute minimum value

$$\begin{aligned} f(\ln 2) &= 1 + \tfrac{1}{2}\ln 2 + e^{-\ln 2} = 1 + \tfrac{1}{2}\ln 2 + (e^{\ln 2})^{-1} \\ &= 1 + \tfrac{1}{2}\ln 2 + \tfrac{1}{2} = \tfrac{3}{2} + \tfrac{1}{2}\ln 2 \approx 1.85, \end{aligned}$$

no local maximum value **G.** $f''(x) = e^{-x} > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

H.



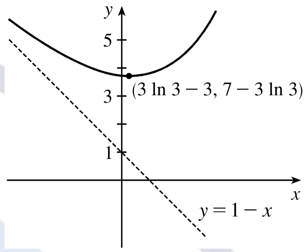
70. $y = f(x) = 1 - x + e^{1+x/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 1 + e$, no x-intercept [see part **F**]

C. No symmetry **D.** No VA or HA $\lim_{x\to-\infty}[f(x) - (1-x)] = \lim_{x\to-\infty} e^{1+x/3} = 0$, so $y = 1 - x$ is a SA.

E. $f'(x) = -1 + \frac{1}{3}e^{1+x/3} > 0 \;\Leftrightarrow\; \frac{1}{3}e^{1+x/3} > 1 \;\Leftrightarrow\; e^{1+x/3} > 3 \;\Leftrightarrow\; 1 + \dfrac{x}{3} > \ln 3 \;\Leftrightarrow\; \dfrac{x}{3} > \ln 3 - 1 \;\Leftrightarrow$

$x > 3(\ln 3 - 1) \approx 0.3$, so f isincreasing on $(3\ln 3 - 3, \infty)$ and decreasing on $(-\infty, 3\ln 3 - 3)$. **F.** Local and absolute minimum value $f(3\ln 3 - 3) = 1 - (3\ln 3 - 3) + e^{1+\ln 3 - 1} = 4 - 3\ln 3 + 3 = 7 - 3\ln 3 \approx 3.7$, no local maximum value **G.** $f''(x) = \frac{1}{9}e^{1+x/3} > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

H.

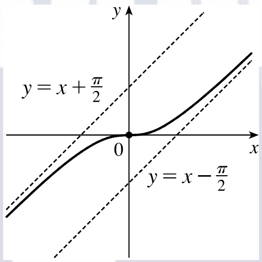


71. $y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

$f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x(1+x^2-x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

$\lim_{x\to\infty}\left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA.

Also, $\lim_{x\to-\infty}\left[f(x) - \left(x + \frac{\pi}{2}\right)\right] = \lim_{x\to-\infty}\left(-\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$, so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.



72. $y = f(x) = \sqrt{x^2 + 4x} = \sqrt{x(x+4)}$. $x(x+4) \geq 0 \;\Leftrightarrow\; x \leq -4$ or $x \geq 0$, so $D = (-\infty, -4] \cup [0, \infty)$.

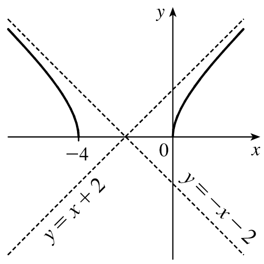
y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = -4, 0$.

$$\sqrt{x^2+4x} \mp (x+2) = \frac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \frac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \frac{(x^2+4x) - (x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)}$$
$$= \frac{-4}{\sqrt{x^2+4x} \pm (x+2)}$$

so $\lim_{x\to\pm\infty}[f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x + 2$ as $x \to \infty$ and it approaches the slant asymptote $y = -(x+2)$ as $x \to -\infty$. $f'(x) = \dfrac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values. $f'(x) = (x+2)(x^2+4x)^{-1/2} \;\Rightarrow$

$$\begin{aligned} f''(x) &= (x+2)\cdot\left(-\tfrac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4) + (x^2+4x)^{-1/2} \\ &= (x^2+4x)^{-3/2}\left[-(x+2)^2 + (x^2+4x)\right] = -4(x^2+4x)^{-3/2} < 0 \text{ on } D \end{aligned}$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP



73. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

$$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2} - \left(-\frac{b}{a}x\right)\right] = -\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

74. $f(x) - x^2 = \dfrac{x^3+1}{x} - x^2 = \dfrac{x^3+1-x^3}{x} = \dfrac{1}{x}$, and $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x} = 0$. Therefore, $\displaystyle\lim_{x\to\pm\infty}\left[f(x)-x^2\right] = 0$,

and so the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

A. $D = \{x \mid x \neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y = 0 \quad \Leftrightarrow \quad x = -1$.

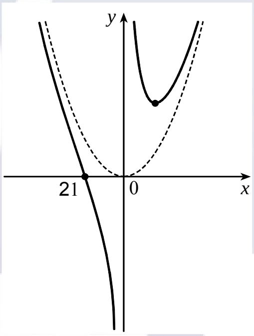
C. No symmetry **D.** $\displaystyle\lim_{x\to 0^+}\frac{x^3+1}{x} = \infty$ and $\displaystyle\lim_{x\to 0^-}\frac{x^3+1}{x} = -\infty$,

so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y = x^2$, as shown above. **E.** $f'(x) = 2x - 1/x^2 > 0 \quad \Leftrightarrow \quad x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\dfrac{1}{\sqrt[3]{2}}, \infty\right)$ and decreasing on $(-\infty, 0)$ and $\left(0, \dfrac{1}{\sqrt[3]{2}}\right)$.

F. Local minimum value $f\left(\dfrac{1}{\sqrt[3]{2}}\right) = \dfrac{3\sqrt[3]{3}}{2}$, no local maximum

G. $f''(x) = 2 + 2/x^3 > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP at $(-1, 0)$

H.



75. $\displaystyle\lim_{x\to\pm\infty}\left[f(x) - x^3\right] = \lim_{x\to\pm\infty}\frac{x^4+1}{x} - \frac{x^4}{x} = \lim_{x\to\pm\infty}\frac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

A. $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\displaystyle\lim_{x\to 0^-}\left(x^3 + \frac{1}{x}\right) = -\infty$ and

$\displaystyle\lim_{x\to 0^+}\left(x^3 + \frac{1}{x}\right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y = x^3$.

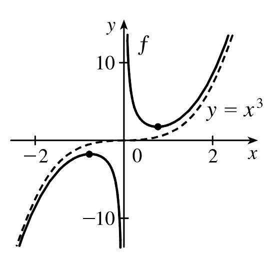
E. $f'(x) = 3x^2 - 1/x^2 > 0 \quad \Leftrightarrow \quad x^4 > \frac{1}{3} \quad \Leftrightarrow \quad |x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty, -\dfrac{1}{\sqrt[4]{3}}\right)$ and $\left(\dfrac{1}{\sqrt[4]{3}}, \infty\right)$ and

decreasing on $\left(-\dfrac{1}{\sqrt[4]{3}}, 0\right)$ and $\left(0, \dfrac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value

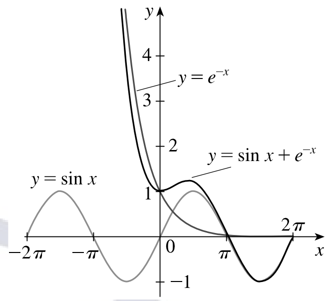
$f\left(-\dfrac{1}{\sqrt[4]{3}}\right) = -4\cdot 3^{-5/4}$, local minimum value $f\left(\dfrac{1}{\sqrt[4]{3}}\right) = 4\cdot 3^{-5/4}$

G. $f''(x) = 6x + 2/x^3 > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.

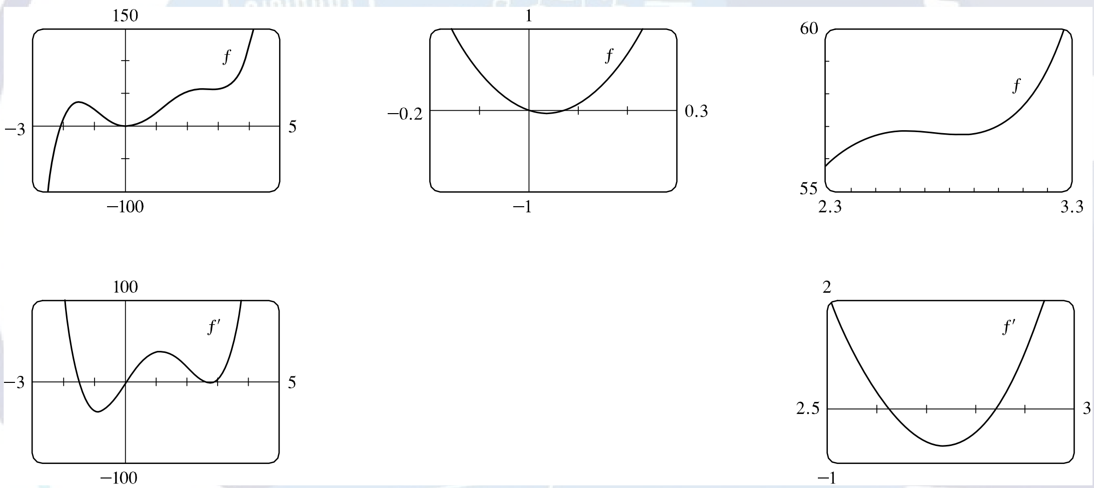


76. $f(x) = \sin x + e^{-x}$. $\lim\limits_{x\to\infty} [f(x) - \sin x] = \lim\limits_{x\to\infty} e^{-x} = 0$, so the graph of f is asymptotic to the graph of $\sin x$ as $x \to \infty$. $\lim\limits_{x\to -\infty} e^{-x} = \infty$, whereas $|\sin x| \le 1$, so for large negative x, the graph of f looks like the graph of e^{-x}.



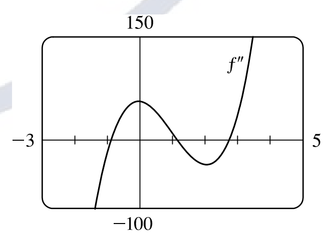
4.6 Graphing with Calculus and Calculators

1. $f(x) = x^5 - 5x^4 - x^3 + 28x^2 - 2x \;\Rightarrow\; f'(x) = 5x^4 - 20x^3 - 3x^2 + 56x - 2 \;\Rightarrow\; f''(x) = 20x^3 - 60x^2 - 6x + 56$. $f(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx -2.09, 0.07$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -1.50, 0.04, 2.62, 2.84$; $f''(x) = 0 \;\Leftrightarrow\; x \approx -0.89, 1.15, 2.74$.



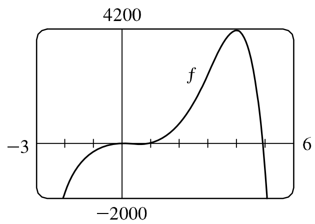
From the graphs of f', we estimate that $f' < 0$ and that f is decreasing on $(-1.50, 0.04)$ and $(2.62, 2.84)$, and that $f' > 0$ and f is increasing on $(-\infty, -1.50)$, $(0.04, 2.62)$, and $(2.84, \infty)$ with local minimum values $f(0.04) \approx -0.04$ and $f(2.84) \approx 56.73$ and local maximum values $f(-1.50) \approx 36.47$ and $f(2.62) \approx 56.83$.

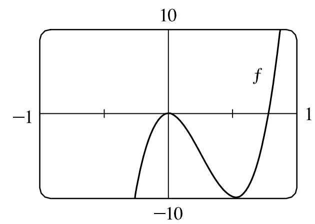
From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-0.89, 1.15)$ and $(2.74, \infty)$, and that $f'' < 0$ and f is CD on $(-\infty, -0.89)$ and $(1.15, 2.74)$. There are inflection points at about $(-0.89, 20.90)$, $(1.15, 26.57)$, and $(2.74, 56.78)$.

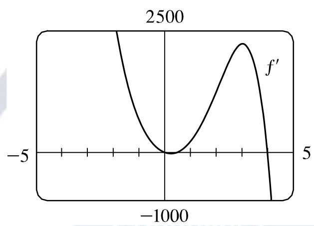


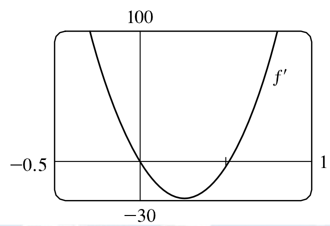
2. $f(x) = -2x^6 + 5x^5 + 140x^3 - 110x^2 \;\Rightarrow\; f'(x) = -12x^5 + 25x^4 + 420x^2 - 220x \;\Rightarrow$ $f''(x) = -60x^4 + 100x^3 + 840x - 220$. $f(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx 0.77, 4.93$; $f'(x) = 0 \;\Leftrightarrow\; x = 0$ or

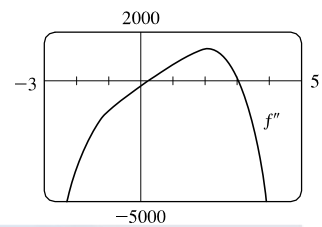
$x \approx 0.52, 3.99$; $f''(x) = 0 \quad \Leftrightarrow \quad x \approx 0.26, 3.05$.











From the graphs of f', we estimate that $f' > 0$ and that f is increasing on $(-\infty, 0)$ and $(0.52, 3.99)$, and that $f' < 0$ and that f is decreasing on $(0, 0.52)$ and $(3.99, \infty)$. f has local maximum values $f(0) = 0$ and $f(3.99) \approx 4128.20$, and f has local minimum value $f(0.52) \approx -9.91$. From the graph of f'', we estimate that $f'' > 0$ and f is CU on $(0.26, 3.05)$, and that $f'' < 0$ and f is CD on $(-\infty, 0.26)$ and $(3.05, \infty)$. There are inflection points at about $(0.26, -4.97)$ and $(3.05, 2649.46)$.

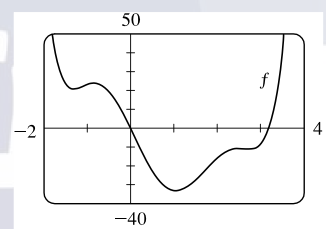
3. $f(x) = x^6 - 5x^5 + 25x^3 - 6x^2 - 48x \quad \Rightarrow$

$f'(x) = 6x^5 - 25x^4 + 75x^2 - 12x - 48 \quad \Rightarrow$

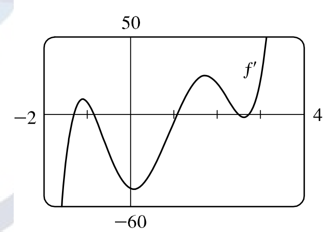
$f''(x) = 30x^4 - 100x^3 + 150x - 12$. $\quad f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx 3.20$;

$f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.31, -0.84, 1.06, 2.50, 2.75$; $f''(x) = 0 \quad \Leftrightarrow$

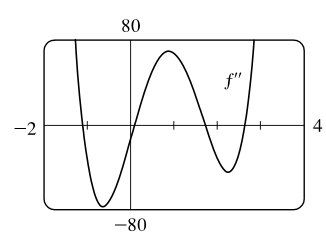
$x \approx -1.10, 0.08, 1.72, 2.64$.



From the graph of f', we estimate that f is decreasing on $(-\infty, -1.31)$, increasing on $(-1.31, -0.84)$, decreasing on $(-0.84, 1.06)$, increasing on $(1.06, 2.50)$, decreasing on $(2.50, 2.75)$, and increasing on $(2.75, \infty)$. f has local minimum values $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, and $f(2.75) \approx -11.33$. f has local maximum values $f(-0.84) \approx 23.71$ and $f(2.50) \approx -11.02$.



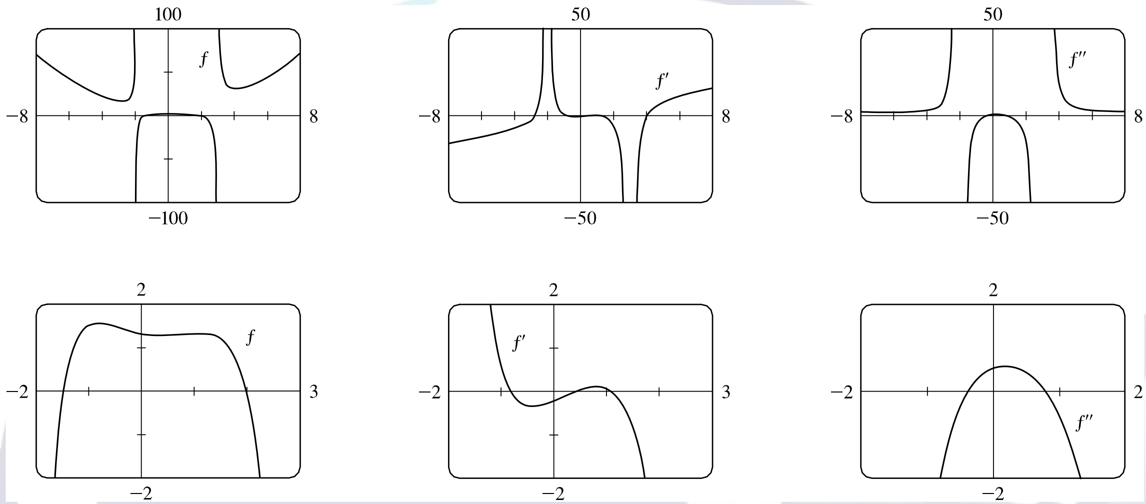
From the graph of f'', we estimate that f is CU on $(-\infty, -1.10)$, CD on $(-1.10, 0.08)$, CU on $(0.08, 1.72)$, CD on $(1.72, 2.64)$, and CU on $(2.64, \infty)$. There are inflection points at about $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, and $(2.64, -11.18)$.



4. $f(x) = \dfrac{x^4 - x^3 - 8}{x^2 - x - 6} \quad \Rightarrow \quad f'(x) = \dfrac{2(x^5 - 2x^4 - 11x^3 + 9x^2 + 8x - 4)}{(x^2 - x - 6)^2} \quad \Rightarrow$

$f''(x) = \dfrac{2(x^6 - 3x^5 - 15x^4 + 41x^3 + 174x^2 - 84x - 56}{(x^2 - x - 6)^3}$. $f(x) = 0 \;\Leftrightarrow\; x \approx -1.48$ or $x = 2$; $f'(x) = 0 \;\Leftrightarrow$

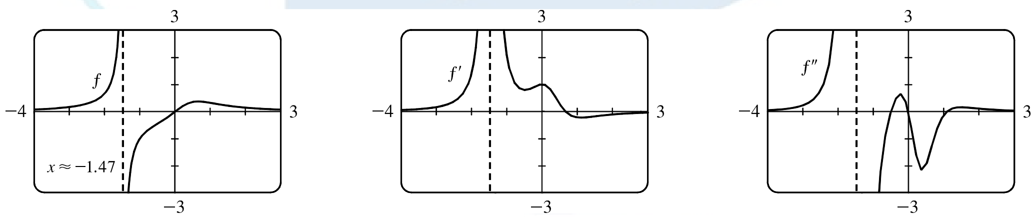
$x \approx -2.74, -0.81, 0.41, 1.08, 4.06$; $f''(x) = 0 \;\Leftrightarrow\; x \approx -0.39, 0.79$. The VAs are $x = -2$ and $x = 3$.



From the graphs of f', we estimate that f is decreasing on $(-\infty, -2.74)$, increasing on $(-2.74, -2)$, increasing on $(-2, -0.81)$, decreasing on $(-0.81, 0.41)$, increasing on $(0.41, 1.08)$, decreasing on $(1.08, 3)$, decreasing on $(3, 4.06)$, and increasing on $(4.06, \infty)$. f has local minimum values $f(-2.74) \approx 16.23$, $f(0.41) \approx 1.29$, and $f(4.06) \approx 30.63$. f has local maximum values $f(-0.81) \approx 1.55$ and $f(1.08) \approx 1.34$.

From the graphs of f'', we estimate that f is CU on $(-\infty, -2)$, CD on $(-2, -0.39)$, CU on $(-0.39, 0.79)$, CD on $(0.79, 3)$, and CU on $(3, \infty)$. There are inflection points at about $(-0.39, 1.45)$ and $(0.79, 1.31)$.

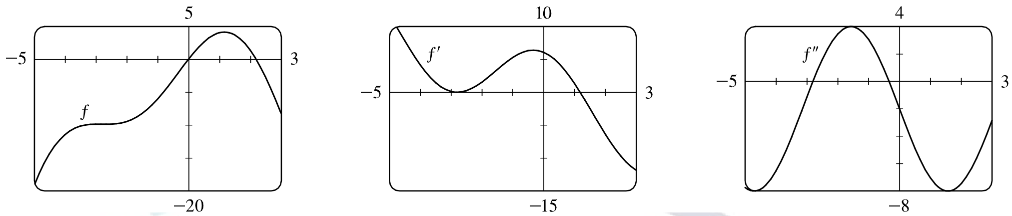
5. $f(x) = \dfrac{x}{x^3 + x^2 + 1} \quad \Rightarrow \quad f'(x) = -\dfrac{2x^3 + x^2 - 1}{(x^3 + x^2 + 1)^2} \quad \Rightarrow \quad f''(x) = \dfrac{2x(3x^4 + 3x^3 + x^2 - 6x - 3)}{(x^3 + x^2 + 1)^3}$



From the graph of f, we see that there is a VA at $x \approx -1.47$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.47)$, increasing on $(-1.47, 0.66)$, and decreasing on $(0.66, \infty)$, with local maximum value $f(0.66) \approx 0.38$.

From the graph of f'', we estimate that f is CU on $(-\infty, -1.47)$, CD on $(-1.47, -0.49)$, CU on $(-0.49, 0)$, CD on $(0, 1.10)$, and CU on $(1.10, \infty)$. There is an inflection point at $(0, 0)$ and at about $(-0.49, -0.44)$ and $(1.10, 0.31)$.

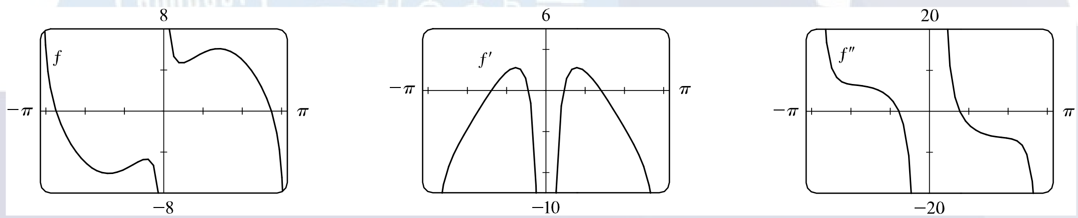
6. $f(x) = 6\sin x - x^2$, $-5 \le x \le 3 \quad \Rightarrow \quad f'(x) = 6\cos x - 2x \quad \Rightarrow \quad f''(x) = -6\sin x - 2$



From the graph of f', which has two negative zeros, we estimate that f is increasing on $(-5, -2.94)$, decreasing on $(-2.94, -2.66)$, increasing on $(-2.66, 1.17)$, and decreasing on $(1.17, 3)$, with local maximum values $f(-2.94) \approx -9.84$ and $f(1.17) \approx 4.16$, and local minimum value $f(-2.66) \approx -9.85$.

From the graph of f'', we estimate that f is CD on $(-5, -2.80)$, CU on $(-2.80, -0.34)$, and CD on $(-0.34, 3)$. There are inflection points at about $(-2.80, -9.85)$ and $(-0.34, -2.12)$.

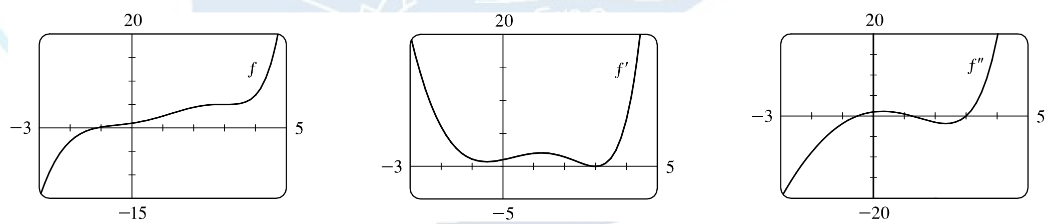
7. $f(x) = 6\sin x + \cot x$, $-\pi \le x \le \pi \quad \Rightarrow \quad f'(x) = 6\cos x - \csc^2 x \quad \Rightarrow \quad f''(x) = -6\sin x + 2\csc^2 x\, \cot x$



From the graph of f, we see that there are VAs at $x = 0$ and $x = \pm\pi$. f is an odd function, so its graph is symmetric about the origin. From the graph of f', we estimate that f is decreasing on $(-\pi, -1.40)$, increasing on $(-1.40, -0.44)$, decreasing on $(-0.44, 0)$, decreasing on $(0, 0.44)$, increasing on $(0.44, 1.40)$, and decreasing on $(1.40, \pi)$, with local minimum values $f(-1.40) \approx -6.09$ and $f(0.44) \approx 4.68$, and local maximum values $f(-0.44) \approx -4.68$ and $f(1.40) \approx 6.09$.

From the graph of f'', we estimate that f is CU on $(-\pi, -0.77)$, CD on $(-0.77, 0)$, CU on $(0, 0.77)$, and CD on $(0.77, \pi)$. There are IPs at about $(-0.77, -5.22)$ and $(0.77, 5.22)$.

8. $f(x) = e^x - 0.186x^4 \quad \Rightarrow \quad f'(x) = e^x - 0.744x^3 \quad \Rightarrow \quad f''(x) = e^x - 2.232x^2$

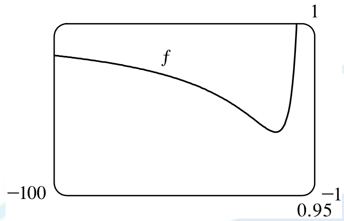


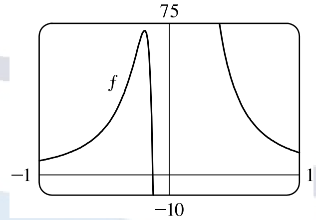
From the graph of f', which has two positive zeros, we estimate that f is increasing on $(-\infty, 2.973)$, decreasing on $(2.973, 3.027)$, and increasing on $(3.027, \infty)$, with local maximum value $f(2.973) \approx 5.01958$ and local minimum value $f(3.027) \approx 5.01949$.

From the graph of f'', we estimate that f is CD on $(-\infty, -0.52)$, CU on $(-0.52, 1.25)$, CD on $(1.25, 3.00)$, and CU on $(3.00, \infty)$. There are inflection points at about $(-0.52, 0.58)$, $(1.25, 3.04)$ and $(3.00, 5.01954)$.

9. $f(x) = 1 + \frac{1}{x} + \frac{8}{x^2} + \frac{1}{x^3} \Rightarrow f'(x) = -\frac{1}{x^2} - \frac{16}{x^3} - \frac{3}{x^4} = -\frac{1}{x^4}(x^2 + 16x + 3) \Rightarrow$

$f''(x) = \frac{2}{x^3} + \frac{48}{x^4} + \frac{12}{x^5} = \frac{2}{x^5}(x^2 + 24x + 6)$.





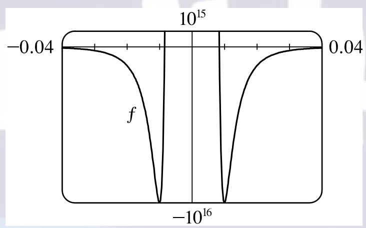
From the graphs, it appears that f increases on $(-15.8, -0.2)$ and decreases on $(-\infty, -15.8)$, $(-0.2, 0)$, and $(0, \infty)$; that f has a local minimum value of $f(-15.8) \approx 0.97$ and a local maximum value of $f(-0.2) \approx 72$; that f is CD on $(-\infty, -24)$ and $(-0.25, 0)$ and is CU on $(-24, -0.25)$ and $(0, \infty)$; and that f has IPs at $(-24, 0.97)$ and $(-0.25, 60)$.

To find the exact values, note that $f' = 0 \Rightarrow x = \frac{-16 \pm \sqrt{256 - 12}}{2} = -8 \pm \sqrt{61}$ $[\approx -0.19 \text{ and } -15.81]$. f' is positive (f is increasing) on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$ and f' is negative (f is decreasing) on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, and $(0, \infty)$. $f'' = 0 \Rightarrow x = \frac{-24 \pm \sqrt{576 - 24}}{2} = -12 \pm \sqrt{138}$ $[\approx -0.25 \text{ and } -23.75]$. f'' is positive (f is CU) on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$ and $(0, \infty)$ and f'' is negative (f is CD) on $\left(-\infty, -12 - \sqrt{138}\right)$ and $\left(-12 + \sqrt{138}, 0\right)$.

10. $f(x) = \frac{1}{x^8} - \frac{c}{x^4}$ $[c = 2 \times 10^8] \Rightarrow$

$f'(x) = -\frac{8}{x^9} + \frac{4c}{x^5} = -\frac{4}{x^9}(2 - cx^4) \Rightarrow$

$f''(x) = \frac{72}{x^{10}} - \frac{20c}{x^6} = \frac{4}{x^{10}}(18 - 5cx^4)$.



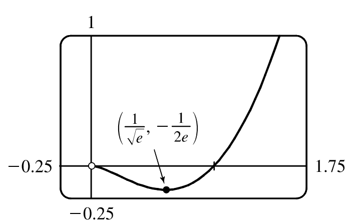
From the graph, it appears that f increases on $(-0.01, 0)$ and $(0.01, \infty)$ and decreases on $(-\infty, -0.01)$ and $(0, 0.01)$; that f has a local minimum value of $f(\pm 0.01) = -10^{16}$; and that f is CU on $(-0.012, 0)$ and $(0, 0.012)$ and f is CD on $(-\infty, -0.012)$ and $(0.012, \infty)$.

To find the exact values, note that $f' = 0 \Rightarrow x^4 = \frac{2}{c} \Rightarrow x \pm \sqrt[4]{\frac{2}{c}} = \pm\frac{1}{100}$ $[c = 2 \times 10^8]$. f' is positive (f is increasing) on $(-0.01, 0)$ and $(0.01, \infty)$ and f' is negative (f is decreasing) on $(-\infty, -0.01)$ and $(0, 0.01)$. $f'' = 0 \Rightarrow x^4 = \frac{18}{5c} \Rightarrow x = \pm\sqrt[4]{\frac{18}{5c}} = \pm\frac{1}{100}\sqrt[4]{1.8}$ $[\approx \pm 0.0116]$. f'' is positive (f is CU) on $\left(-\frac{1}{100}\sqrt[4]{1.8}, 0\right)$ and $\left(0, \frac{1}{100}\sqrt[4]{1.8}\right)$ and f'' is negative (f is CD) on $\left(-\infty, -\frac{1}{100}\sqrt[4]{1.8}\right)$ and $\left(\frac{1}{100}\sqrt[4]{1.8}, \infty\right)$.

11. (a) $f(x) = x^2 \ln x$. The domain of f is $(0, \infty)$.

(b) $\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \left(-\frac{x^2}{2}\right) = 0$.

There is a hole at $(0, 0)$.

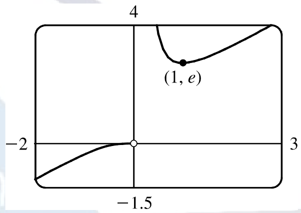


(c) It appears that there is an IP at about $(0.2, -0.06)$ and a local minimum at $(0.6, -0.18)$. $f(x) = x^2 \ln x \Rightarrow$

$f'(x) = x^2(1/x) + (\ln x)(2x) = x(2\ln x + 1) > 0 \Leftrightarrow \ln x > -\frac{1}{2} \Leftrightarrow x > e^{-1/2}$, so f is increasing on $\left(1/\sqrt{e}, \infty\right)$, decreasing on $\left(0, 1/\sqrt{e}\right)$. By the FDT, $f\left(1/\sqrt{e}\right) = -1/(2e)$ is a local minimum value. This point is approximately $(0.6065, -0.1839)$, which agrees with our estimate.

$f''(x) = x(2/x) + (2\ln x + 1) = 2\ln x + 3 > 0 \Leftrightarrow \ln x > -\frac{3}{2} \Leftrightarrow x > e^{-3/2}$, so f is CU on $(e^{-3/2}, \infty)$ and CD on $(0, e^{-3/2})$. IP is $(e^{-3/2}, -3/(2e^3)) \approx (0.2231, -0.0747)$.

12. (a) $f(x) = xe^{1/x}$. The domain of f is $(-\infty, 0) \cup (0, \infty)$.



(b) $\displaystyle\lim_{x\to 0^+} xe^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty$,

so $x = 0$ is a VA.

Also $\displaystyle\lim_{x\to 0^-} xe^{1/x} = 0$ since $1/x \to -\infty \Rightarrow e^{1/x} \to 0$.

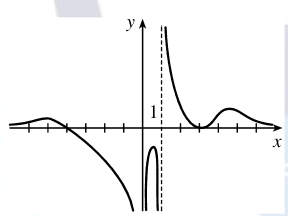
(c) It appears that there is a local minimum at $(1, 2.7)$. There are no IP and f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$.

$f(x) = xe^{1/x} \Rightarrow f'(x) = xe^{1/x}\left(-\dfrac{1}{x^2}\right) + e^{1/x} = e^{1/x}\left(1 - \dfrac{1}{x}\right) > 0 \Leftrightarrow \dfrac{1}{x} < 1 \Leftrightarrow x < 0$ or $x > 1$,

so f is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1)$. By the FDT, $f(1) = e$ is a local minimum value, which agrees with our estimate.

$f''(x) = e^{1/x}(1/x^2) + (1 - 1/x)e^{1/x}(-1/x^2) = (e^{1/x}/x^2)(1 - 1 + 1/x) = e^{1/x}/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

13.

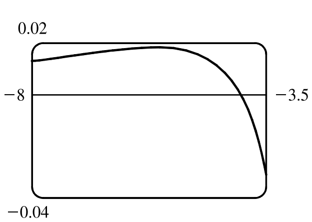


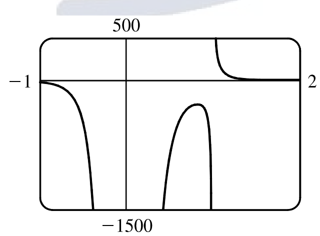
$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\displaystyle\lim_{x\to 0} f(x) = -\infty$, $\displaystyle\lim_{x\to 1^-} f(x) = -\infty$ and $\displaystyle\lim_{x\to 1^+} f(x) = \infty$.

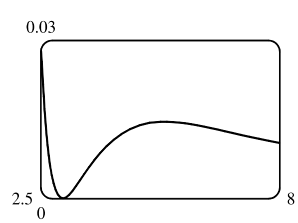
$$f(x) = \frac{\dfrac{x+4}{x} \cdot \dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3} \cdot (x-1)} \quad \left[\begin{array}{c}\text{dividing numerator}\\ \text{and denominator by } x^3\end{array}\right] = \frac{(1 + 4/x)(1 - 3/x)^2}{x(x-1)} \to 0$$

as $x \to \pm\infty$, so f is asymptotic to the x-axis.

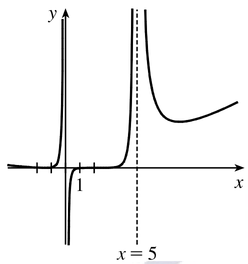
Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \Rightarrow (x+4)(x-3)^2 = 0 \Rightarrow x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.







From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

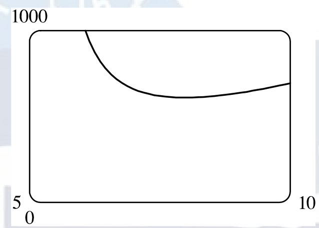
14. 

$f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$ has VAs at $x = 0$ and $x = 5$ since $\lim\limits_{x \to 0^-} f(x) = \infty$, $\lim\limits_{x \to 0^+} f(x) = -\infty$, and $\lim\limits_{x \to 5} f(x) = \infty$. No HA since $\lim\limits_{x \to \pm\infty} f(x) = \infty$.

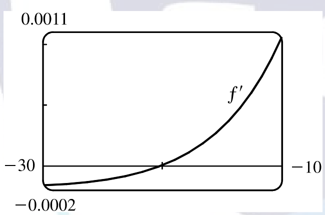
Since f is undefined at $x = 0$, it has no y-intercept.

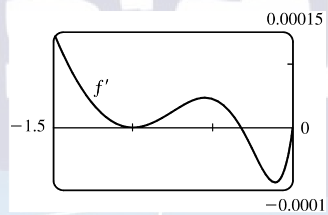
$f(x) = 0 \quad \Leftrightarrow \quad (2x+3)^2 (x-2)^5 = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2}$ or $x = 2$, so f has x-intercepts at $-\frac{3}{2}$ and 2. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = -\frac{3}{2}$, since f is positive as $x \to \left(-\frac{3}{2}\right)^-$ and as $x \to \left(-\frac{3}{2}\right)^+$. There is a local minimum value of $f\left(-\frac{3}{2}\right) = 0$.

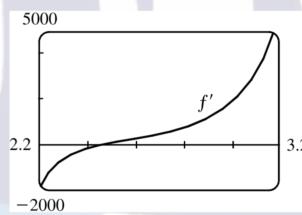
The only "mystery" feature is the local minimum to the right of the VA $x = 5$. From the graph, we see that $f(7.98) \approx 609$ is a local minimum value.



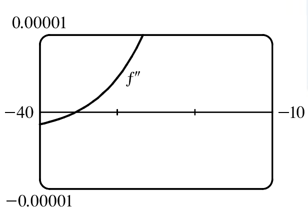
15. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad \Rightarrow \quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ [from CAS].

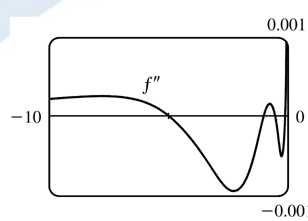


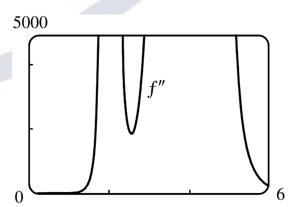




From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20, -0.3$, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.) We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.



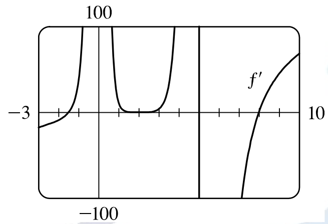




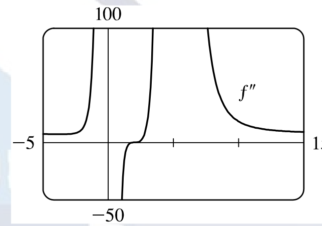
From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$ and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

16. From a CAS, $$f'(x) = \frac{2(x-2)^4(2x+3)(2x^3-14x^2-10x-45)}{x^4(x-5)^3}$$

and $$f''(x) = \frac{2(x-2)^3(4x^6-56x^5+216x^4+460x^3+805x^2+1710x+5400)}{x^5(x-5)^4}$$



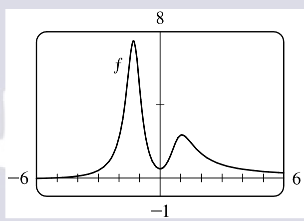
From Exercise 14 and $f'(x)$ above, we know that the zeros of f' are -1.5, 2, and 7.98. From the graph of f', we conclude that f is decreasing on $(-\infty, -1.5)$, increasing on $(-1.5, 0)$ and $(0, 5)$, decreasing on $(5, 7.98)$, and increasing on $(7.98, \infty)$.

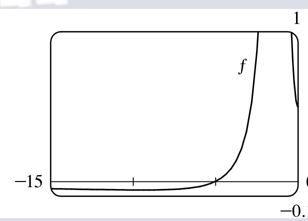


From $f''(x)$, we know that $x = 2$ is a zero, and the graph of f'' shows us that $x = 2$ is the only zero of f''. Thus, f is CU on $(-\infty, 0)$, CD on $(0, 2)$, CU on $(2, 5)$, and CU on $(5, \infty)$.

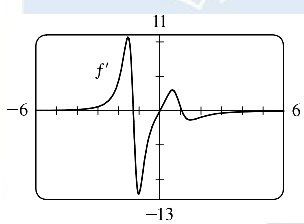
17. $f(x) = \dfrac{x^3+5x^2+1}{x^4+x^3-x^2+2}$. From a CAS, $f'(x) = \dfrac{-x(x^5+10x^4+6x^3+4x^2-3x-22)}{(x^4+x^3-x^2+2)^2}$ and

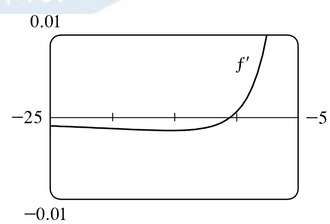
$$f''(x) = \frac{2(x^9+15x^8+18x^7+21x^6-9x^5-135x^4-76x^3+21x^2+6x+22)}{(x^4+x^3-x^2+2)^3}$$





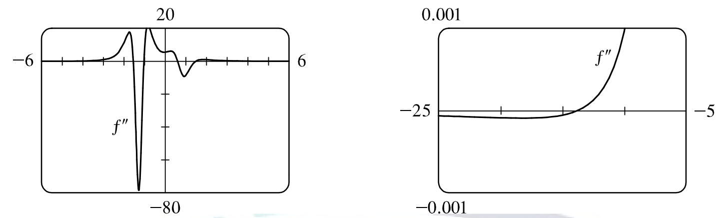
The first graph of f shows that $y = 0$ is a HA. As $x \to \infty$, $f(x) \to 0$ through positive values. As $x \to -\infty$, it is not clear if $f(x) \to 0$ through positive or negative values. The second graph of f shows that f has an x-intercept near -5, and will have a local minimum and inflection point to the left of -5.





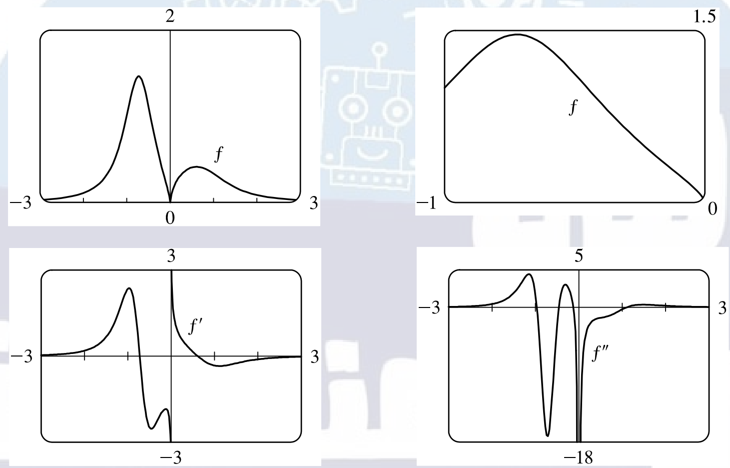
From the two graphs of f', we see that f' has four zeros. We conclude that f is decreasing on $(-\infty, -9.41)$, increasing on $(-9.41, -1.29)$, decreasing on $(-1.29, 0)$, increasing on $(0, 1.05)$, and decreasing on $(1.05, \infty)$. We have local minimum values $f(-9.41) \approx -0.056$ and $f(0) = 0.5$, and local maximum values $f(-1.29) \approx 7.49$ and $f(1.05) \approx 2.35$.

[continued]



From the two graphs of f'', we see that f'' has five zeros. We conclude that f is CD on $(-\infty, -13.81)$, CU on $(-13.81, -1.55)$, CD on $(-1.55, -1.03)$, CU on $(-1.03, 0.60)$, CD on $(0.60, 1.48)$, and CU on $(1.48, \infty)$. There are five inflection points: $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, and $(1.48, 1.93)$.

18. $y = f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$. From a CAS, $y' = -\dfrac{10x^4 + x - 2}{3x^{1/3}(x^4 + x + 1)^2}$ and $y'' = \dfrac{2(65x^8 - 14x^5 - 80x^4 + 2x^2 - 8x - 1)}{9x^{4/3}(x^4 + x + 1)^3}$

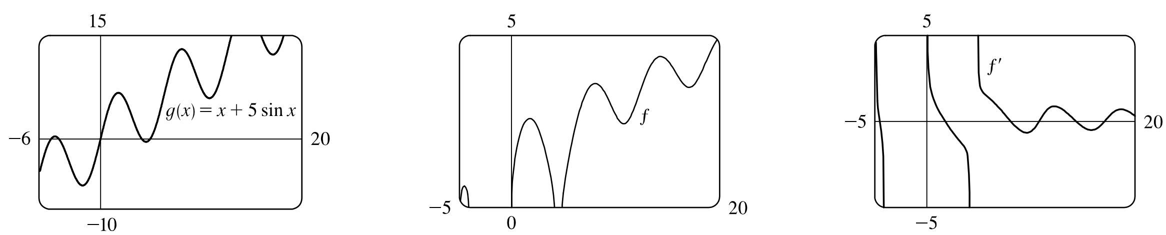


$f'(x)$ does not exist at $x = 0$ and $f'(x) = 0 \quad\Leftrightarrow\quad x \approx -0.72$ and 0.61, so f is increasing on $(-\infty, -0.72)$, decreasing on $(-0.72, 0)$, increasing on $(0, 0.61)$, and decreasing on $(0.61, \infty)$. There is a local maximum value of $f(-0.72) \approx 1.46$ and a local minimum value of $f(0.61) \approx 0.41$. $f''(x)$ does not exist at $x = 0$ and $f''(x) = 0 \quad\Leftrightarrow\quad x \approx -0.97, -0.46, -0.12$, and 1.11, so f is CU on $(-\infty, -0.97)$, CD on $(-0.97, -0.46)$, CU on $(-0.46, -0.12)$, CD on $(-0.12, 0)$, CD on $(0, 1.11)$, and CU on $(1.11, \infty)$. There are inflection points at $(-0.97, 1.08)$, $(-0.46, 1.01)$, $(-0.12, 0.28)$, and $(1.11, 0.29)$.

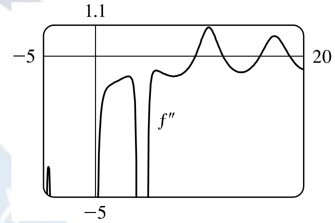
19. $y = f(x) = \sqrt{x + 5\sin x}$, $x \leq 20$.

From a CAS, $y' = \dfrac{5\cos x + 1}{2\sqrt{x + 5\sin x}}$ and $y'' = -\dfrac{10\cos x + 25\sin^2 x + 10x\sin x + 26}{4(x + 5\sin x)^{3/2}}$.

We'll start with a graph of $g(x) = x + 5\sin x$. Note that $f(x) = \sqrt{g(x)}$ is only defined if $g(x) \geq 0$. $g(x) = 0 \quad\Leftrightarrow\quad x = 0$ or $x \approx -4.91, -4.10, 4.10$, and 4.91. Thus, the domain of f is $[-4.91, -4.10] \cup [0, 4.10] \cup [4.91, 20]$.

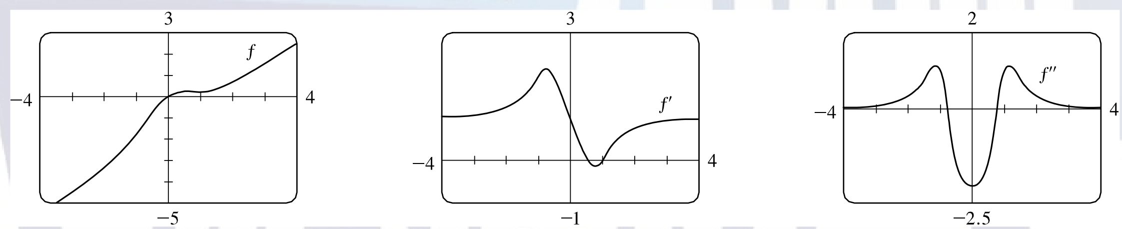


From the expression for y', we see that $y' = 0 \Leftrightarrow 5\cos x + 1 = 0 \Rightarrow x_1 = \cos^{-1}\left(-\frac{1}{5}\right) \approx 1.77$ and $x_2 = 2\pi - x_1 \approx -4.51$ (not in the domain of f). The leftmost zero of f' is $x_1 - 2\pi \approx -4.51$. Moving to the right, the zeros of f' are x_1, $x_1 + 2\pi$, $x_2 + 2\pi$, $x_1 + 4\pi$, and $x_2 + 4\pi$. Thus, f is increasing on $(-4.91, -4.51)$, decreasing on $(-4.51, -4.10)$, increasing on $(0, 1.77)$, decreasing on $(1.77, 4.10)$, increasing on $(4.91, 8.06)$, decreasing on $(8.06, 10.79)$, increasing on $(10.79, 14.34)$, decreasing on $(14.34, 17.08)$, and increasing on $(17.08, 20)$. The local maximum values are $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, and $f(14.34) \approx 4.39$. The local minimum values are $f(10.79) \approx 2.43$ and $f(17.08) \approx 3.49$.



f is CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, CU on $(9.60, 12.25)$, CD on $(12.25, 15.81)$, CU on $(15.81, 18.65)$, and CD on $(18.65, 20)$. There are inflection points at $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, and $(18.65, 4.20)$.

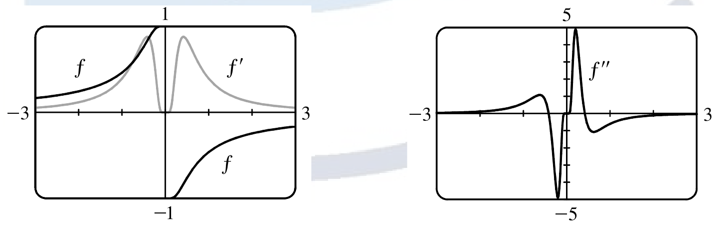
20. $y = f(x) = x - \tan^{-1} x^2$. From a CAS, $y' = \dfrac{x^4 - 2x + 1}{x^4 + 1}$ and $y'' = \dfrac{2(3x^4 - 1)}{(x^4 + 1)^2}$. $y' = 0 \Leftrightarrow x \approx 0.54$ or $x = 1$. $y'' = 0 \Leftrightarrow x \approx \pm 0.76$.



From the graphs of f and f', we estimate that f is increasing on $(-\infty, 0.54)$, decreasing on $(0.54, 1)$, and increasing on $(1, \infty)$. f has local maximum value $f(0.54) \approx 0.26$ and local minimum value $f(1) \approx 0.21$.

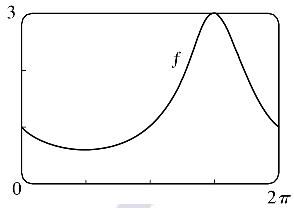
From the graph of f'', we estimate that f is CU on $(-\infty, -0.76)$, CD on $(-0.76, 0.76)$, and CU on $(0.76, \infty)$. There are inflection points at about $(-0.76, -1.28)$ and $(0.76, 0.24)$.

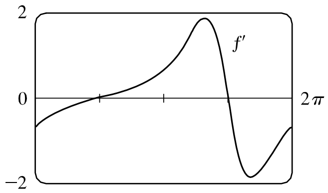
21. $y = f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$. From a CAS, $y' = \dfrac{2e^{1/x}}{x^2(1 + e^{1/x})^2}$ and $y'' = \dfrac{-2e^{1/x}(1 - e^{1/x} + 2x + 2xe^{1/x})}{x^4(1 + e^{1/x})^3}$.

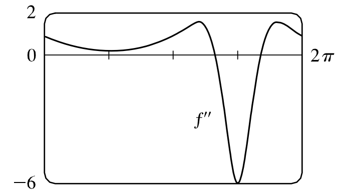


f is an odd function defined on $(-\infty, 0) \cup (0, \infty)$. Its graph has no *x*- or *y*-intercepts. Since $\lim\limits_{x \to \pm\infty} f(x) = 0$, the *x*-axis is a HA. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. It has no local extreme values. $f''(x) = 0$ for $x \approx \pm 0.417$, so f is CU on $(-\infty, -0.417)$, CD on $(-0.417, 0)$, CU on $(0, 0.417)$, and CD on $(0.417, \infty)$. f has IPs at $(-0.417, 0.834)$ and $(0.417, -0.834)$.

22. $y = f(x) = \dfrac{3}{3 + 2\sin x}$. From a CAS, $y' = -\dfrac{6\cos x}{(3 + 2\sin x)^2}$ and $y'' = \dfrac{6(2\sin^2 x + 4\cos^2 x + 3\sin x)}{(3 + 2\sin x)^3}$. Since f is periodic with period 2π, we'll restrict our attention to the interval $[0, 2\pi)$. $y' = 0 \Leftrightarrow 6\cos x = 0 \Leftrightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.
$y'' = 0 \Leftrightarrow x \approx 4.16$ or 5.27.





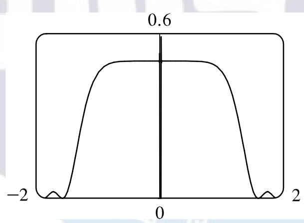


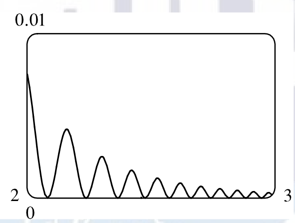
From the graphs of f and f', we conclude that f is decreasing on $\left(0, \frac{\pi}{2}\right)$, increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and decreasing on $\left(\frac{3\pi}{2}, 2\pi\right)$. f has local minimum value $f\left(\frac{\pi}{2}\right) = \frac{3}{5}$ and local maximum value $f\left(\frac{3\pi}{2}\right) = 3$.

From the graph of f'', we conclude that f is CU on $(0, 4.16)$, CD on $(4.16, 5.27)$, and CU on $(5.27, 2\pi)$. There are inflection points at about $(4.16, 2.31)$ and $(5.27, 2.31)$.

23. $f(x) = \dfrac{1 - \cos(x^4)}{x^8} \geq 0$. f is an even function, so its graph is symmetric with respect to the y-axis. The first graph shows that f levels off at $y = \frac{1}{2}$ for $|x| < 0.7$. It also shows that f then drops to the x-axis. Your graphing utility may show some severe oscillations near the origin, but there are none. See the discussion in Section 2.2 after Example 2, as well as "Lies My Calculator and Computer Told Me" on the website.

The second graph indicates that as $|x|$ increases, f has progressively smaller humps.

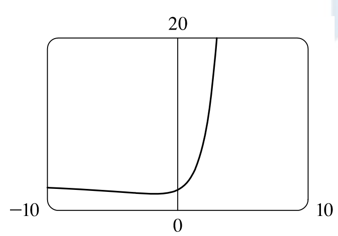


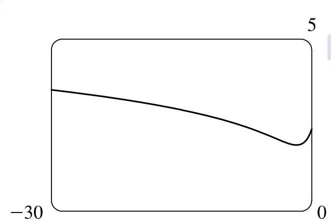


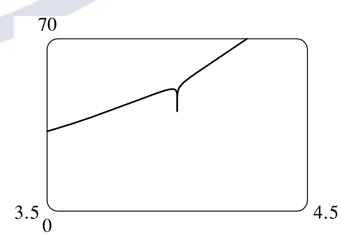
24. $f(x) = e^x + \ln|x - 4|$. The first graph shows the big picture of f but conceals hidden behavior.

The second graph shows that for large negative values of x, f looks like $g(x) = \ln|x|$. It also shows a minimum value and a point of inflection.

The third graph hints at the vertical asymptote that we know exists at $x = 4$ because $\lim\limits_{x \to 4} (e^x + \ln|x - 4|) = -\infty$.

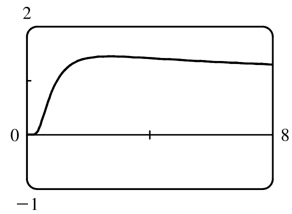






A graphing calculator is unable to show much of the dip of the curve toward the vertical asymptote because of limited resolution. A computer can show more if we restrict ourselves to a narrow interval around $x = 4$. See the solution to Exercise 2.2.48 for a hand-drawn graph of this function.

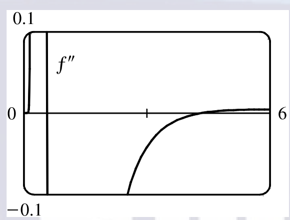
25. (a) $f(x) = x^{1/x}$



(b) Recall that $a^b = e^{b \ln a}$. $\lim\limits_{x\to 0^+} x^{1/x} = \lim\limits_{x\to 0^+} e^{(1/x)\ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x)\ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{(1/x)\ln x}$, but $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so $\lim\limits_{x\to\infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

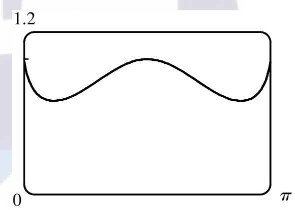
(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln x \Rightarrow \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \Rightarrow y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \Rightarrow$ $\ln x = 1 \Rightarrow x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

26. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x$, so

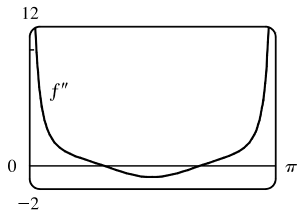
$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x \ln \sin x = \lim_{x\to 0^+} \frac{\ln \sin x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\cot x}{-\csc x \cot x}$$

$$= \lim_{x\to 0^+} (-\sin x) = 0 \quad\Rightarrow\quad \lim_{x\to 0^+} y = e^0 = 1.$$

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$.

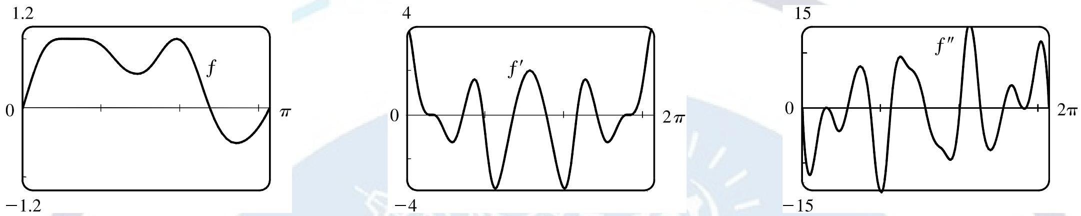
$y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x \Rightarrow \dfrac{y'}{y} = (\sin x)\left(\dfrac{\cos x}{\sin x}\right) + (\ln \sin x)\cos x = \cos x\,(1 + \ln \sin x) \Rightarrow$ $y' = (\sin x)^{\sin x}(\cos x)(1 + \ln \sin x)$. $y' = 0 \Rightarrow \cos x = 0$ or $\ln \sin x = -1 \Rightarrow x_2 = \frac{\pi}{2}$ or $\sin x = e^{-1}$. On $(0, \pi)$, $\sin x = e^{-1} \Rightarrow x_1 = \sin^{-1}(e^{-1})$ and $x_3 = \pi - \sin^{-1}(e^{-1})$. Approximating these points gives us $(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_2, f(x_2)) \approx (1.5708, 1)$, and $(x_3, f(x_3)) \approx (2.7649, 0.6922)$. The approximations confirm our estimates.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

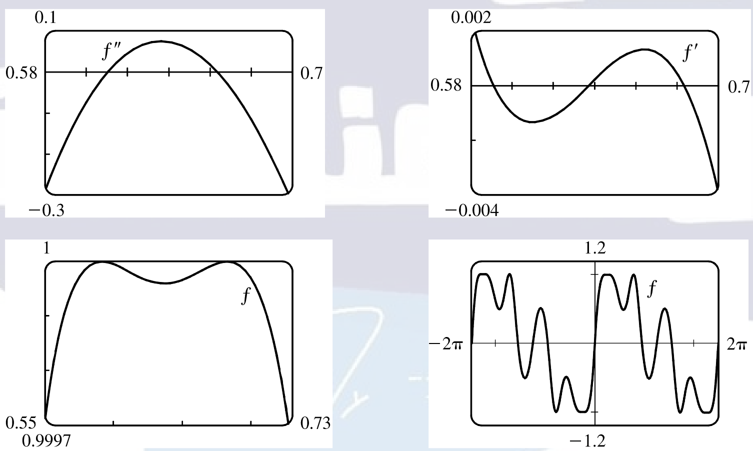
27.



From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of

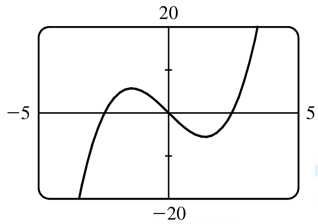
$$f'' = -\left[\sin(x + \sin 3x)\right](1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$$

is even more interesting near this x-value: it seems to just touch the x-axis.

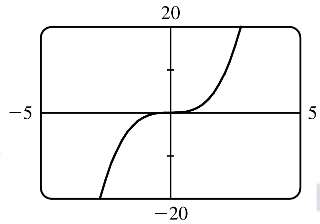


If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

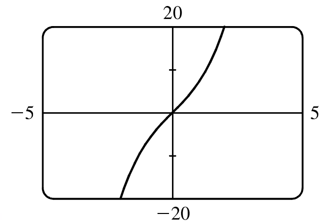
28. $f(x) = x^3 + cx = x(x^2 + c) \quad \Rightarrow \quad f'(x) = 3x^2 + c \quad \Rightarrow \quad f''(x) = 6x$



$c = -6$



$c = 0$



$c = 6$

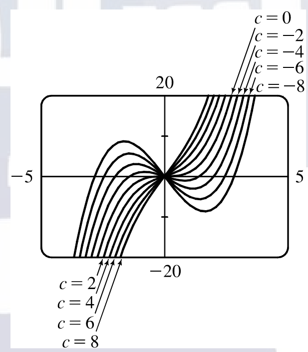
x-intercepts: When $c \geq 0$, 0 is the only x-intercept. When $c < 0$, the x-intercepts are 0 and $\pm\sqrt{-c}$.

y-intercept $= f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x, so f is increasing and has no local maximum or minimum.

If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

If $c < 0$, then $f'(x) = 3[x^2 - (-c/3)] = 3\left(x + \sqrt{-c/3}\right)\left(x - \sqrt{-c/3}\right)$, so $f'(x) > 0$ on $\left(-\infty, -\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3}, \infty\right)$; $f'(x) < 0$ on $\left(-\sqrt{-c/3}, \sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right) = -\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right) = \frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart.



There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.

29. $f(x) = x^2 + 6x + c/x \quad \Rightarrow \quad f'(x) = 2x + 6 - c/x^2 \quad \Rightarrow \quad f''(x) = 2 + 2c/x^3$

$\mathbf{c = 0}$: The graph is the parabola $y = x^2 + 6x$, which has x-intercepts -6 and 0, vertex $(-3, -9)$, and opens upward.

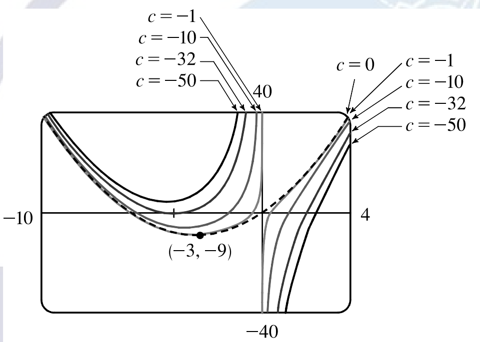
$\mathbf{c \neq 0}$: The parabola $y = x^2 + 6x$ is an asymptote that the graph of f approaches as $x \to \pm\infty$. The y-axis is a vertical asymptote.

$\mathbf{c < 0}$**:** The x-intercepts are found by solving $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 6x^2 + c = g(x) = 0$. Now $g'(x) = 0 \quad \Leftrightarrow \quad x = -4$ or 0, and g (*not* f) has a local maximum at $x = -4$. $\quad g(-4) = 32 + c$, so if $c < -32$, the maximum is negative and there are no negative x-intercepts; if $c = -32$, the maximum is 0 and there is one negative x-intercept; if $-32 < c < 0$, the maximum is positive and there are two negative x-intercepts. In all cases, there is one positive x-intercept.

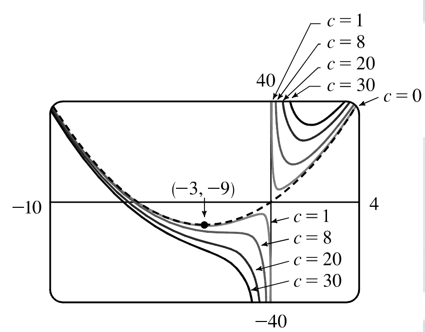
As $c \to 0^-$, the local minimum point moves down and right, approaching $(-3, -9)$. [Note that since $f'(x) = \dfrac{2x^3 + 6x^2 - c}{x^2}$, Descartes' Rule of Signs implies that f' has no positive roots and one negative root when $c < 0$. $f''(x) = \dfrac{2(x^3 + c)}{x^3} > 0$ at that negative root, so that critical point yields a local minimum value. This tells us that there are no local maximums when $c < 0$.] $\quad f'(x) > 0$ for $x > 0$, so f is increasing on $(0, \infty)$. From $f''(x) = \dfrac{2(x^3 + c)}{x^3}$, we see that f

has an inflection point at $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$. This inflection point moves down and left, approaching the origin as $c \to 0^-$. f is CU on $(-\infty, 0)$, CD on $\left(0, \sqrt[3]{-c}\right)$, and CU on $\left(\sqrt[3]{-c}, \infty\right)$.

$\mathbf{c > 0}$: The inflection point $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$ is now in the third quadrant and moves up and right, approaching the origin as $c \to 0^+$. f is CU on $\left(-\infty, \sqrt[3]{-c}\right)$, CD on $\left(\sqrt[3]{-c}, 0\right)$, and CU on $(0, \infty)$. f has a local minimum point in the first quadrant. It moves down and left, approaching the origin as $c \to 0^+$. $f'(x) = 0 \iff 2x^3 + 6x^2 - c = h(x) = 0$. Now $h'(x) = 0 \iff x = -2$ or 0, and h (*not* f) has a local maximum at $x = -2$. $h(-2) = 8 - c$, so $c = 8$ makes $h(x) = 0$, and hence, $f'(x) = 0$. When $c > 8$, $f'(x) < 0$ and f is decreasing on $(-\infty, 0)$. For $0 < c < 8$, there is a local minimum that moves toward $(-3, -9)$ and a local maximum that moves toward the origin as c decreases.



$\mathbf{c \le 0}$



$\mathbf{c \ge 0}$

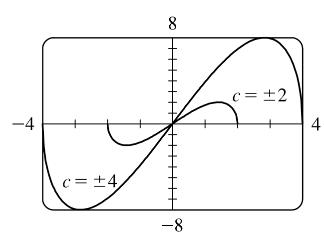
30. With $c = 0$ in $y = f(x) = x\sqrt{c^2 - x^2}$, the graph of f is just the point $(0, 0)$. Since $(-c)^2 = c^2$, we only consider $c > 0$. Since $f(-x) = -f(x)$, the graph is symmetric about the origin. The domain of f is found by solving $c^2 - x^2 \ge 0 \iff x^2 \le c^2 \iff |x| \le c$, which gives us $[-c, c]$.

$$f'(x) = x \cdot \tfrac{1}{2}(c^2 - x^2)^{-1/2}(-2x) + (c^2 - x^2)^{1/2}(1) = (c^2 - x^2)^{-1/2}[-x^2 + (c^2 - x^2)] = \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}}.$$

$f'(x) > 0 \iff c^2 - 2x^2 > 0 \iff x^2 < c^2/2 \iff |x| < c/\sqrt{2}$, so f is increasing on $\left(-c/\sqrt{2}, c/\sqrt{2}\right)$ and decreasing on $\left(-c, -c/\sqrt{2}\right)$ and $\left(c/\sqrt{2}, c\right)$. There is a local minimum value of $f\left(-c/\sqrt{2}\right) = \left(-c/\sqrt{2}\right)\sqrt{c^2 - c^2/2} = \left(-c/\sqrt{2}\right)\left(c/\sqrt{2}\right) = -c^2/2$ and a local maximum value of $f\left(c/\sqrt{2}\right) = c^2/2$.

$$f''(x) = \frac{(c^2 - x^2)^{1/2}(-4x) - (c^2 - 2x^2)\frac{1}{2}\left(c^2 - x^2\right)^{-1/2}(-2x)}{[(c^2 - x^2)^{1/2}]^2}$$
$$= \frac{x(c^2 - x^2)^{-1/2}[(c^2 - x^2)(-4) + (c^2 - 2x^2)]}{(c^2 - x^2)^1} = \frac{2x\left(2x^2 - 3c^2\right)}{(c^2 - x^2)^{3/2}},$$

so $f''(x) = 0 \iff x = 0$ or $x = \pm\sqrt{\frac{3}{2}}\,c$, but only 0 is in the domain of f. $f''(x) < 0$ for $0 < x < c$ and $f''(x) > 0$ for $-c < x < 0$, so f is CD on $(0, c)$ and CU on $(-c, 0)$. There is an IP at $(0, 0)$. So as $|c|$ gets larger, the maximum and minimum values increase in magnitude. The value of c does not affect the concavity of f.

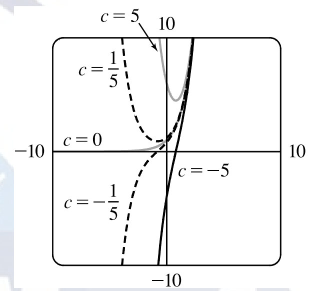


31. $f(x) = e^x + ce^{-x}$. $f = 0 \;\Rightarrow\; ce^{-x} = -e^x \;\Rightarrow\; c = -e^{2x} \;\Rightarrow\; 2x = \ln(-c) \;\Rightarrow\; x = \frac{1}{2}\ln(-c)$.

$f'(x) = e^x - ce^{-x}$. $f' = 0 \;\Rightarrow\; ce^{-x} = e^x \;\Rightarrow\; c = e^{2x} \;\Rightarrow\; 2x = \ln c \;\Rightarrow\; x = \frac{1}{2}\ln c$.

$f''(x) = e^x + ce^{-x} = f(x)$.

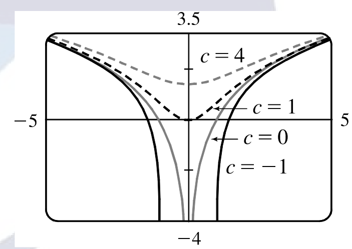
The only transitional value of c is 0. As c increases from $-\infty$ to 0, $\frac{1}{2}\ln(-c)$ is both the the x-intercept and inflection point, and this decreases from ∞ to $-\infty$. Also $f' > 0$, so f is increasing. When $c = 0$, $f(x) = f'(x) = f''(x) = e^x$, f is positive, increasing, and concave upward. As c increases from 0 to ∞, the absolute minimum occurs at $x = \frac{1}{2}\ln c$, which increases from $-\infty$ to ∞. Also, $f = f'' > 0$, so f is positive and concave upward. The value of the y-intercept is $f(0) = 1 + c$, and this increases as c increases from $-\infty$ to ∞.



Note: The minimum point $\left(\frac{1}{2}\ln c, 2\sqrt{c}\right)$ can be parameterized by $x = \frac{1}{2}\ln c$, $y = 2\sqrt{c}$, and after eliminating the parameter c, we see that the minimum point lies on the graph of $y = 2e^x$.

32. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and $\lim\limits_{x \to \sqrt{-c}^+} f(x) = \lim\limits_{x \to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x \to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there are no asymptotes. To find the extrema and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get $f''(x) = \dfrac{1}{x^2 + c}(2) + 2x\left[-(x^2 + c)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2 + c)^2}$.

Now if $c \le 0$, f'' is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow\; x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $x = \pm\sqrt{c}$, and as c increases, the inflection points get further apart.

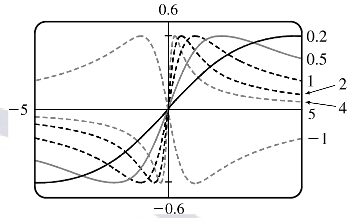


33. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate $f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \;\Leftrightarrow\; c^2x^2 - 1 = 0 \;\Leftrightarrow$

[continued]

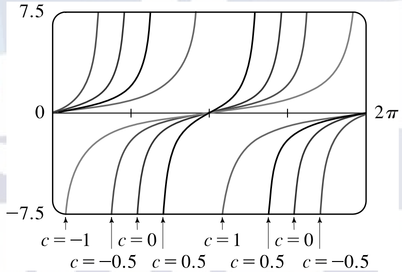
$x = \pm 1/c$. So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

$$f''(x) = \frac{(-2c^3x)(1+c^2x^2)^2 - (-c^3x^2+c)[2(1+c^2x^2)(2c^2x)]}{(1+c^2x^2)^4}$$
$$= \frac{(-2c^3x)(1+c^2x^2) + (c^3x^2-c)(4c^2x)}{(1+c^2x^2)^3} = \frac{2c^3x(c^2x^2-3)}{(1+c^2x^2)^3}$$

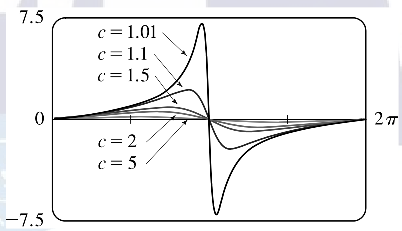


$f''(x) = 0 \iff x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$. Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

34. $f(x) = \dfrac{\sin x}{c + \cos x} \Rightarrow f'(x) = \dfrac{1 + c\cos x}{\cos^2 x + 2c\cos x + c^2} \Rightarrow f''(x) = \dfrac{\sin x(c\cos x - c^2 + 2)}{\cos^3 x + 3c\cos^2 x + 3c^2\cos x + c^3}$. Notice that f is an odd function and has period 2π. We will graph f for $0 \le x \le 2\pi$.



$|\mathbf{c}| \le \mathbf{1}$**:** See the first figure. f has VAs when the denominator is zero, that is, at $x = \cos^{-1}(-c)$ and $x = 2\pi - \cos^{-1}(-c)$. So for $c = -1$, there are VAs at $x = 0$ and $x = 2\pi$, and as c increases, they move closer to $x = \pi$, which is the single VA when $c = 1$. Note that if $c = 0$, then $f(x) = \tan x$. There are no extreme points (on the entire domain) and inflection points occur at multiples of π.



$\mathbf{c > 1}$**:** See the second figure. $f'(x) = 0 \iff x = \cos^{-1}\left(\dfrac{-1}{c}\right)$ or $x = 2\pi - \cos^{-1}\left(\dfrac{-1}{c}\right)$. The VA disappears and there is now a local maximum and a local minimum. As $c \to 1^+$, the coordinates of the local maximum approach π and ∞, and the coordinates of the local minimum approach π and $-\infty$.

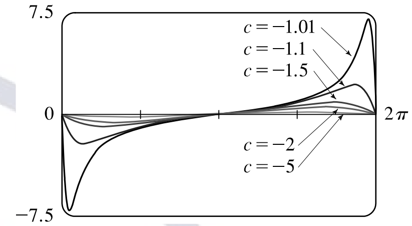
As $c \to \infty$, the graph of f looks like a graph of $y = \sin x$ that is vertically compressed, and the local maximum and local minimum approach $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, respectively.

$f''(x) = 0 \iff \sin x = 0$ (IPs at $x = n\pi$) or $c\cos x - c^2 + 2 = 0$. The second condition is true if $\cos x = \dfrac{c^2 - 2}{c}$ $[c \ne 0]$. The last equation has two solutions if $-1 < \dfrac{c^2-2}{c} < 1 \Rightarrow -c < c^2 - 2 < c \Rightarrow -c < c^2 - 2$ and $c^2 - 2 < c \Rightarrow c^2 + c - 2 > 0$ and $c^2 - c - 2 < 0 \Rightarrow (c+2)(c-1) > 0$ and $(c-2)(c+1) < 0 \Rightarrow c - 1 > 0$

[continued]

[since $c > 1$] and $c - 2 < 0 \quad \Rightarrow \quad c > 1$ and $c < 2$. Thus, for $1 < c < 2$, we have 2 nontrivial IPs at $x = \cos^{-1}\left(\dfrac{c^2 - 2}{c}\right)$ and $x = 2\pi - \cos^{-1}\left(\dfrac{c^2 - 2}{c}\right)$.

$\mathbf{c < -1}$: See the third figure. The VAs for $c = -1$ at $x = 0$ and $x = 2\pi$ in the first figure disappear and we now have a local minimum and a local maximum. As $c \to -1^+$, the coordinates of the local minimum approach 0 and $-\infty$, and the coordinates of the local maximum approach 2π and ∞. As $c \to -\infty$, the graph of f looks like a graph of $y = \sin x$ that is vertically compressed, and the local minimum and local maximum approach $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, respectively. As above, we have two nontrivial IPs for $-2 < c < -1$.



35. $f(x) = cx + \sin x \quad \Rightarrow \quad f'(x) = c + \cos x \quad \Rightarrow \quad f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

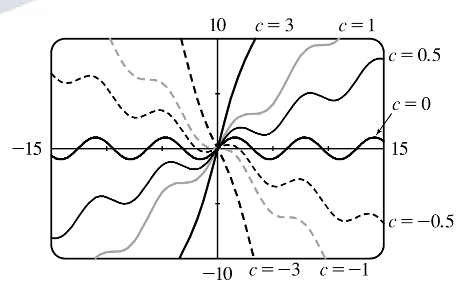
$f(x) = 0 \quad \Leftrightarrow \quad \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \le 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = -1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = 1$, $x_1 = \pi$.)

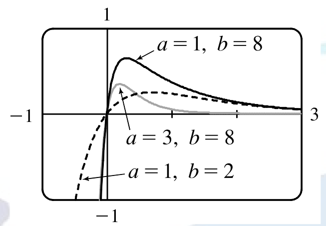
$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(n\pi, n\pi c)$, where n is an integer.

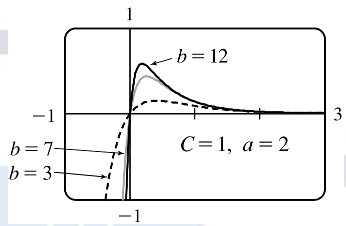
If $c \ge 1$, then $f'(x) \ge 0$ for all x, so f is increasing and has no extremum. If $c \le -1$, then $f'(x) \le 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1 - c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1 - c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \ge 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.



36. For $f(t) = C(e^{-at} - e^{-bt})$, C affects only vertical stretching, so we let $C = 1$. From the first figure, we notice that the graphs all pass through the origin, approach the t-axis as t increases, and approach $-\infty$ as $t \to -\infty$. Next we let $a = 2$ and produce the second figure.





Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases.

$f(t) = e^{-2t} - e^{-bt} \quad \Rightarrow \quad f'(t) = be^{-bt} - 2e^{-2t}$. $f'(0) = b - 2$, which increases as b increases.

$f'(t) = 0 \quad \Rightarrow \quad be^{-bt} = 2e^{-2t} \quad \Rightarrow \quad \dfrac{b}{2} = e^{(b-2)t} \quad \Rightarrow \quad \ln \dfrac{b}{2} = (b-2)t \quad \Rightarrow \quad t = t_1 = \dfrac{\ln b - \ln 2}{b - 2}$, which decreases as b increases (the maximum is getting closer to the y-axis). $f(t_1) = \dfrac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b, which is always positive.

37. If $c < 0$, then $\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} xe^{-cx} = \lim_{x\to-\infty} \frac{x}{e^{cx}} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{1}{ce^{cx}} = 0$, and $\displaystyle\lim_{x\to\infty} f(x) = \infty$.

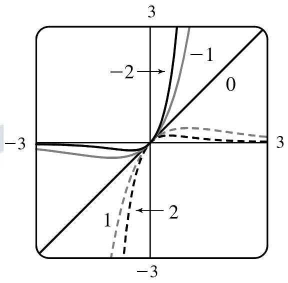
If $c > 0$, then $\displaystyle\lim_{x\to-\infty} f(x) = -\infty$, and $\displaystyle\lim_{x\to\infty} f(x) \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\displaystyle\lim_{x\to\pm\infty} f(x) = \pm\infty$, respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \quad \Rightarrow$

$f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \quad \Leftrightarrow \quad x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$

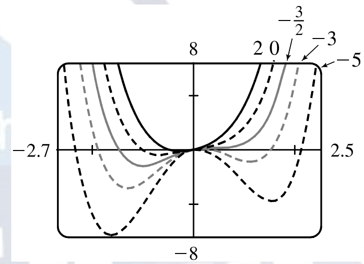
$f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



38. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative:

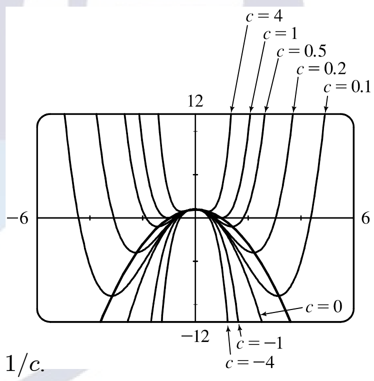
$f(x) = x^4 + cx^2 + x \quad \Rightarrow \quad f'(x) = 4x^3 + 2cx + 1 \quad \Rightarrow \quad f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$.

After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$. To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$. This has a double root at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.



39. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \to 0$, the minimum points spread apart and move downward; they are below the x-axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

(b) $f'(x) = 4cx^3 - 4x = 4cx(x^2 - 1/c)$ $[c \neq 0]$. If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''\left(\pm 1/\sqrt{c}\right) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f\left(\pm 1/\sqrt{c}\right) = c(1/c^2) - 2/c + 1 = -1/c + 1$. But $\left(\pm 1/\sqrt{c}, -1/c + 1\right)$ lies on $y = 1 - x^2$ since $1 - \left(\pm 1/\sqrt{c}\right)^2 = 1 - 1/c$.



40. (a) $f(x) = 2x^3 + cx^2 + 2x \quad \Rightarrow \quad f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$. $f'(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{-c \pm \sqrt{c^2 - 12}}{6}$.

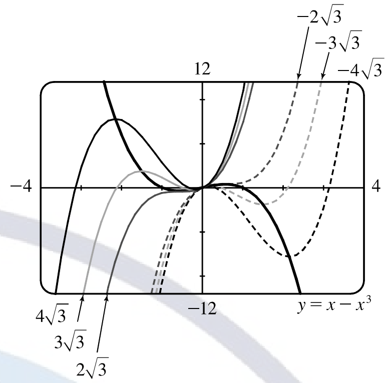
So f has critical points $\quad \Leftrightarrow \quad c^2 - 12 \geq 0 \quad \Leftrightarrow \quad |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \Rightarrow$ $3x_0^2 + cx_0 + 1 = 0 \Leftrightarrow c = \dfrac{-1 - 3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2\left(\frac{-1 - 3x_0^2}{x_0}\right) + 2x_0$$

$$= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3$$

So the point is $(x_0, y_0) = (x_0, x_0 - x_0^3)$; that is, the point lies on the curve $y = x - x^3$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \Leftrightarrow x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

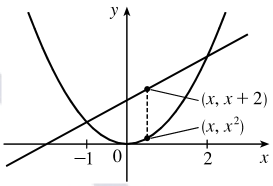
Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \Rightarrow x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

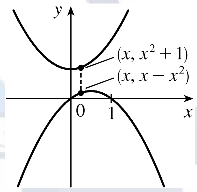
3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

4. Call the two numbers x and y. Then $x + y = 16$, so $y = 16 - x$. Call the sum of their squares S. Then $S = x^2 + y^2 = x^2 + (16 - x)^2 \Rightarrow S' = 2x + 2(16 - x)(-1) = 2x - 32 + 2x = 4x - 32$. $S' = 0 \Rightarrow x = 8$. Since $S'(x) < 0$ for $0 < x < 8$ and $S'(x) > 0$ for $x > 8$, there is an absolute minimum at $x = 8$. Thus, $y = 16 - 8 = 8$ and $S = 8^2 + 8^2 = 128$.

5. Let the vertical distance be given by $v(x) = (x + 2) - x^2$, $-1 \le x \le 2$. $v'(x) = 1 - 2x = 0 \Leftrightarrow x = \frac{1}{2}$. $v(-1) = 0$, $v\left(\frac{1}{2}\right) = \frac{9}{4}$, and $v(2) = 0$, so there is an absolute maximum at $x = \frac{1}{2}$. The maximum distance is $v\left(\frac{1}{2}\right) = \frac{1}{2} + 2 - \frac{1}{4} = \frac{9}{4}$.



6. Let the vertical distance be given by $v(x) = (x^2 + 1) - (x - x^2) = 2x^2 - x + 1$. $v'(x) = 4x - 1 = 0 \Leftrightarrow x = \frac{1}{4}$. $v'(x) < 0$ for $x < \frac{1}{4}$ and $v'(x) > 0$ for $x > \frac{1}{4}$, so there is an absolute minimum at $x = \frac{1}{4}$. The minimum distance is $v\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{1}{4} + 1 = \frac{7}{8}$.



7. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625$ m^2. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

8. If the rectangle has dimensions x and y, then its area is $xy = 1000$ m^2, so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

9. We need to maximize Y for $N \ge 0$. $Y(N) = \dfrac{kN}{1 + N^2} \Rightarrow$

$$Y'(N) = \frac{(1 + N^2)k - kN(2N)}{(1 + N^2)^2} = \frac{k(1 - N^2)}{(1 + N^2)^2} = \frac{k(1 + N)(1 - N)}{(1 + N^2)^2}.$$

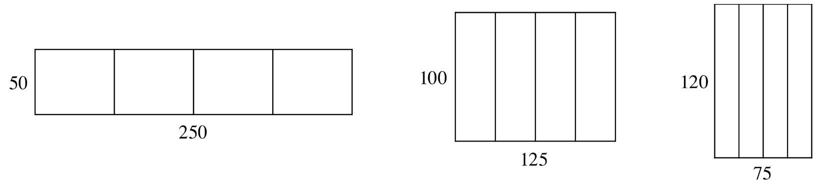
$Y'(N) > 0$ for $0 < N < 1$ and $Y'(N) < 0$ for $N > 1$. Thus, Y has an absolute maximum of $Y(1) = \frac{1}{2}k$ at $N = 1$.

10. We need to maximize P for $I \ge 0$. $P(I) = \dfrac{100I}{I^2 + I + 4} \Rightarrow$

$$P'(I) = \frac{(I^2 + I + 4)(100) - 100I(2I + 1)}{(I^2 + I + 4)^2} = \frac{100(I^2 + I + 4 - 2I^2 - I)}{(I^2 + I + 4)^2} = \frac{-100(I^2 - 4)}{(I^2 + I + 4)^2} = \frac{-100(I + 2)(I - 2)}{(I^2 + I + 4)^2}.$$

$P'(I) > 0$ for $0 < I < 2$ and $P'(I) < 0$ for $I > 2$. Thus, P has an absolute maximum of $P(2) = 20$ at $I = 2$.

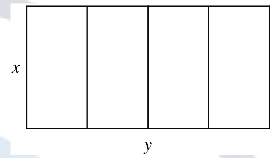
11. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



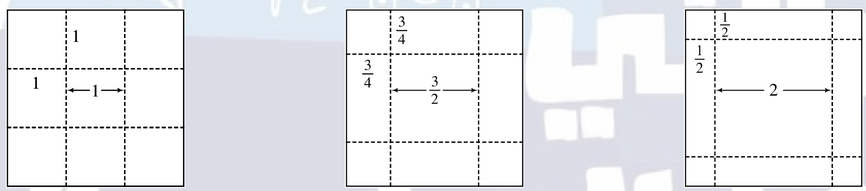
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

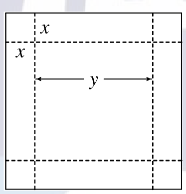
(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5\ \text{ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

12. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume $V = \text{length} \times \text{width} \times \text{height} \quad \Rightarrow \quad V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3 \quad \Rightarrow \quad x + y + x = 3 \quad \Rightarrow \quad y + 2x = 3$

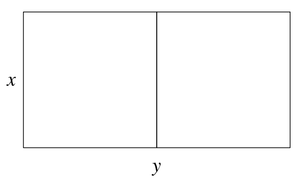
(e) $y + 2x = 3 \quad \Rightarrow \quad y = 3 - 2x \quad \Rightarrow \quad V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2 \quad \Rightarrow$

$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3)$,

so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2\ \text{ft}^3$, which is the value found from our third figure in part (a).

13.



$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$.

$F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

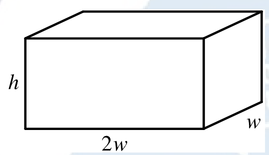
14. Let b be the length of the base of the box and h the height. The volume is $32{,}000 = b^2 h \quad \Rightarrow \quad h = 32{,}000/b^2$.

The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32{,}000/b^2)b = b^2 + 4(32{,}000)/b$.

So $S'(b) = 2b - 4(32{,}000)/b^2 = 2\left(b^3 - 64{,}000\right)/b^2 = 0 \quad \Leftrightarrow \quad b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

15. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad \Rightarrow \quad h = (1200 - b^2)/(4b)$. The volume is $V = b^2 h = b^2(1200 - b^2)/4b = 300b - b^3/4 \quad \Rightarrow \quad V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \quad \Rightarrow \quad 300 = \frac{3}{4}b^2 \quad \Rightarrow \quad b^2 = 400 \quad \Rightarrow \quad b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 328). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000 \text{ cm}^3$.

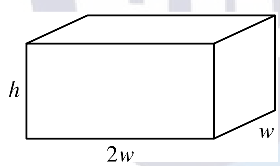
16. 

$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$.

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

$C(w) = 20w^2 + 36w\left(5/w^2\right) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = (40w^3 - 180)/w^2 = 40\left(w^3 - \frac{9}{2}\right)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. The minimum cost is $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \dfrac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

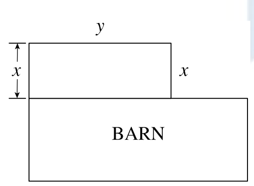
17. 

$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$.

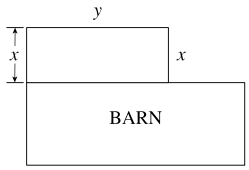
The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] + 6(2w^2) = 32w^2 + 36wh$, so

$C(w) = 32w^2 + 36w(5/w^2) = 32w^2 + 180/w$.

$C'(w) = 64w - 180/w^2 = (64w^3 - 180)/w^2 = 4(16w^3 - 45)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{45}{16}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{45}{16}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32\left(\sqrt[3]{\frac{45}{16}}\right)^2 + \dfrac{180}{\sqrt[3]{45/16}} \approx \$191.28$.

18. 

See the figure. The fencing cost \$20 per linear foot to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(20x) + 20y + 20x = 20y + 30x$. The area A will be maximized when $C = 5000$, so $5000 = 20y + 30x \quad \Leftrightarrow \quad 20y = 5000 - 30x \quad \Leftrightarrow$ $y = 250 - \frac{3}{2}x$. Now $A = xy = x\left(250 - \frac{3}{2}x\right) = 250x - \frac{3}{2}x^2 \quad \Rightarrow \quad A' = 250 - 3x$. $\quad A' = 0 \quad \Leftrightarrow \quad x = \frac{250}{3}$ and since $A'' = -3 < 0$, we have a maximum for A when $x = \frac{250}{3}$ ft and $y = 250 - \frac{3}{2}\left(\frac{250}{3}\right) = 125$ ft. [The maximum area is $125\left(\frac{250}{3}\right) = 10{,}416.\overline{6} \text{ ft}^2$.]

19. 

See the figure. The fencing cost \$20 per linear foot to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(20x) + 20y + 20x = 20y + 30x$. The area A to be enclosed is 8000 ft^2, so $A = xy = 8000 \Rightarrow y = \dfrac{8000}{x}$.

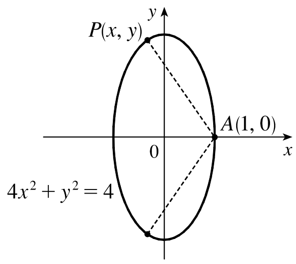
Now $C = 20y + 30x = 20\left(\dfrac{8000}{x}\right) + 30x = \dfrac{160{,}000}{x} + 30x \Rightarrow C' = -\dfrac{160{,}000}{x^2} + 30$. $C' = 0 \Leftrightarrow$ $30 = \dfrac{160{,}000}{x^2} \Leftrightarrow x^2 = \dfrac{16{,}000}{3} \Rightarrow x = \sqrt{\dfrac{16{,}000}{3}} = 40\sqrt{\dfrac{10}{3}} = \dfrac{40}{3}\sqrt{30}$. Since $C'' = \dfrac{320{,}000}{x^3} > 0$ [for $x > 0$], we have a minimum for C when $x = \dfrac{40}{3}\sqrt{30}$ ft and $y = \dfrac{8000}{x} = \dfrac{8000}{40} \cdot \dfrac{3}{\sqrt{30}} \cdot \dfrac{\sqrt{30}}{\sqrt{30}} = 20\sqrt{30}$ ft. [The minimum cost is $20(20\sqrt{30}) + 30\left(\frac{40}{3}\sqrt{30}\right) = 800\sqrt{30} \approx \$4381.78$.]

20. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2\left(x^2 - A\right)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \Rightarrow 2y = p - 2x \Rightarrow y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $A'(x) = 0 \Rightarrow \frac{1}{2}p - 2x = 0 \Rightarrow 2x = \frac{1}{2}p \Rightarrow x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum for A when $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

21. The distance d from the origin $(0,0)$ to a point $(x, 2x+3)$ on the line is given by $d = \sqrt{(x-0)^2 + (2x+3-0)^2}$ and the square of the distance is $S = d^2 = x^2 + (2x+3)^2$. $S' = 2x + 2(2x+3)2 = 10x + 12$ and $S' = 0 \Leftrightarrow x = -\frac{6}{5}$. Now $S'' = 10 > 0$, so we know that S has a minimum at $x = -\frac{6}{5}$. Thus, the y-value is $2\left(-\frac{6}{5}\right) + 3 = \frac{3}{5}$ and the point is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

22. The distance d from the point $(3,0)$ to a point $(x, \sqrt{x}\,)$ on the curve is given by $d = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$ and the square of the distance is $S = d^2 = (x-3)^2 + x$. $S' = 2(x-3) + 1 = 2x - 5$ and $S' = 0 \Leftrightarrow x = \frac{5}{2}$. Now $S'' = 2 > 0$, so we know that S has a minimum at $x = \frac{5}{2}$. Thus, the y-value is $\sqrt{\frac{5}{2}}$ and the point is $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$.

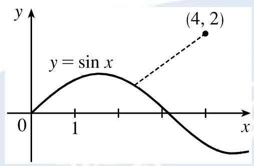
23. 

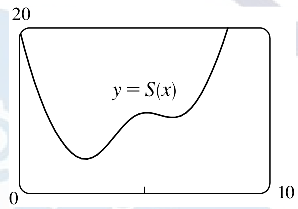
From the figure, we see that there are two points that are farthest away from $A(1,0)$. The distance d from A to an arbitrary point $P(x,y)$ on the ellipse is $d = \sqrt{(x-1)^2 + (y-0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \le x \le 1$, $S(-1) = 4$,

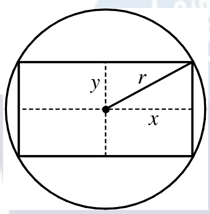
$S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are

$y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

24. The distance d from the point $(4, 2)$ to a point $(x, \sin x)$ on the curve is given by $d = \sqrt{(x-4)^2 + (\sin x - 2)^2}$ and the square of the distance is $S = d^2 = (x-4)^2 + (\sin x - 2)^2$. $S' = 2(x-4) + 2(\sin x - 2)\cos x$. Using a calculator, it is clear that S has a minimum between 0 and 5, and from a graph of S', we find that $S' = 0 \;\Rightarrow\; x \approx 2.65$, so the point is about $(2.65, 0.47)$.





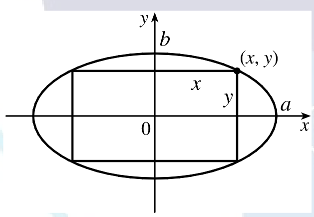
25. 

The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now

$A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\,\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is

$x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

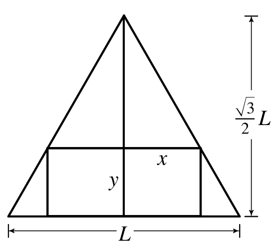
26. 

The area of the rectangle is $(2x)(2y) = 4xy$. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives

$y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\dfrac{b}{a}x\sqrt{a^2 - x^2}$.

$$A'(x) = \frac{4b}{a}\left[x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2} \cdot 1\right]$$
$$= \frac{4b}{a}(a^2 - x^{2-1/2}[-x^2 + a^2 - x^2] = \frac{4b}{a\sqrt{a^2 - x^2}}[a^2 - 2x^2]$$

So the critical number is $x = \frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right) = 2ab$.

27. 

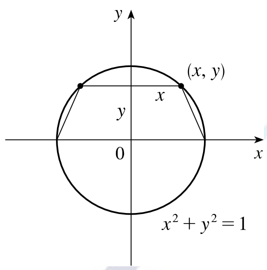
The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2 + (L/2)^2 = L^2 \;\Rightarrow\; h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \;\Rightarrow$

$h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \;\Rightarrow$

$\sqrt{3}\,x = \frac{\sqrt{3}}{2}L - y \;\Rightarrow\; y = \frac{\sqrt{3}}{2}L - \sqrt{3}\,x \;\Rightarrow\; y = \frac{\sqrt{3}}{2}(L - 2x)$.

[continued]

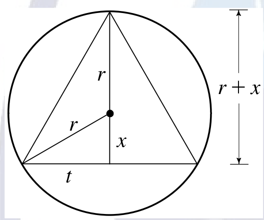
The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \quad \Rightarrow \quad x = \sqrt{3}\,L/(4\sqrt{3}) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

28. 

The area A of a trapezoid is given by $A = \frac{1}{2}h(B + b)$. From the diagram, $h = y$, $B = 2$, and $b = 2x$, so $A = \frac{1}{2}y(2 + 2x) = y(1 + x)$. Since it's easier to substitute for y^2, we'll let $T = A^2 = y^2(1 + x)^2 = (1 - x^2)(1 + x)^2$. Now

$$\begin{aligned} T' &= (1 - x^2)2(1 + x) + (1 + x)^2(-2x) = -2(1 + x)[-(1 - x^2) + (1 + x)x] \\ &= -2(1 + x)(2x^2 + x - 1) = -2(1 + x)(2x - 1)(x + 1) \end{aligned}$$

$T' = 0 \quad \Leftrightarrow \quad x = -1$ or $x = \frac{1}{2}$. $T' > 0$ if $x < \frac{1}{2}$ and $T' < 0$ if $x > \frac{1}{2}$, so we get a maximum at $x = \frac{1}{2}$ $[x = -1$ gives us $A = 0]$. Thus, $y = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ and the maximum area is $A = y(1 + x) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$.

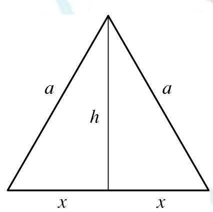
29. 

The area of the triangle is

$A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}(r + x)$. Then

$$\begin{aligned} 0 = A'(x) &= r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}} \\ &= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \quad \Rightarrow \end{aligned}$$

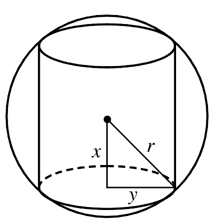
$\dfrac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \quad \Rightarrow \quad x^2 + rx = r^2 - x^2 \quad \Rightarrow \quad 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \quad \Rightarrow$ $x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \quad \Rightarrow \quad$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

30. 

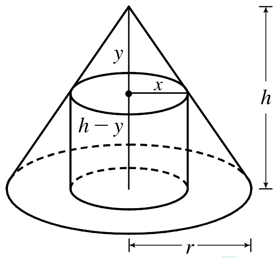
From the figure, we have $x^2 + h^2 = a^2 \quad \Rightarrow \quad h = \sqrt{a^2 - x^2}$. The area of the isosceles triangle is $A = \frac{1}{2}(2x)h = xh = x\sqrt{a^2 - x^2}$ with $0 \le x \le a$. Now

$$\begin{aligned} A' &= x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2}(1) \\ &= (a^2 - x^2)^{-1/2}[-x^2 + (a^2 - x^2)] = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \end{aligned}$$

$A' = 0 \quad \Leftrightarrow \quad x^2 = \frac{1}{2}a^2 \quad \Rightarrow \quad x = a/\sqrt{2}$. Since $A(0) = 0$, $A(a) = 0$, and $A(a/\sqrt{2}) = (a/\sqrt{2})\sqrt{a^2/2} = \frac{1}{2}a^2$, we see that $x = a/\sqrt{2}$ gives us the maximum area and the length of the base is $2x = 2(a/\sqrt{2}) = \sqrt{2}\,a$. Note that the triangle has sides a, a, and $\sqrt{2}\,a$, which form a *right* triangle, with the right angle between the two sides of equal length.

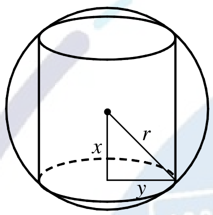
31. 

The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2$, so $V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$. $V'(x) = 2\pi\left(r^2 - 3x^2\right) = 0 \quad \Rightarrow \quad x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right) = \pi(r^2 - r^2/3)\left(2r/\sqrt{3}\right) = 4\pi r^3/\left(3\sqrt{3}\right)$.

32. 

By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$. So $V'(x) = 0 \quad \Rightarrow \quad x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is

$$\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\tfrac{2}{3}r\right)^2 h\left(1 - \tfrac{2}{3}\right) = \tfrac{4}{27}\pi r^2 h.$$

33. 

The cylinder has surface area

$$2(\text{area of the base}) + (\text{lateral surface area}) = 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height})$$
$$= 2\pi y^2 + 2\pi y(2x)$$

Now $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2 \quad \Rightarrow \quad y = \sqrt{r^2 - x^2}$, so the surface area is

$$S(x) = 2\pi(r^2 - x^2) + 4\pi x\sqrt{r^2 - x^2}, \quad 0 \le x \le r$$
$$= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\right)$$

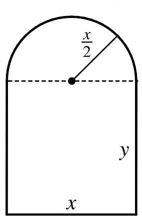
Thus,

$$S'(x) = 0 - 4\pi x + 4\pi\left[x \cdot \tfrac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1\right]$$
$$= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}$$

$S'(x) = 0 \quad \Rightarrow \quad x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \quad \Rightarrow \quad \left(x\sqrt{r^2 - x^2}\right)^2 = (r^2 - 2x^2)^2 \quad \Rightarrow$
$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad 5x^4 - 5r^2x^2 + r^4 = 0.$

This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy $(\star)$. [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\, r$. Since $S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10}r^2 \quad \Rightarrow \quad y^2 = r^2 - \frac{5 - \sqrt{5}}{10}r^2 = \frac{5 + \sqrt{5}}{10}r^2 \quad \Rightarrow$ the surface area is

$$2\pi\left(\tfrac{5 + \sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\tfrac{5 - \sqrt{5}}{10}}\sqrt{\tfrac{5 + \sqrt{5}}{10}}\,r^2 = \pi r^2\left[2 \cdot \tfrac{5+\sqrt{5}}{10} + 4\tfrac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right] = \pi r^2\left[\tfrac{5+\sqrt{5}}{5} + \tfrac{2\sqrt{20}}{5}\right]$$
$$= \pi r^2\left[\tfrac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\tfrac{5+5\sqrt{5}}{5}\right] = \pi r^2\left(1 + \sqrt{5}\right).$$

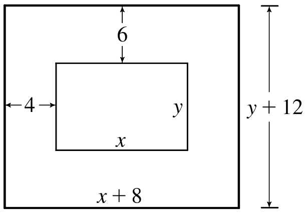
34. 

Perimeter $= 30 \quad \Rightarrow \quad 2y + x + \pi\left(\dfrac{x}{2}\right) = 30 \quad \Rightarrow$

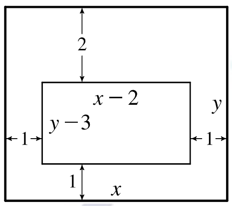
$y = \dfrac{1}{2}\left(30 - x - \dfrac{\pi x}{2}\right) = 15 - \dfrac{x}{2} - \dfrac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$, so $A(x) = x\left(15 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \quad \Rightarrow \quad x = \dfrac{15}{1 + \pi/4} = \dfrac{60}{4 + \pi}$. $A''(x) = -\left(1 + \dfrac{\pi}{4}\right) < 0$, so this gives a maximum.

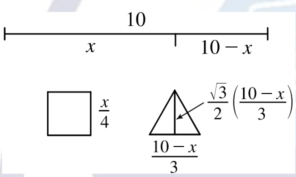
The dimensions are $x = \dfrac{60}{4 + \pi}$ ft and $y = 15 - \dfrac{30}{4 + \pi} - \dfrac{15\pi}{4 + \pi} = \dfrac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \dfrac{30}{4 + \pi}$ ft, so the height of the rectangle is half the base.

35. 

$xy = 384 \Rightarrow y = 384/x$. Total area is
$A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so
$A'(x) = 12(1 - 256/x^2) = 0 \Rightarrow x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$.
When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

36. 

$xy = 180$, so $y = 180/x$. The printed area is
$(x - 2)(y - 3) = (x - 2)(180/x - 3) = 186 - 3x - 360/x = A(x)$.
$A'(x) = -3 + 360/x^2 = 0$ when $x^2 = 120 \Rightarrow x = 2\sqrt{30}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and $A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

37. 

Let x be the length of the wire used for the square. The total area is

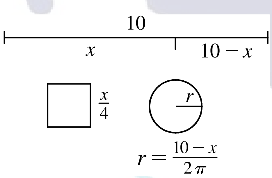
$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2,\ 0 \le x \le 10$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10 - x) = 0 \Leftrightarrow \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \Leftrightarrow x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}$.

Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

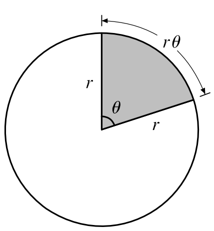
(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

38. 

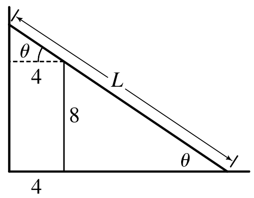
Total area is $A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi}$, $0 \le x \le 10$.

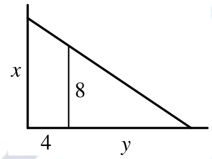
$$A'(x) = \frac{x}{8} - \frac{10-x}{2\pi} = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{5}{\pi} = 0 \Rightarrow x = 40/(4 + \pi).$$

$A(0) = 25/\pi \approx 7.96$, $A(10) = 6.25$, and $A(40/(4 + \pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4 + \pi)$ m.

39. 

From the figure, the perimeter of the slice is $2r + r\theta = 32$, so $\theta = \dfrac{32 - 2r}{r}$. The area A of the slice is $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\dfrac{32-2r}{r}\right) = r(16 - r) = 16r - r^2$ for $0 \le r \le 16$. $A'(r) = 16 - 2r$, so $A' = 0$ when $r = 8$. Since $A(0) = 0$, $A(16) = 0$, and $A(8) = 64\text{ in.}^2$, the largest piece comes from a pizza with radius 8 in. and diameter 16 in. Note that $\theta = 2$ radians $\approx 114.6°$, which is about 32% of the whole pizza.

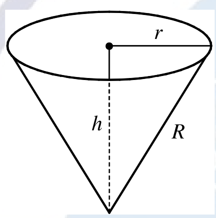
40. 



$L = 8\csc\theta + 4\sec\theta$, $0 < \theta < \frac{\pi}{2}$, $\dfrac{dL}{d\theta} = -8\csc\theta\,\cot\theta + 4\sec\theta\,\tan\theta = 0$ when $\sec\theta\,\tan\theta = 2\csc\theta\,\cot\theta \quad\Leftrightarrow\quad \tan^3\theta = 2 \quad\Leftrightarrow\quad \tan\theta = \sqrt[3]{2} \quad\Leftrightarrow\quad \theta = \tan^{-1}\sqrt[3]{2}$. $dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length

$$L = 8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65 \text{ ft.}$$

Another method: Minimize $L^2 = x^2 + (4+y)^2$, where $\dfrac{x}{4+y} = \dfrac{8}{y}$.

41. 

$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2h - h^3)$. $V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is $V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3$.

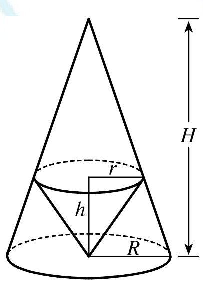
42. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \quad\Rightarrow\quad \frac{1}{3}\pi r^2 h = 27 \quad\Rightarrow\quad r^2 = \dfrac{81}{\pi h}$ **(1)**.

$A = \pi^2 r^2(r^2 + h^2) = \pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h} + h^2\right) = \dfrac{81^2}{h^2} + 81\pi h$, so $A' = 0 \quad\Rightarrow\quad \dfrac{-2\cdot 81^2}{h^3} + 81\pi = 0 \quad\Rightarrow$

$81\pi = \dfrac{2\cdot 81^2}{h^3} \quad\Rightarrow\quad h^3 = \dfrac{162}{\pi} \quad\Rightarrow\quad h = \sqrt[3]{\dfrac{162}{\pi}} = 3\sqrt[3]{\dfrac{6}{\pi}} \approx 3.722$. From **(1)**, $r^2 = \dfrac{81}{\pi h} = \dfrac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \dfrac{27}{\sqrt[3]{6\pi^2}} \quad\Rightarrow$

$r = \dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$. $A'' = 6\cdot 81^2/h^4 > 0$, so A and hence S has an absolute minimum at these values of r and h.

43. 

By similar triangles, $\dfrac{H}{R} = \dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h. $\dfrac{Hr}{R} = H - h \quad\Rightarrow$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R}(R - r) \quad \textbf{(2)}.$$

Thus, $V(r) = \dfrac{\pi}{3}r^2\cdot\dfrac{H}{R}(R - r) = \dfrac{\pi H}{3R}(Rr^2 - r^3) \quad\Rightarrow$

$$V'(r) = \frac{\pi H}{3R}(2Rr - 3r^2) = \frac{\pi H}{3R}r(2R - 3r).$$

$V'(r) = 0 \quad\Rightarrow\quad r = 0$ or $2R = 3r \quad\Rightarrow\quad r = \frac{2}{3}R$ and from **(2)**, $h = \dfrac{H}{R}\left(R - \frac{2}{3}R\right) = \dfrac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

44. We need to minimize F for $0 \le \theta < \pi/2$. $F(\theta) = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \;\Rightarrow\; F'(\theta) = \dfrac{-\mu W\,(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$ [by the Reciprocal Rule]. $F'(\theta) > 0 \;\Rightarrow\; \mu\cos\theta - \sin\theta < 0 \;\Rightarrow\; \mu\cos\theta < \sin\theta \;\Rightarrow\; \mu < \tan\theta \;\Rightarrow\; \theta > \tan^{-1}\mu$. So F is decreasing on $\left(0, \tan^{-1}\mu\right)$ and increasing on $\left(\tan^{-1}\mu, \frac{\pi}{2}\right)$. Thus, F attains its minimum value at $\theta = \tan^{-1}\mu$. This maximum value is $F(\tan^{-1}\mu) = \dfrac{\mu W}{\sqrt{\mu^2+1}}$.

45. $P(R) = \dfrac{E^2 R}{(R+r)^2} \;\Rightarrow$

$$P'(R) = \frac{(R+r)^2 \cdot E^2 - E^2 R \cdot 2(R+r)}{[(R+r)^2]^2} = \frac{(R^2 + 2Rr + r^2)E^2 - 2E^2R^2 - 2E^2Rr}{(R+r)^4}$$
$$= \frac{E^2r^2 - E^2R^2}{(R+r)^4} = \frac{E^2(r^2 - R^2)}{(R+r)^4} = \frac{E^2(r+R)(r-R)}{(R+r)^4} = \frac{E^2(r-R)}{(R+r)^3}$$

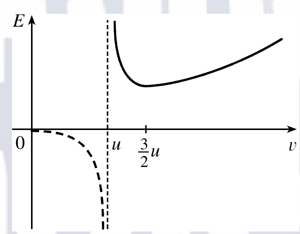
$P'(R) = 0 \;\Rightarrow\; R = r \;\Rightarrow\; P(r) = \dfrac{E^2 r}{(r+r)^2} = \dfrac{E^2 r}{4r^2} = \dfrac{E^2}{4r}$.

The expression for $P'(R)$ shows that $P'(R) > 0$ for $R < r$ and $P'(R) < 0$ for $R > r$. Thus, the maximum value of the power is $E^2/(4r)$, and this occurs when $R = r$.

46. (a) $E(v) = \dfrac{aLv^3}{v-u} \;\Rightarrow\; E'(v) = aL\dfrac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$ when $2v^3 = 3uv^2 \;\Rightarrow\; 2v = 3u \;\Rightarrow\; v = \frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E.

(b)

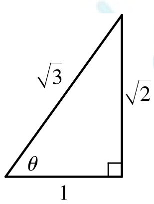


47. $S = 6sh - \frac{3}{2}s^2\cot\theta + 3s^2\frac{\sqrt{3}}{2}\csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - 3s^2\frac{\sqrt{3}}{2}\csc\theta\,\cot\theta$ or $\frac{3}{2}s^2\csc\theta\left(\csc\theta - \sqrt{3}\cot\theta\right)$.

(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \;\Rightarrow\; \dfrac{1}{\sin\theta} - \sqrt{3}\dfrac{\cos\theta}{\sin\theta} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

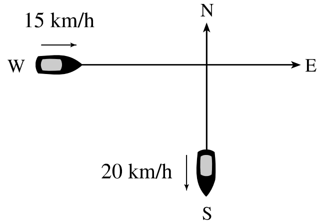
(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \tfrac{3}{2}s^2\tfrac{1}{\sqrt{2}} + 3s^2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{3}}{\sqrt{2}} = 6sh - \tfrac{3}{2\sqrt{2}}s^2 + \tfrac{9}{2\sqrt{2}}s^2$$
$$= 6sh + \tfrac{6}{2\sqrt{2}}s^2 = 6s\left(h + \tfrac{1}{2\sqrt{2}}s\right)$$

48.



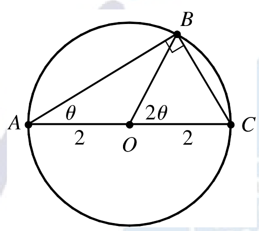
Let t be the time, in hours, after 2:00 PM. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize $f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.

$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when $t = \frac{450}{1250} = 0.36$ h.

$0.36 \text{ h} \times \frac{60 \text{ min}}{\text{h}} = 21.6 \text{ min} = 21 \text{ min } 36 \text{ s}$. Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 PM.

49. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5 \quad \Rightarrow \quad T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0 \quad \Leftrightarrow \quad 8x = 6\sqrt{x^2+25} \quad \Leftrightarrow$ $16x^2 = 9(x^2+25) \quad \Leftrightarrow \quad x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, he should row directly to B.

50.



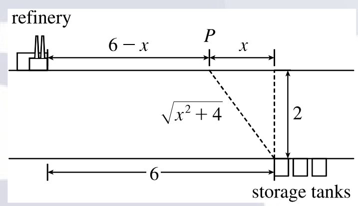
In isosceles triangle AOB, $\angle O = 180° - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by $T(\theta) = \dfrac{4\cos\theta}{2} + \dfrac{4\theta}{4} = 2\cos\theta + \theta$, $0 \le \theta \le \frac{\pi}{2}$.

$T'(\theta) = -2\sin\theta + 1 = 0 \quad \Leftrightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$.

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$.

$T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

51. There are $(6-x)$ km over land and $\sqrt{x^2+4}$ km under the river.



We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6-x)(4) + \left(\sqrt{x^2+4}\right)(8) \quad \Rightarrow$

$C'(x) = -4 + 8 \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0 \quad \Rightarrow \quad 4 = \dfrac{8x}{\sqrt{x^2+4}} \quad \Rightarrow \quad \sqrt{x^2+4} = 2x \quad \Rightarrow \quad x^2 + 4 = 4x^2 \quad \Rightarrow \quad 4 = 3x^2 \quad \Rightarrow \quad x^2 = \frac{4}{3} \quad \Rightarrow$

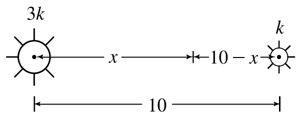
$x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0$, $2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$, $C\left(2/\sqrt{3}\right) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

52. The distance from the refinery to P is now $\sqrt{(6-x)^2 + 1^2} = \sqrt{x^2 - 12x + 37}$.

Thus, $C(x) = 4\sqrt{x^2 - 12x + 37} + 8\sqrt{x^2+4} \quad \Rightarrow$

$C'(x) = 4 \cdot \frac{1}{2}(x^2 - 12x + 37)^{-1/2}(2x - 12) + 8 \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x) = \dfrac{4(x-6)}{\sqrt{x^2 - 12x + 37}} + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0 \quad \Rightarrow \quad x \approx 1.12$ [from a graph of C' or a numerical rootfinder]. $C(0) \approx 40.3$, $C(1.12) \approx 38.3$, and $C(6) \approx 54.6$. So the minimum cost is slightly higher (than in the previous exercise) at about \$3.83 million when P is approximately 4.88 km from the point on the bank 1 km south of the refinery.

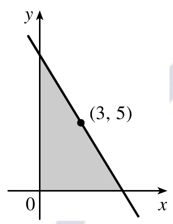
53. 

The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

$$I'(x) = \frac{-6k}{x^3} + \frac{2k}{(10-x)^3} = 0 \quad\Rightarrow\quad 6k(10-x)^3 = 2kx^3 \quad\Rightarrow$$

$3(10-x)^3 = x^3 \quad\Rightarrow\quad \sqrt[3]{3}\,(10-x) = x \quad\Rightarrow\quad 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x \quad\Rightarrow\quad 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \quad\Rightarrow$

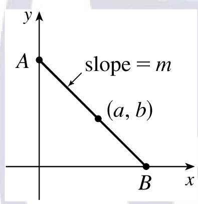
$10\sqrt[3]{3} = \left(1+\sqrt[3]{3}\right)x \quad\Rightarrow\quad x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

54. 

The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x-3)$ or $y = mx + (5 - 3m)$. The y-intercept is $5 - 3m$ and the x-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5-3m)(-5/m+3) = 15 - 25/(2m) - \frac{9}{2}m$. Now

$A'(m) = \dfrac{25}{2m^2} - \dfrac{9}{2} = 0 \quad\Leftrightarrow\quad m^2 = \frac{25}{9} \quad\Rightarrow\quad m = -\frac{5}{3}$ (since $m < 0$).

$A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Thus, an equation of the line is $y - 5 = -\frac{5}{3}(x-3)$ or $y = -\frac{5}{3}x + 10$.

55. 

Every line segment in the first quadrant passing through (a, b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x-a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\left(a - \frac{b}{m}, 0\right)$. The distance d from A to B is given by $d = \sqrt{\left[\left(a - \frac{b}{m}\right) - 0\right]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$S(m) = \left(a - \dfrac{b}{m}\right)^2 + (am-b)^2 = a^2 - \dfrac{2ab}{m} + \dfrac{b^2}{m^2} + a^2m^2 - 2abm + b^2$. Thus,

$$\begin{aligned} S'(m) &= \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3) \\ &= \frac{2}{m^3}[b(am-b) + am^3(am-b)] = \frac{2}{m^3}(am-b)(b+am^3) \end{aligned}$$

Thus, $S'(m) = 0 \quad\Leftrightarrow\quad m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$.

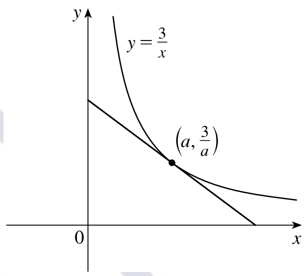
That value is

$$\begin{aligned} S\left(-\sqrt[3]{\tfrac{b}{a}}\right) &= \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2 \\ &= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \end{aligned}$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 \quad [= (x+y)^3]$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

56. $y = 1 + 40x^3 - 3x^5 \quad\Rightarrow\quad y' = 120x^2 - 15x^4$, so the tangent line to the curve at $x = a$ has slope $m(a) = 120a^2 - 15a^4$. Now $m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a+2)(a-2)$, so $m'(a) > 0$ for $a < -2$, and $0 < a < 2$, and

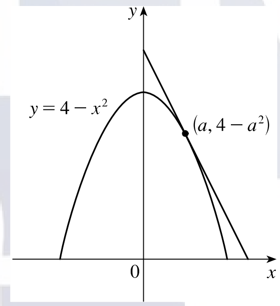
$m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. Clearly, $m(a) \to -\infty$ as $a \to \pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240$. So 240 is the largest slope, and it occurs at the points $(-2, -223)$ and $(2, 225)$. *Note:* $a = 0$ corresponds to a local *minimum* of m.



57. $y = \dfrac{3}{x} \quad \Rightarrow \quad y' = -\dfrac{3}{x^2}$, so an equation of the tangent line at the point $(a, \frac{3}{a})$ is $y - \dfrac{3}{a} = -\dfrac{3}{a^2}(x - a)$, or $y = -\dfrac{3}{a^2}x + \dfrac{6}{a}$. The y-intercept $[x = 0]$ is $6/a$. The x-intercept $[y = 0]$ is $2a$. The distance d of the line segment that has endpoints at the intercepts is $d = \sqrt{(2a - 0)^2 + (0 - 6/a)^2}$. Let $S = d^2$, so $S = 4a^2 + \dfrac{36}{a^2} \quad \Rightarrow$

$S' = 8a - \dfrac{72}{a^3}$. $S' = 0 \quad \Leftrightarrow \quad \dfrac{72}{a^3} = 8a \quad \Leftrightarrow \quad a^4 = 9 \quad \Leftrightarrow \quad a^2 = 3 \quad \Rightarrow \quad a = \sqrt{3}$.

$S'' = 8 + \dfrac{216}{a^4} > 0$, so there is an absolute minimum at $a = \sqrt{3}$. Thus, $S = 4(3) + \frac{36}{3} = 12 + 12 = 24$ and hence, $d = \sqrt{24} = 2\sqrt{6}$.



58. $y = 4 - x^2 \quad \Rightarrow \quad y' = -2x$, so an equation of the tangent line at $(a, 4 - a^2)$ is $y - (4 - a^2) = -2a(x - a)$, or $y = -2ax + a^2 + 4$. The y-intercept $[x = 0]$ is $a^2 + 4$. The x-intercept $[y = 0]$ is $\dfrac{a^2 + 4}{2a}$. The area A of the triangle is

$A = \dfrac{1}{2}(\text{base})(\text{height}) = \dfrac{1}{2} \cdot \dfrac{a^2 + 4}{2a}(a^2 + 4) = \dfrac{1}{4}\dfrac{a^4 + 8a^2 + 16}{a} = \dfrac{1}{4}\left(a^3 + 8a + \dfrac{16}{a}\right)$.

$A' = 0 \quad \Rightarrow \quad \dfrac{1}{4}\left(3a^2 + 8 - \dfrac{16}{a^2}\right) = 0 \quad \Rightarrow \quad 3a^4 + 8a^2 - 16 = 0 \quad \Rightarrow$

$(3a^2 - 4)(a^2 + 4) = 0 \quad \Rightarrow \quad a^2 = \dfrac{4}{3} \quad \Rightarrow \quad a = \dfrac{2}{\sqrt{3}}$. $A'' = \dfrac{1}{4}\left(6a + \dfrac{32}{a^3}\right) > 0$, so there is an absolute minimum at $a = \dfrac{2}{\sqrt{3}}$. Thus, $A = \dfrac{1}{2} \cdot \dfrac{4/3 + 4}{2(2/\sqrt{3})}\left(\dfrac{4}{3} + 4\right) = \dfrac{1}{2} \cdot \dfrac{4\sqrt{3}}{3} \cdot \dfrac{16}{3} = \dfrac{32}{9}\sqrt{3}$.

59. (a) If $c(x) = \dfrac{C(x)}{x}$, then, by the Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when $xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad \Leftrightarrow \quad 2x^{3/2} = 16{,}000 \quad \Leftrightarrow$ $x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate

$c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}\,(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

60. (a) The total profit is $P(x) = R(x) - C(x)$. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if $P'(x) = R'(x) - C'(x) = 0$, then $R'(x) = C'(x)$. Therefore, if the profit is a maximum, then the marginal revenue equals the marginal cost.

(b) $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \quad\Leftrightarrow\quad 1700 - 14x = 500 - 3.2x + 0.012x^2 \quad\Leftrightarrow\quad 0.012x^2 + 10.8x - 1200 = 0 \quad\Leftrightarrow\quad x^2 + 900x - 100{,}000 = 0 \quad\Leftrightarrow\quad (x + 1000)(x - 100) = 0 \quad\Leftrightarrow\quad x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

61. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is $\frac{10 - 8}{27{,}000 - 33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \quad\Rightarrow\quad y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \quad\Rightarrow\quad R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \quad\Rightarrow\quad$ the price is $p(28{,}500) = \$9.50$.

62. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \quad\Leftrightarrow\quad y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \quad\Rightarrow\quad x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \$13$.

63. (a) As in Example 6, we see that the demand function p is linear. We are given that $p(1200) = 350$ and deduce that $p(1280) = 340$, since a \$10 reduction in price increases sales by 80 per week. The slope for p is $\dfrac{340 - 350}{1280 - 1200} = -\dfrac{1}{8}$, so an equation is $p - 350 = -\frac{1}{8}(x - 1200)$ or $p(x) = -\frac{1}{8}x + 500$, where $x \geq 1200$.

4.7.63: added text at end of last line.

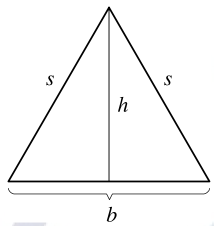
(b) $R(x) = x\,p(x) = -\frac{1}{8}x^2 + 500x$. $\quad R'(x) = -\frac{1}{4}x + 500 = 0$ when $x = 4(500) = 2000$. $\quad p(2000) = 250$, so the price should be set at \$250 to maximize revenue.

(c) $C(x) = 35{,}000 + 120x \quad\Rightarrow\quad P(x) = R(x) - C(x) = -\frac{1}{8}x^2 + 500x - 35{,}000 - 120x = -\frac{1}{8}x^2 + 380x - 35{,}000$. $P'(x) = -\frac{1}{4}x + 380 = 0$ when $x = 4(380) = 1520$. $\quad p(1520) = 310$, so the price should be set at \$310 to maximize profit.

4.7.64: Added condition on w on line 2 (w >= 16).

64. Let w denote the number of operating wells. Then the amount of daily oil production for each well is $240 - 8(w - 16) = 368 - 8w$, where $w \geq 16$. The total daily oil production P for all wells is given by $P(w) = w(368 - 8w) = 368w - 8w^2$. Now $P'(w) = 368 - 16w$ and $P'(w) = 0 \iff w = \frac{368}{16} = 23$. $P''(w) = -16 < 0$, so the daily production is maximized when the company adds $23 - 16 = 7$ wells.

65.



Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$.

Let the perimeter be p, so $2s + b = p$ or $s = (p - b)/2 \Rightarrow$

$A(b) = \frac{1}{2}b\sqrt{(p-b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

$$A'(b) = \frac{\sqrt{p^2 - 2pb}}{4} - \frac{bp/4}{\sqrt{p^2 - 2pb}} = \frac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}.$$

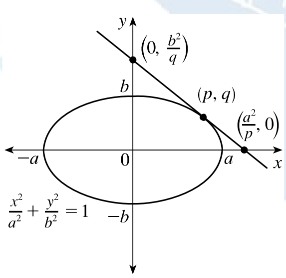
Therefore, $A'(b) = 0 \Rightarrow -3pb + p^2 = 0 \Rightarrow b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \Rightarrow s = b \Rightarrow$ the triangle is equilateral.

66. From Exercise 51, with K replacing 8 for the "under river" cost (measured in \$100,000), we see that $C'(x) = 0 \iff$ $4\sqrt{x^2 + 4} = Kx \iff 16x^2 + 64 = K^2x^2 \iff 64 = (K^2 - 16)x^2 \iff x = \dfrac{8}{\sqrt{K^2 - 16}}$. Also from Exercise 51, we have $C(x) = (6 - x)4 + \sqrt{x^2 + 4}\,K$. We now compare costs for using the minimum distance possible under the river $[x = 0]$ and using the critical number above. $C(0) = 24 + 2K$ and

$$C\left(\frac{8}{\sqrt{K^2-16}}\right) = 24 - \frac{32}{\sqrt{K^2-16}} + \sqrt{\frac{64}{K^2-16} + 4}\,K = 24 - \frac{32}{\sqrt{K^2-16}} + \sqrt{\frac{4K^2}{K^2-16}}\,K$$
$$= 24 - \frac{32}{\sqrt{K^2-16}} + \frac{2K^2}{\sqrt{K^2-16}} = 24 + \frac{2(K^2-16)}{\sqrt{K^2-16}} = 24 + 2\sqrt{K^2-16}$$

Since $\sqrt{K^2 - 16} < K$, we see that $C\left(\dfrac{8}{\sqrt{K^2-16}}\right) < C(0)$ for any cost K, so the minimum distance possible for the "under river" portion of the pipeline should *never* be used.

67. (a)



Using implicit differentiation, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2y\,y'}{b^2} = 0 \Rightarrow$ $\dfrac{2y\,y'}{b^2} = -\dfrac{2x}{a^2} \Rightarrow y' = -\dfrac{b^2x}{a^2y}$. At (p, q), $y' = -\dfrac{b^2p}{a^2q}$, and an equation of the tangent line is $y - q = -\dfrac{b^2p}{a^2q}(x - p) \iff y = -\dfrac{b^2p}{a^2q}x + \dfrac{b^2p^2}{a^2q} + q \iff$ $y = -\dfrac{b^2p}{a^2q}x + \dfrac{b^2p^2 + a^2q^2}{a^2q}$. The last term is the y-intercept, but not the term we want, namely b^2/q. Since (p, q) is on the ellipse, we know $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} = 1$. To use that relationship we must divide b^2p^2 in the y-intercept by a^2b^2, so divide all terms by a^2b^2. $\dfrac{(b^2p^2 + a^2q^2)/a^2b^2}{(a^2q)/a^2b^2} = \dfrac{p^2/a^2 + q^2/b^2}{q/b^2} = \dfrac{1}{q/b^2} = \dfrac{b^2}{q}$. So the tangent line has equation $y = -\dfrac{b^2p}{a^2q}x + \dfrac{b^2}{q}$. Let $y = 0$ and solve for x to find that x-intercept: $\dfrac{b^2p}{a^2q}x = \dfrac{b^2}{q} \iff$ $x = \dfrac{b^2a^2q}{qb^2p} = \dfrac{a^2}{p}$.

(b) The portion of the tangent line cut off by the coordinate axes is the distance between the intercepts, $\left(a^2/p, 0\right)$ and $\left(0, b^2/q\right)$: $\sqrt{\left(\dfrac{a^2}{p}\right)^2 + \left(-\dfrac{b^2}{q}\right)^2} = \sqrt{\dfrac{a^4}{p^2} + \dfrac{b^4}{q^2}}$. To eliminate p or q, we turn to the relationship $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} = 1 \quad \Leftrightarrow$ $\dfrac{q^2}{b^2} = 1 - \dfrac{p^2}{a^2} \quad \Leftrightarrow \quad q^2 = b^2 - \dfrac{b^2 p^2}{a^2} \quad \Leftrightarrow \quad q^2 = \dfrac{b^2(a^2 - p^2)}{a^2}$. Now substitute for q^2 and use the square S of the distance. $\quad S(p) = \dfrac{a^4}{p^2} + \dfrac{b^4 a^2}{b^2(a^2 - p^2)} = \dfrac{a^4}{p^2} + \dfrac{a^2 b^2}{a^2 - p^2}$ for $0 < p < a$. Note that as $p \to 0$ or $p \to a$, $S(p) \to \infty$, so the minimum value of S must occur at a critical number. Now $S'(p) = -\dfrac{2a^4}{p^3} + \dfrac{2a^2 b^2 p}{(a^2 - p^2)^2}$ and $S'(p) = 0 \quad \Leftrightarrow$

$\dfrac{2a^4}{p^3} = \dfrac{2a^2 b^2 p}{(a^2 - p^2)^2} \quad \Leftrightarrow \quad a^2(a^2 - p^2)^2 = b^2 p^4 \quad \Rightarrow \quad a(a^2 - p^2) = bp^2 \quad \Leftrightarrow \quad a^3 = (a+b)p^2 \quad \Leftrightarrow \quad p^2 = \dfrac{a^3}{a+b}$.

Substitute for p^2 in $S(p)$:

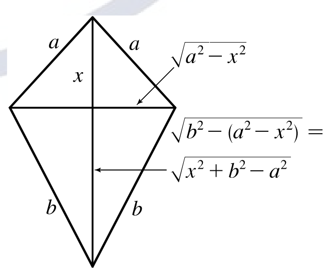
$$\frac{a^4}{\dfrac{a^3}{a+b}} + \frac{a^2 b^2}{a^2 - \dfrac{a^3}{a+b}} = \frac{a^4(a+b)}{a^3} + \frac{a^2 b^2 (a+b)}{a^2(a+b) - a^3} = \frac{a(a+b)}{1} + \frac{a^2 b^2 (a+b)}{a^2 b}$$
$$= a(a+b) + b(a+b) = (a+b)(a+b) = (a+b)^2$$

Taking the square root gives us the desired minimum length of $a + b$.

(c) The triangle formed by the tangent line and the coordinate axes has area $A = \dfrac{1}{2}\left(\dfrac{a^2}{p}\right)\left(\dfrac{b^2}{q}\right)$. As in part (b), we'll use the square of the area and substitute for q^2. $\quad S = \dfrac{a^4 b^4}{4p^2 q^2} = \dfrac{a^4 b^4 a^2}{4p^2 b^2 (a^2 - p^2)} = \dfrac{a^6 b^2}{4p^2(a^2 - p^2)}$. Minimizing S (and hence A) is equivalent to maximizing $p^2(a^2 - p^2)$. Let $f(p) = p^2(a^2 - p^2) = a^2 p^2 - p^4$ for $0 < p < a$. As in part (b), the minimum value of S must occur at a critical number. Now $f'(p) = 2a^2 p - 4p^3 = 2p(a^2 - 2p^2)$. $\quad f'(p) = 0 \quad \Rightarrow$ $p^2 = a^2/2 \quad \Rightarrow \quad p = a/\sqrt{2}$ $[p > 0]$. Substitute for p^2 in $S(p)$: $\dfrac{a^6 b^2}{4\left(\dfrac{a^2}{2}\right)\left(a^2 - \dfrac{a^2}{2}\right)} = \dfrac{a^6 b^2}{a^4} = a^2 b^2 = (ab)^2$. Taking the square root gives us the desired minimum area of ab.

68. See the figure. The area is given by

$$A(x) = \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\,\right)x + \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\,\right)\left(\sqrt{x^2 + b^2 - a^2}\,\right)$$
$$= \sqrt{a^2 - x^2}\left(x + \sqrt{x^2 + b^2 - a^2}\,\right)$$



for $0 \le x \le a$. Now

$$A'(x) = \sqrt{a^2 - x^2}\left(1 + \frac{x}{\sqrt{x^2 + b^2 - a^2}}\right) + \left(x + \sqrt{x^2 + b^2 - a^2}\,\right)\frac{-x}{\sqrt{a^2 - x^2}}$$
$$= 0 \quad \Leftrightarrow$$

$$\frac{x}{\sqrt{a^2-x^2}}\left(x+\sqrt{x^2+b^2-a^2}\,\right)=\sqrt{a^2-x^2}\left(\frac{x+\sqrt{x^2+b^2-a^2}}{\sqrt{x^2+b^2-a^2}}\right).$$

Except for the trivial case where $x=0$, $a=b$ and $A(x)=0$, we have $x+\sqrt{x^2+b^2-a^2}>0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2-x^2}}=\dfrac{\sqrt{a^2-x^2}}{\sqrt{x^2+b^2-a^2}} \Rightarrow x\sqrt{x^2+b^2-a^2}=a^2-x^2 \Rightarrow$

$$x^2(x^2+b^2-a^2)=a^4-2a^2x^2+x^4 \Rightarrow x^2(b^2-a^2)=a^4-2a^2x^2 \Rightarrow x^2(b^2+a^2)=a^4 \Rightarrow x=\frac{a^2}{\sqrt{a^2+b^2}}.$$

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0)=a\sqrt{b^2-a^2}$, $A(a)=0$ and

$$A\!\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)=\sqrt{a^2-\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+b^2-a^2}\,\right]$$

$$=\frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\frac{b^2}{\sqrt{a^2+b^2}}\right]=\frac{ab(a^2+b^2)}{a^2+b^2}=ab$$

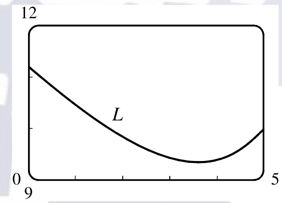
Since $b\ge\sqrt{b^2-a^2}$, $A\!\left(a^2/\sqrt{a^2+b^2}\,\right)\ge A(0)$. So there is an absolute maximum when $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$. In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be $\dfrac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}$.

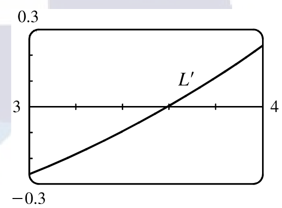
69. Note that $|AD|=|AP|+|PD| \Rightarrow 5=x+|PD| \Rightarrow |PD|=5-x$.

Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

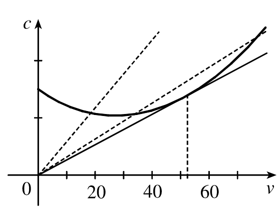
$$L(x)=|AP|+|BP|+|CP|=x+\sqrt{(5-x)^2+2^2}+\sqrt{(5-x)^2+3^2}$$
$$=x+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+34} \Rightarrow$$

$L'(x)=1+\dfrac{x-5}{\sqrt{x^2-10x+29}}+\dfrac{x-5}{\sqrt{x^2-10x+34}}$. From the graphs of L and L', it seems that the minimum value of L is about $L(3.59)=9.35$ m.

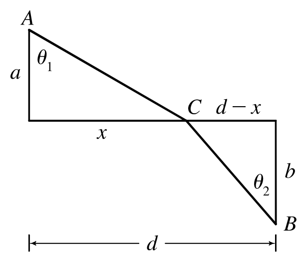




70. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\dfrac{c}{v}=\dfrac{\text{gallons/hour}}{\text{miles/hour}}=\dfrac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\dfrac{dG}{dv}=\dfrac{d}{dv}\left(\dfrac{c}{v}\right)=\dfrac{v\,\dfrac{dc}{dv}-c\,\dfrac{dv}{dv}}{v^2}=\dfrac{v\,\dfrac{dc}{dv}-c}{v^2}$.



This is 0 when $v\,\dfrac{dc}{dv}-c=0 \Leftrightarrow \dfrac{dc}{dv}=\dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v\approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v,c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

71. 

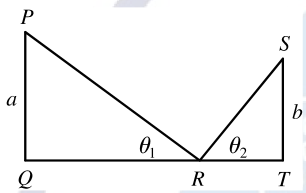
The total time is

$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d - x)^2}}{v_2},\ 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{d - x}{v_2\sqrt{b^2 + (d - x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad \Rightarrow \quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

72. 

If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$.

Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\,\frac{d\theta_2}{d\theta_1}.$$

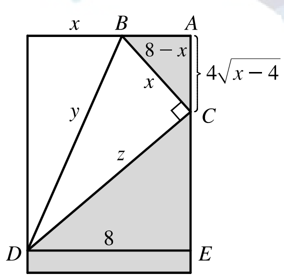
So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\,\dfrac{d\theta_2}{d\theta_1} = 0 \quad \Rightarrow$

$\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \quad\Leftrightarrow\quad -a\csc\theta_1\,\cot\theta_1 + a\frac{\csc^2\theta_1\,\cot\theta_2}{\csc\theta_2} = 0 \quad\Leftrightarrow$$

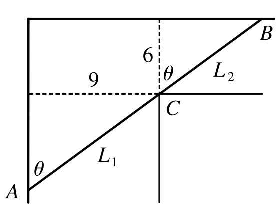
$\cot\theta_1\,\csc\theta_2 = \csc\theta_1\,\cot\theta_2 \quad\Leftrightarrow\quad \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \quad\Leftrightarrow\quad \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

73. 

$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/8 = x/\left(4\sqrt{x-4}\right) \quad\Rightarrow\quad z = 2x/\sqrt{x-4}$. Thus, we minimize $f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4),\ 4 < x \le 8$.

$$f'(x) = \frac{(x-4)(3x^2) - x^3}{(x-4)^2} = \frac{x^2[3(x-4) - x]}{(x-4)^2} = \frac{2x^2(x-6)}{(x-4)^2} = 0$$

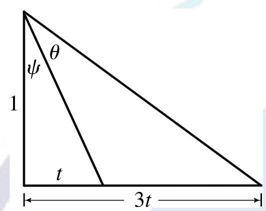
when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

74. 

Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \quad\Rightarrow\quad dL/d\theta = -9\csc\theta\,\cot\theta + 6\sec\theta\,\tan\theta = 0$ when $6\sec\theta\,\tan\theta = 9\csc\theta\,\cot\theta \quad\Leftrightarrow\quad \tan^3\theta = \frac{9}{6} = 1.5 \quad\Leftrightarrow\quad \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.853 \quad\Rightarrow\quad L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.

75. 

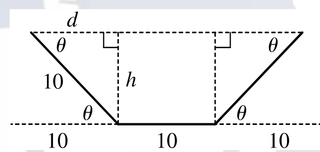
$\theta = (\theta + \psi) - \psi = \arctan\dfrac{3t}{1} - \arctan\dfrac{t}{1} \quad\Rightarrow\quad \theta' = \dfrac{3}{1 + 9t^2} - \dfrac{1}{1 + t^2}$.

$\theta' = 0 \quad\Rightarrow\quad \dfrac{3}{1 + 9t^2} = \dfrac{1}{1 + t^2} \quad\Rightarrow\quad 3 + 3t^2 = 1 + 9t^2 \quad\Rightarrow\quad 2 = 6t^2 \quad\Rightarrow$

$t^2 = \frac{1}{3} \quad\Rightarrow\quad t = 1/\sqrt{3}$. Thus,

$\theta = \arctan 3/\sqrt{3} - \arctan 1/\sqrt{3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

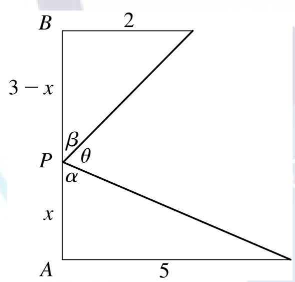
76. We maximize the cross-sectional area



$$\begin{aligned} A(\theta) &= 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\ &= 100(\sin\theta + \sin\theta\,\cos\theta),\ \ 0 \le \theta \le \tfrac{\pi}{2} \end{aligned}$$

$$\begin{aligned} A'(\theta) &= 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1) \\ &= 100(2\cos\theta - 1)(\cos\theta + 1) = 0 \text{ when } \cos\theta = \tfrac{1}{2} \quad\Leftrightarrow\quad \theta = \tfrac{\pi}{3} \qquad [\cos\theta \ne -1 \text{ since } 0 \le \theta \le \tfrac{\pi}{2}.] \end{aligned}$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

77. 

From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3 - x}$. Since

$\alpha + \beta + \theta = 180^\circ = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3 - x}\right) \quad\Rightarrow$

$$\begin{aligned} \frac{d\theta}{dx} &= -\frac{1}{1 + \left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1 + \left(\frac{2}{3 - x}\right)^2}\left[\frac{2}{(3 - x)^2}\right] \\ &= \frac{x^2}{x^2 + 25}\cdot\frac{5}{x^2} - \frac{(3 - x)^2}{(3 - x)^2 + 4}\cdot\frac{2}{(3 - x)^2}. \end{aligned}$$

Now $\dfrac{d\theta}{dx} = 0 \quad\Rightarrow\quad \dfrac{5}{x^2 + 25} = \dfrac{2}{x^2 - 6x + 13} \quad\Rightarrow\quad 2x^2 + 50 = 5x^2 - 30x + 65 \quad\Rightarrow$

$3x^2 - 30x + 15 = 0 \quad\Rightarrow\quad x^2 - 10x + 5 = 0 \quad\Rightarrow\quad x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

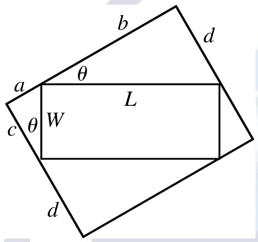
78. Let x be the distance from the observer to the wall. Then, from the given figure,

$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right), \quad x > 0 \quad \Rightarrow$$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{d[x^2+(h+d)^2]-(h+d)(x^2+d^2)}{[x^2+(h+d)^2](x^2+d^2)} = \frac{h^2d+hd^2-hx^2}{[x^2+(h+d)^2](x^2+d^2)} = 0 \quad \Leftrightarrow$$

$hx^2 = h^2d + hd^2 \quad \Leftrightarrow \quad x^2 = hd + d^2 \quad \Leftrightarrow \quad x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

79. 

In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\ &= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta \\ &= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\,\cos\theta \\ &= LW(\sin^2\theta + \cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta = LW + \tfrac{1}{2}(L^2+W^2)\sin 2\theta,\ 0 \le \theta \le \tfrac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \quad \Leftrightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow$ $\theta = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2+2LW+W^2) = \frac{1}{2}(L+W)^2$.

80. (a) Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \quad \Rightarrow \quad |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a-|AB|}{|BC|} \quad \Rightarrow \quad |BC| = (a-|AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a-|AB|)\sec\theta = b\csc\theta \quad \Rightarrow \quad b\cot\theta = a - |AB| \quad \Rightarrow \quad |AB| = a - b\cot\theta$. The total resistance is

$$R(\theta) = C\frac{|AB|}{r_1^4} + C\frac{|BC|}{r_2^4} = C\left(\frac{a-b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right).$$

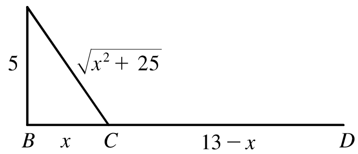
(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\,\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right).$

$R'(\theta) = 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \quad \Leftrightarrow \quad \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta.$

$R'(\theta) > 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \quad \Rightarrow \quad \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79^\circ$.

81. (a)



If $k =$ energy/km over land, then energy/km over water $= 1.4k$.

So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x),\ 0 \le x \le 13$,

and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \quad\Rightarrow\quad 1.96x^2 = x^2+25 \quad\Rightarrow\quad 0.96x^2 = 25 \quad\Rightarrow\quad x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$.

Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight.

$E = W\sqrt{25+x^2} + L(13-x) \quad\Rightarrow\quad \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+} (W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1 \cdot (25+4^2)^{1/2} \quad\Rightarrow\quad c = \sqrt{41}/4 \approx 1.6$.

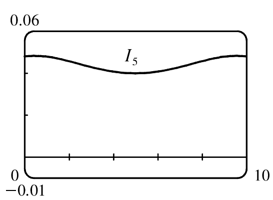
82. (a) $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

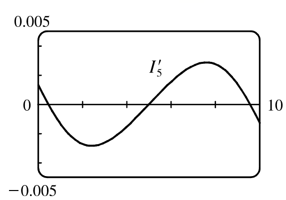
$$I(x) = \frac{k}{x^2+d^2} + \frac{k}{(10-x)^2+d^2} = \frac{k}{x^2+d^2} + \frac{k}{x^2-20x+100+d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$. $I'(x) = -\dfrac{2x}{(x^2+d^2)^2} - \dfrac{2(x-10)}{(x^2-20x+100+d^2)^2}$.

Substituting $d = 5$ into the equations for $I(x)$and $I'(x)$, we get

$$I_5(x) = \frac{1}{x^2+25} + \frac{1}{x^2-20x+125} \quad\text{and}\quad I_5'(x) = -\frac{2x}{(x^2+25)^2} - \frac{2(x-10)}{(x^2-20x+125)^2}$$

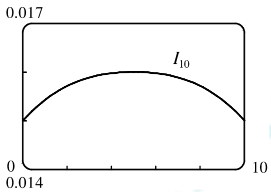


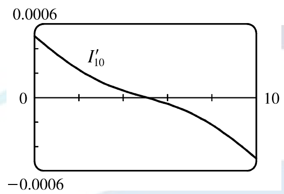


From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives

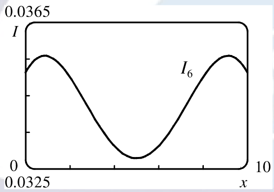
$$I_{10}(x) = \frac{1}{x^2 + 100} + \frac{1}{x^2 - 20x + 200} \quad \text{and} \quad I'_{10}(x) = -\frac{2x}{(x^2 + 100)^2} - \frac{2\,(x - 10)}{(x^2 - 20x + 200)^2}$$

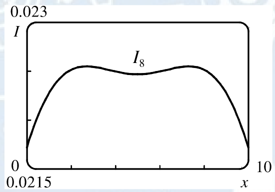


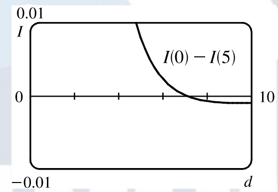


From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).







So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$ (with $k = 1$):

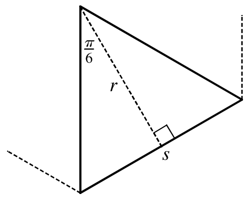
$$\frac{1}{d^2} + \frac{1}{100 + d^2} = \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2} \quad \Rightarrow \quad (25 + d^2)(100 + d^2) + d^2(25 + d^2) = 2d^2(100 + d^2) \quad \Rightarrow$$

$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \quad \Rightarrow \quad 2500 = 50d^2 \quad \Rightarrow \quad d^2 = 50 \quad \Rightarrow \quad d = 5\sqrt{2} \approx 7.071$$

[for $0 \le d \le 10$]. The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that $I(0) - I(5) = 0$, that is, $I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

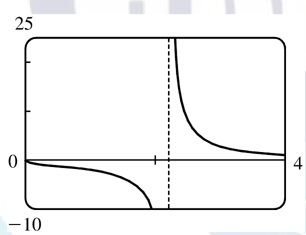
APPLIED PROJECT The Shape of a Can

1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2(4r^2)$. But $V = \pi r^2 h \quad \Leftrightarrow \quad h = V/\pi r^2$. Substituting this expression for h in A gives $A = 2V/r + 8r^2$. Differentiating A with respect to r, we get $dA/dr = -2V/r^2 + 16r = 0 \quad \Rightarrow$ $16r^3 = 2V = 2\pi r^2 h \quad \Leftrightarrow \quad \dfrac{h}{r} = \dfrac{8}{\pi} \approx 2.55$. This gives a minimum because $\dfrac{d^2A}{dr^2} = 16 + \dfrac{4V}{r^3} > 0$.

2. 

We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}\,r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}\,r^2$. Substituting for h as in Problem 1 and differentiating, we get $\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\sqrt{3}\,r$. Setting this equal to 0, we get $8\sqrt{3}\,r^3 = 2V = 2\pi r^2 h \quad\Rightarrow\quad \dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi} \approx 2.21$. Again this minimizes A because $\dfrac{d^2A}{dr^2} = 8\sqrt{3} + \dfrac{4V}{r^3} > 0$.

3. Let $C = 4\sqrt{3}\,r^2 + 2\pi rh + k\,(4\pi r + h) = 4\sqrt{3}\,r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right) + k\left(4\pi r + \dfrac{V}{\pi r^2}\right)$. Then $\dfrac{dC}{dr} = 8\sqrt{3}\,r - \dfrac{2V}{r^2} + 4k\pi - \dfrac{2kV}{\pi r^3}$. Setting this equal to 0, dividing by 2 and substituting $\dfrac{V}{r^2} = \pi h$ and $\dfrac{V}{\pi r^3} = \dfrac{h}{r}$ in the second and fourth terms respectively, we get $0 = 4\sqrt{3}\,r - \pi h + 2k\pi - \dfrac{kh}{r} \quad\Leftrightarrow$
$k\left(2\pi - \dfrac{h}{r}\right) = \pi h - 4\sqrt{3}\,r \quad\Rightarrow\quad \dfrac{k}{r}\,\dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1$. We now multiply by $\dfrac{\sqrt[3]{V}}{k}$, noting that $\dfrac{\sqrt[3]{V}}{k}\,\dfrac{k}{r} = \sqrt[3]{\dfrac{V}{r^3}} = \sqrt[3]{\dfrac{\pi h}{r}}$, and get $\dfrac{\sqrt[3]{V}}{k} = \sqrt[3]{\dfrac{\pi h}{r}} \cdot \dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$.

4. 

Let $\sqrt[3]{V}/k = T$ and $h/r = x$ so that $T(x) = \sqrt[3]{\pi x} \cdot \dfrac{2\pi - x}{\pi x - 4\sqrt{3}}$. We see from the graph of T that when the ratio $\sqrt[3]{V}/k$ is large; that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently shaped can in special circumstances.

APPLIED PROJECT Planes and Birds: Minimizing Energy

1. $P(v) = Av^3 + \dfrac{BL^2}{v} \quad \Rightarrow \quad P'(v) = 3Av^2 - \dfrac{BL^2}{v^2}$. $\quad P'(v) = 0 \quad \Leftrightarrow \quad 3Av^2 = \dfrac{BL^2}{v^2} \quad \Leftrightarrow \quad v^4 = \dfrac{BL^2}{3A} \quad \Rightarrow$
$v = \sqrt[4]{\dfrac{BL^2}{3A}}$. $\quad P''(v) = 6Av + \dfrac{2BL^2}{v^3} > 0$, so the speed that minimizes the required power is $v_P = \left(\dfrac{BL^2}{3A}\right)^{1/4}$.

2. $E(v) = \dfrac{P(v)}{v} = Av^2 + \dfrac{BL^2}{v^2} \quad \Rightarrow \quad E'(v) = 2Av - \dfrac{2BL^2}{v^3}$. $\quad E'(v) = 0 \quad \Leftrightarrow \quad 2Av = \dfrac{2BL^2}{v^3} \quad \Leftrightarrow \quad v^4 = \dfrac{BL^2}{A} \quad \Rightarrow$
$v = \sqrt[4]{\dfrac{BL^2}{A}}$. $\quad E''(v) = 2A + \dfrac{6BL^2}{v^4} > 0$, so the speed that minimizes the energy needed to propel the plane is
$v_E = \left(\dfrac{BL^2}{A}\right)^{1/4}$.

3. $\dfrac{v_E}{v_P} = \dfrac{\left(\dfrac{BL^2}{A}\right)^{1/4}}{\left(\dfrac{BL^2}{3A}\right)^{1/4}} = \left(\dfrac{\dfrac{BL^2}{A}}{\dfrac{BL^2}{3A}}\right)^{1/4} = 3^{1/4} \approx 1.316$. Thus, $v_E \approx 1.316\, v_P$, so the speed for minimum energy is about 31.6% greater (faster) than the speed for minimum power.

4. Since x is the fraction of flying time spent in flapping mode, $1 - x$ is the fraction of time spent in folded mode. The average power $\overline{P}$ is the weighted average of P_{flap} and P_{fold}, so

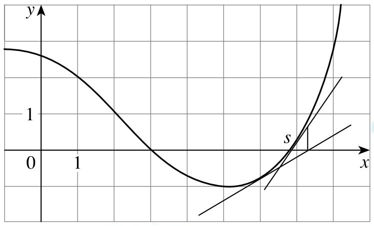
$$\begin{aligned}\overline{P} &= xP_{\text{flap}} + (1-x)P_{\text{fold}} = x\left[(A_b + A_w)\,v^3 + \frac{B(mg/x)^2}{v}\right] + (1-x)A_bv^3 \\ &= xA_bv^3 + xA_wv^3 + x\,\frac{Bm^2g^2}{x^2v} + A_bv^3 - xA_bv^3 = A_bv^3 + xA_wv^3 + \frac{Bm^2g^2}{xv}\end{aligned}$$

5. $\overline{P}(x) = A_bv^3 + xA_wv^3 + \dfrac{Bm^2g^2}{xv} \quad \Rightarrow \quad \overline{P}'(x) = A_wv^3 - \dfrac{Bm^2g^2}{x^2v}$. $\quad \overline{P}'(x) = 0 \quad \Leftrightarrow \quad A_wv^3 = \dfrac{Bm^2g^2}{x^2v} \quad \Leftrightarrow$
$x^2 = \dfrac{Bm^2g^2}{A_wv^4} \quad \Rightarrow \quad x = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$. Since $\overline{P}''(x) = \dfrac{2Bm^2g^2}{x^3v} > 0$, this critical number, call it $x_{\overline{P}}$, gives an absolute minimum for the average power. If the bird flies slowly, then v is smaller and $x_{\overline{P}}$ increases, and the bird spends a larger fraction of its flying time flapping. If the bird flies faster and faster, then v is larger and $x_{\overline{P}}$ decreases, and the bird spends a smaller fraction of its flying time flapping, while still minimizing average power.

6. $\overline{E}(x) = \dfrac{\overline{P}(x)}{v} \quad \Rightarrow \quad \overline{E}'(x) = \dfrac{1}{v}\overline{P}'(x)$, so $\overline{E}'(x) = 0 \quad \Leftrightarrow \quad \overline{P}'(x) = 0$. The value of x that minimizes $\overline{E}$ is the same value of x that minimizes $\overline{P}$, namely $x_{\overline{P}} = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$.

4.8 Newton's Method

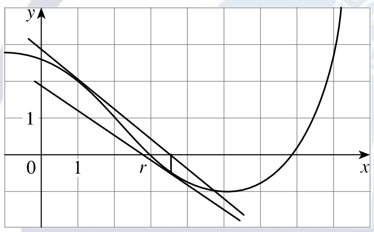
1. (a)



The tangent line at $x_1 = 6$ intersects the x-axis at $x \approx 7.3$, so $x_2 = 7.3$. The tangent line at $x = 7.3$ intersects the x-axis at $x \approx 6.8$, so $x_3 \approx 6.8$.

(b) $x_1 = 8$ would be a better first approximation because the tangent line at $x = 8$ intersects the x-axis closer to s than does the first approximation $x_1 = 6$.

2.



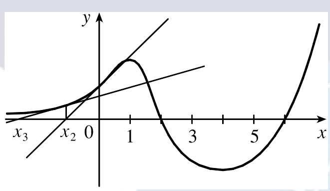
The tangent line at $x_1 = 1$ intersects the x-axis at $x \approx 3.5$, so $x_2 = 3.5$. The tangent line at $x = 3.5$ intersects the x-axis at $x \approx 2.8$, so $x_3 = 2.8$.

3. Since the tangent line $y = 9 - 2x$ is tangent to the curve $y = f(x)$ at the point $(2, 5)$, we have $x_1 = 2$, $f(x_1) = 5$, and $f'(x_1) = -2$ [the slope of the tangent line]. Thus, by Equation 2,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{5}{-2} = \frac{9}{2}$$

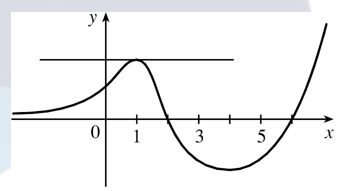
Note that geometrically $\frac{9}{2}$ represents the x-intercept of the tangent line $y = 9 - 2x$.

4. (a)



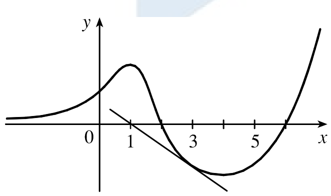
If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



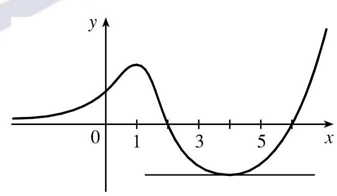
If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

(c)

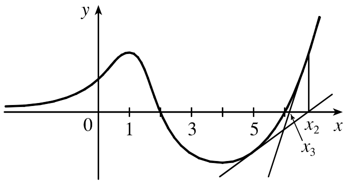


If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)

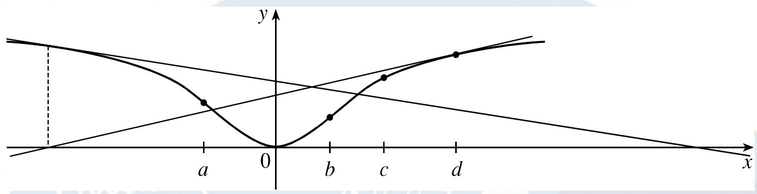


If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

(e) 

If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

5. The initial approximations $x_1 = a, b$, and c will work, resulting in a second approximation closer to the origin, and lead to the root of the equation $f(x) = 0$, namely, $x = 0$. The initial approximation $x_1 = d$ will not work because it will result in successive approximations farther and farther from the origin.



6. $f(x) = 2x^3 - 3x^2 + 2 \quad\Rightarrow\quad f'(x) = 6x^2 - 6x$, so $x_{n+1} = x_n - \dfrac{2x_n^3 - 3x_n^2 + 2}{6x_n^2 - 6x_n}$. Now $x_1 = -1 \quad\Rightarrow$

$$x_2 = -1 - \frac{2(-1)^3 - 3(-1)^2 + 2}{6(-1)^2 - 6(-1)} = -1 - \frac{-3}{12} = -\frac{3}{4} \quad\Rightarrow$$

$$x_3 = -\frac{3}{4} - \frac{2\left(-\frac{3}{4}\right)^3 - 3\left(-\frac{3}{4}\right)^2 + 2}{6\left(-\frac{3}{4}\right)^2 - 6\left(-\frac{3}{4}\right)} = -\frac{3}{4} - \frac{-17/32}{63/8} = -\frac{43}{63} \approx -0.6825.$$

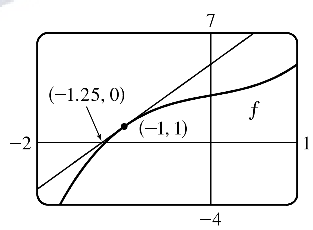
7. $f(x) = \dfrac{2}{x} - x^2 + 1 \quad\Rightarrow\quad f'(x) = -\dfrac{2}{x^2} - 2x$, so $x_{n+1} = x_n - \dfrac{2/x_n - x_n^2 + 1}{-2/x_n^2 - 2x_n}$. Now $x_1 = 2 \quad\Rightarrow$

$$x_2 = 2 - \frac{1 - 4 + 1}{-1/2 - 4} = 2 - \frac{-2}{-9/2} = \frac{14}{9} \quad\Rightarrow\quad x_3 = \frac{14}{9} - \frac{2/(14/9) - (14/9)^2 + 1}{-2(14/9)^2 - 2(14/9)} \approx 1.5215.$$

8. $f(x) = x^7 + 4 \quad\Rightarrow\quad f'(x) = 7x^6$, so $x_{n+1} = x_n - \dfrac{x_n^7 + 4}{7x_n^6}$. Now $x_1 = -1 \quad\Rightarrow$

$$x_2 = -1 - \frac{(-1)^7 + 4}{7(-1)^6} = -1 - \frac{3}{7} = -\frac{10}{7} \quad\Rightarrow\quad x_3 = -\frac{10}{7} - \frac{\left(-\frac{10}{7}\right)^7 + 4}{7\left(-\frac{10}{7}\right)^6} \approx -1.2917.$$

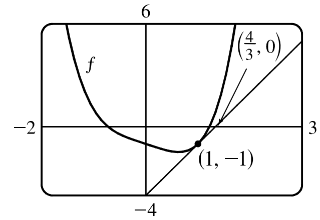
9. $f(x) = x^3 + x + 3 \quad\Rightarrow\quad f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 3}{3x_n^2 + 1}$.



Now $x_1 = -1 \quad\Rightarrow$

$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$$

Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.



10. $f(x) = x^4 - x - 1 \quad \Rightarrow \quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1 - \dfrac{1^4 - 1 - 1}{4 \cdot 1^3 - 1} = 1 - \dfrac{-1}{3} = \dfrac{4}{3}$. Newton's method follows the tangent line at $(1, -1)$ up to its intersection with the x-axis at $\left(\frac{4}{3}, 0\right)$, giving the second approximation $x_2 = \frac{4}{3}$.

11. To approximate $x = \sqrt[4]{75}$ (so that $x^4 = 75$), we can take $f(x) = x^4 - 75$. So $f'(x) = 4x^3$, and thus, $x_{n+1} = x_n - \dfrac{x_n^4 - 75}{4x_n^3}$. Since $\sqrt[4]{81} = 3$ and 81 is reasonably close to 75, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad \Rightarrow \quad x_2 = 2.9\overline{4}$, $x_3 \approx 2.94283228$, $x_4 \approx 2.94283096 \approx x_5$. So $\sqrt[4]{75} \approx 2.94283096$, to eight decimal places.

To use Newton's method on a calculator, assign f to Y_1 and f' to Y_2. Then store x_1 in X and enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \to \mathrm{X}$ to get x_2 and further approximations (repeatedly press ENTER).

12. $f(x) = x^8 - 500 \quad \Rightarrow \quad f'(x) = 8x^7$, so $x_{n+1} = x_n - \dfrac{x_n^8 - 500}{8x_n^7}$. Since $\sqrt[8]{256} = 2$ and 256 is reasonably close to 500, we'll use $x_1 = 2$. We need to find approximations until they agree to eight decimal places. $x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.23828125$, $x_3 \approx 2.18055972$, $x_4 \approx 2.17461675$, $x_5 \approx 2.17455928 \approx x_6$. So $\sqrt[8]{500} \approx 2.17455928$, to eight decimal places.

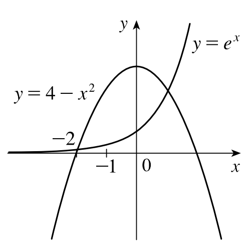
13. (a) Let $f(x) = 3x^4 - 8x^3 + 2$. The polynomial f is continuous on $[2, 3]$, $f(2) = -14 < 0$, and $f(3) = 29 > 0$, so by the Intermediate Value Theorem, there is a number c in $(2, 3)$ such that $f(c) = 0$. In other words, the equation $3x^4 - 8x^3 + 2 = 0$ has a root in $[2, 3]$.

(b) $f'(x) = 12x^3 - 24x^2 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{3x_n^4 - 8x_n^3 + 2}{12x_n^3 - 24x_n^2}$. Taking $x_1 = 2.5$, we get $x_2 = 2.655$, $x_3 \approx 2.630725$, $x_4 \approx 2.630021$, $x_5 \approx 2.630020 \approx x_6$. To six decimal places, the root is 2.630020. Note that taking $x_1 = 2$ is not allowed since $f'(2) = 0$.

14. (a) Let $f(x) = -2x^5 + 9x^4 - 7x^3 - 11x$. The polynomial f is continuous on $[3, 4]$, $f(3) = 21 > 0$, and $f(4) = -236 < 0$, so by the Intermediate Value Theorem, there is a number c in $(3, 4)$ such that $f(c) = 0$. In other words, the equation $-2x^5 + 9x^4 - 7x^3 - 11x = 0$ has a root in $[3, 4]$.

(b) $f'(x) = -10x^4 + 36x^3 - 21x^2 - 11. \quad x_{n+1} = x_n - \dfrac{-2x_n^5 + 9x_n^4 - 7x_n^3 - 11x_n}{-10x_n^4 + 36x_n^3 - 21x_n^2 - 11}$. Taking $x_1 = 3.5$, we get $x_2 \approx 3.329174$, $x_3 = 3.278706$, $x_4 \approx 3.274501$, and $x_5 \approx 3.274473 \approx x_6$. To six decimal places, the root is 3.274473.

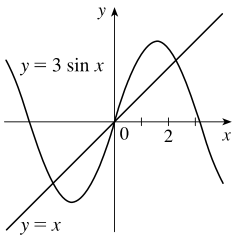
15.



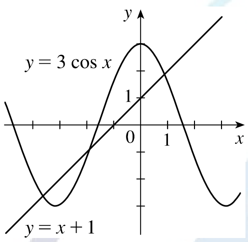
$e^x = 4 - x^2$, so $f(x) = e^x - 4 + x^2 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{e^{x_n} - 4 + x_n^2}{e^{x_n} + 2x_n}$.

From the figure, the negative root of $e^x = 4 - x^2$ is near -2.

$x_1 = -2 \quad \Rightarrow \quad x_2 \approx -1.964981$, $x_3 \approx -1.964636 \approx x_4$. So the negative root is -1.964636, to six decimal places.

16. 

$3\sin x = x$, so $f(x) = 3\sin x - x \Rightarrow f'(x) = 3\cos x - 1 \Rightarrow$ $x_{n+1} = x_n - \dfrac{3\sin x_n - x_n}{3\cos x_n - 1}$. From the figure, the positive root of $3\sin x = x$ is near 2. $x_1 = 2 \Rightarrow x_2 \approx 2.323732$, $x_3 \approx 2.279595$, $x_4 \approx 2.278863 \approx x_5$. So the positive root is 2.278863, to six decimal places.

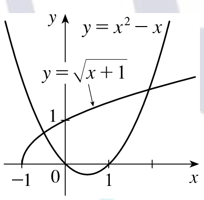
17. 

From the graph, we see that there appear to be points of intersection near $x = -4$, $x = -2$, and $x = 1$. Solving $3\cos x = x + 1$ is the same as solving $f(x) = 3\cos x - x - 1 = 0$. $f'(x) = -3\sin x - 1$, so

$$x_{n+1} = x_n - \frac{3\cos x_n - x_n - 1}{-3\sin x_n - 1}.$$

$x_1 = -4$	$x_1 = -2$	$x_1 = 1$
$x_2 \approx -3.682281$	$x_2 \approx -1.856218$	$x_2 \approx 0.892438$
$x_3 \approx -3.638960$	$x_3 \approx -1.862356$	$x_3 \approx 0.889473$
$x_4 \approx -3.637959$	$x_4 \approx -1.862365 \approx x_5$	$x_4 \approx 0.889470 \approx x_5$
$x_5 \approx -3.637958 \approx x_6$		

To six decimal places, the roots of the equation are -3.637958, -1.862365, and 0.889470.

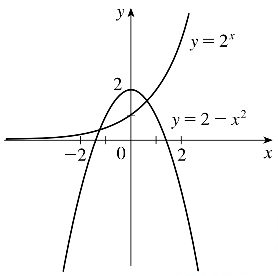
18. 

From the graph, we see that there appear to be points of intersection near $x = -0.5$ and $x = 2$. Solving $\sqrt{x+1} = x^2 - x$ is the same as solving $f(x) = \sqrt{x+1} - x^2 + x = 0$. $f'(x) = \dfrac{1}{2\sqrt{x+1}} - 2x + 1$, so

$$x_{n+1} = x_n - \frac{\sqrt{x_n + 1} - x_n^2 + x_n}{\dfrac{1}{2\sqrt{x_n+1}} - 2x_n + 1}.$$

$x_1 = -0.5$	$x_1 = 2$
$x_2 \approx -0.484155$	$x_2 \approx 1.901174$
$x_3 \approx -0.484028 \approx x_4$	$x_3 \approx 1.897186$
	$x_4 \approx 1.897179 \approx x_5$

To six decimal places, the roots of the equation are -0.484028 and 0.897179.

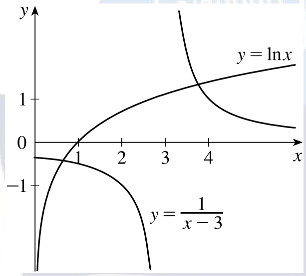
19. 

From the figure, we see that the graphs intersect between -2 and -1 and between 0 and 1. Solving $2^x = 2 - x^2$ is the same as solving $f(x) = 2^x - 2 + x^2 = 0$. $f'(x) = 2^x \ln 2 + 2x$, so

$$x_{n+1} = x_n - \frac{2^{x_n} - 2 + x_n^2}{2^{x_n} \ln 2 + 2x_n}.$$

$x_1 = -1$	$x_1 = 1$
$x_2 \approx -1.302402$	$x_2 \approx 0.704692$
$x_3 \approx -1.258636$	$x_3 \approx 0.654915$
$x_4 \approx -1.257692$	$x_4 \approx 0.653484$
$x_5 \approx -1.257691 \approx x_6$	$x_5 \approx 0.653483 \approx x_6$

To six decimal places, the roots of the equation are -1.257691 and 0.653483.

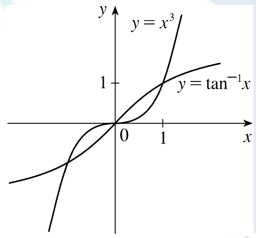
20. 

From the figure, we see that the graphs intersect between 0 and 1 and between 3 and 4. Solving $\ln x = \dfrac{1}{x-3}$ is the same as solving $f(x) = \ln x - \dfrac{1}{x-3} = 0$.

$f'(x) = \dfrac{1}{x} + \dfrac{1}{(x-3)^2}$, so $x_{n+1} = x_n - \dfrac{\ln x_n - 1/(x_n - 3)}{(1/x_n) + 1/(x_n - 3)^2}$.

$x_1 = 1$	$x_1 = 4$
$x_2 \approx 0.6$	$x_2 \approx 3.690965$
$x_3 \approx 0.651166$	$x_3 \approx 3.750726$
$x_4 \approx 0.653057$	$x_4 \approx 3.755672$
$x_5 \approx 0.653060 \approx x_6$	$x_5 \approx 3.755701 \approx x_6$

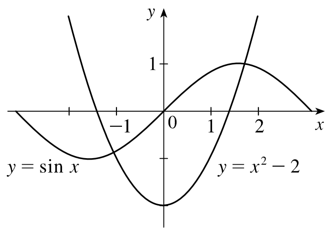
To six decimal places, the roots of the equation are 0.653060 and 3.755701.

21. 

From the figure, we see that the graphs intersect at 0 and at $x = \pm a$, where $a \approx 1$. [Both functions are odd, so the roots are negatives of each other.] Solving $x^3 = \tan^{-1} x$ is the same as solving $f(x) = x^3 - \tan^{-1} x = 0$.

$f'(x) = 3x^2 - \dfrac{1}{1+x^2}$, so $x_{n+1} = x_n - \dfrac{x_n^3 - \tan^{-1} x_n}{3x_n^2 - \dfrac{1}{1+x_n^2}}$.

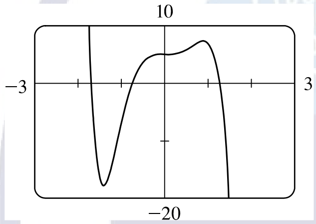
Now $x_1 = 1 \;\Rightarrow\; x_2 \approx 0.914159$, $x_3 \approx 0.902251$, $x_4 \approx 0.902026$, $x_5 \approx 0.902025 \approx x_6$. To six decimal places, the nonzero roots of the equation are ± 0.902025.

22. 

From the graph, we see that there appear to be points of intersection near $x = -1$ and $x = 2$. Solving $\sin x = x^2 - 2$ is the same as solving $f(x) = \sin x - x^2 + 2 = 0$. $f'(x) = \cos x - 2x$, so $x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 2}{\cos x_n - 2x_n}$.

$x_1 = -1$	$x_1 = 2$
$x_2 \approx -1.062406$	$x_2 \approx 1.753019$
$x_3 \approx -1.061550 \approx x_4$	$x_3 \approx 1.728710$
	$x_4 \approx 1.728466 \approx x_5$

To six decimal places, the roots of the equation are -1.061550 and 1.728466.

23. 

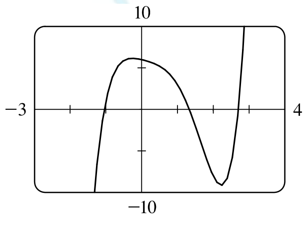
$f(x) = -2x^7 - 5x^4 + 9x^3 + 5 \quad \Rightarrow \quad f'(x) = -14x^6 - 20x^3 + 27x^2 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{-2x_n^7 - 5x_n^4 + 9x_n^3 + 5}{-14x_n^6 - 20x_n^3 + 27x_n^2}$.

From the graph of f, there appear to be roots near -1.7, -0.7, and 1.3.

$x_1 = -1.7$	$x_1 = -0.7$	$x_1 = 1.3$
$x_2 = -1.693255$	$x_2 \approx -0.74756345$	$x_2 = 1.268776$
$x_3 \approx -1.69312035$	$x_3 \approx -0.74467752$	$x_3 \approx 1.26589387$
$x_4 \approx -1.69312029 \approx x_5$	$x_4 \approx -0.74466668 \approx x_5$	$x_4 \approx 1.26587094 \approx x_5$

To eight decimal places, the roots of the equation are -1.69312029, -0.74466668, and 1.26587094.

24. 

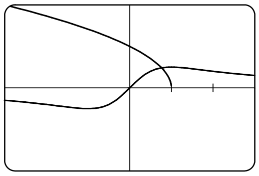
$f(x) = x^5 - 3x^4 + x^3 - x^2 - x + 6 \quad \Rightarrow$

$f'(x) = 5x^4 - 12x^3 + 3x^2 - 2x - 1 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^5 - 3x_n^4 + x_n^3 - x_n^2 - x_n + 6}{5x_n^4 - 12x_n^3 + 3x_n^2 - 2x_n - 1}$. From the graph of f, there appear to be roots near -1, 1.3, and 2.7.

$x_1 = -1$	$x_1 = 1.3$	$x_1 = 2.7$
$x_2 \approx -1.04761905$	$x_2 \approx 1.33313045$	$x_2 \approx 2.70556135$
$x_3 \approx -1.04451724$	$x_3 \approx 1.33258330$	$x_3 \approx 2.70551210$
$x_4 \approx -1.04450307 \approx x_5$	$x_4 \approx 1.33258316 \approx x_5$	$x_4 \approx 2.70551209 \approx x_5$

To eight decimal places, the roots of the equation are -1.04450307, 1.33258316, and 2.70551209.

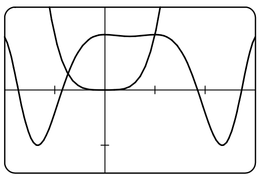
25. 

Solving $\dfrac{x}{x^2+1} = \sqrt{1-x}$ is the same as solving

$f(x) = \dfrac{x}{x^2+1} - \sqrt{1-x} = 0$. $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} + \dfrac{1}{2\sqrt{1-x}} \Rightarrow$

$$x_{n+1} = x_n - \frac{\dfrac{x_n}{x_n^2+1} - \sqrt{1-x_n}}{\dfrac{1-x_n^2}{(x_n^2+1)^2} + \dfrac{1}{2\sqrt{1-x_n}}}.$$

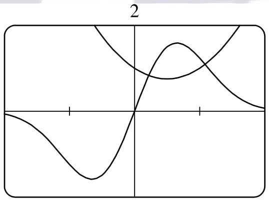
From the graph, we see that the curves intersect at about 0.8. $x_1 = 0.8 \Rightarrow x_2 \approx 0.76757581$, $x_3 \approx 0.76682610$, $x_4 \approx 0.76682579 \approx x_5$. To eight decimal places, the root of the equation is 0.76682579.

26. 

Solving $\cos(x^2 - x) = x^4$ is the same as solving

$f(x) = \cos(x^2 - x) - x^4 = 0$. $f'(x) = -(2x-1)\sin(x^2 - x) - 4x^3 \Rightarrow$

$x_{n+1} = x_n - \dfrac{\cos(x_n^2 - x_n) - x_n^4}{-(2x_n - 1)\sin(x_n^2 - x_n) - 4x_n^3}$. From the equations $y = \cos(x^2 - x)$ and $y = x^4$ and the graph, we deduce that one root of the equation $\cos(x^2 - x) = x^4$ is $x = 1$. We also see that the graphs intersect at approximately $x = -0.7$. $x_1 = -0.7 \Rightarrow x_2 \approx -0.73654354$, $x_3 \approx -0.73486274$, $x_4 \approx -0.73485910 \approx x_5$. To eight decimal places, one root of the equation is -0.73485910; the other root is 1.

27. 

Solving $4e^{-x^2}\sin x = x^2 - x + 1$ is the same as solving

$f(x) = 4e^{-x^2}\sin x - x^2 + x - 1 = 0$.

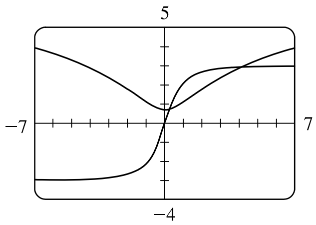
$f'(x) = 4e^{-x^2}(\cos x - 2x\sin x) - 2x + 1 \Rightarrow$

$$x_{n+1} = x_n - \frac{4e^{-x_n^2}\sin x_n - x_n^2 + x_n - 1}{4e^{-x_n^2}(\cos x_n - 2x_n\sin x_n) - 2x_n + 1}.$$

From the figure, we see that the graphs intersect at approximately $x = 0.2$ and $x = 1.1$.

$x_1 = 0.2$	$x_1 = 1.1$
$x_2 \approx 0.21883273$	$x_2 \approx 1.08432830$
$x_3 \approx 0.21916357$	$x_3 \approx 1.08422462 \approx x_4$
$x_4 \approx 0.21916368 \approx x_5$	

To eight decimal places, the roots of the equation are 0.21916368 and 1.08422462.

28. 

Solving $\ln(x^2+2) = \dfrac{3x}{\sqrt{x^2+1}}$ is the same as solving

$$f(x) = \ln(x^2+2) - \frac{3x}{\sqrt{x^2+1}} = 0.$$

$$f'(x) = \frac{2x}{x^2+2} - \frac{(x^2+1)^{1/2}(3) - (3x)\frac{1}{2}(x^2+1)^{-1/2}(2x)}{[(x^2+1)^{1/2}]^2}$$

$$= \frac{2x}{x^2+2} - \frac{(x^2+1)^{-1/2}[3(x^2+1) - 3x^2]}{(x^2+1)^1}$$

$$= \frac{2x}{x^2+2} - \frac{3}{(x^2+1)^{3/2}} \quad\Rightarrow\quad x_{n+1} = x_n - \frac{\ln(x_n^2+2) - \dfrac{3x_n}{\sqrt{x_n^2+1}}}{\dfrac{2x_n}{x_n^2+2} - \dfrac{3}{(x_n^2+1)^{3/2}}}.$$

4.8.28: changed the last term in last line of display

From the figure, we see that the graphs intersect at approximately $x = 0.2$ and $x = 4$.

$x_1 = 0.2$	$x_1 = 4$
$x_2 \approx 0.24733161$	$x_2 \approx 4.04993412$
$x_3 \approx 0.24852333$	$x_3 \approx 4.05010983$
$x_4 \approx 0.24852414 \approx x_5$	$x_4 \approx 4.05010984 \approx x_5$

To eight decimal places, the roots of the equation are 0.24852414 and 4.05010984.

29. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

30. (a) $f(x) = \dfrac{1}{x} - a \Rightarrow f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6984 \approx 0.588789$.

31. $f(x) = x^3 - 3x + 6 \Rightarrow f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

32. $x^3 - x = 1 \Leftrightarrow x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \Rightarrow f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

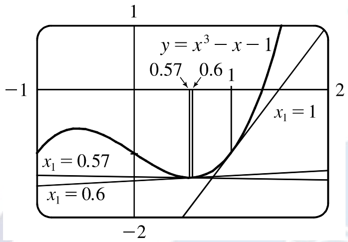
(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$,

$x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

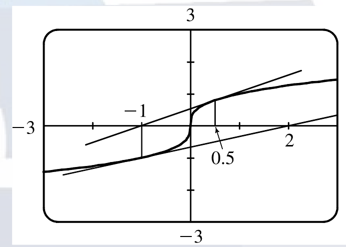
(d)



From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

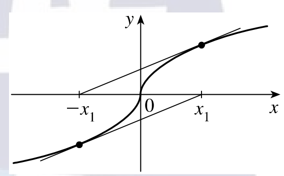


Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.

34. According to Newton's Method, for $x_n > 0$,

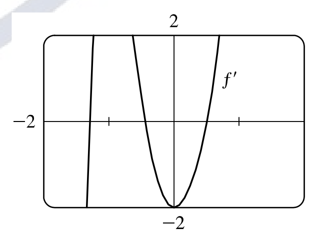
$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/\left(2\sqrt{x_n}\right)} = x_n - 2x_n = -x_n \text{ and for } x_n < 0,$$

$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)} = x_n - [-2(-x_n)] = -x_n.$$ So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



35. (a) $f(x) = x^6 - x^4 + 3x^3 - 2x \quad\Rightarrow\quad f'(x) = 6x^5 - 4x^3 + 9x^2 - 2 \quad\Rightarrow$ $f''(x) = 30x^4 - 12x^2 + 18x$. To find the critical numbers of f, we'll find the zeros of f'. From the graph of f', it appears there are zeros at approximately $x = -1.3$, -0.4, and 0.5. Try $x_1 = -1.3 \quad\Rightarrow$

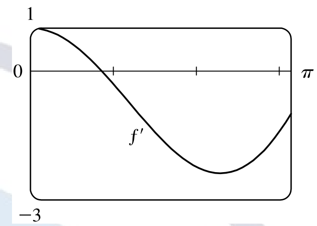
$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} \approx -1.293344 \quad\Rightarrow\quad x_3 \approx -1.293227 \approx x_4.$$



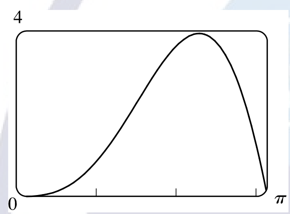
Now try $x_1 = -0.4 \quad\Rightarrow\quad x_2 \approx -0.443755 \quad\Rightarrow\quad x_3 \approx -0.441735 \quad\Rightarrow\quad x_4 \approx -0.441731 \approx x_5$. Finally try $x_1 = 0.5 \quad\Rightarrow\quad x_2 \approx 0.507937 \quad\Rightarrow\quad x_3 \approx 0.507854 \approx x_4$. Therefore, $x = -1.293227$, -0.441731, and 0.507854 are all the critical numbers correct to six decimal places.

(b) There are two critical numbers where f' changes from negative to positive, so f changes from decreasing to increasing. $f(-1.293227) \approx -2.0212$ and $f(0.507854) \approx -0.6721$, so -2.0212 is the absolute minimum value of f correct to four decimal places.

36. $f(x) = x\cos x \quad \Rightarrow \quad f'(x) = \cos x - x\sin x$. $f'(x)$ exists for all x, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = 0.9$. Use $g(x) = \cos x - x\sin x$ and $g'(x) = -2\sin x - x\cos x$ to obtain $x_2 \approx 0.860781$, $x_3 \approx 0.860334 \approx x_4$. Now we have $f(0) = 0$, $f(\pi) = -\pi$, and $f(0.860334) \approx 0.561096$, so 0.561096 is the absolute maximum value of f correct to six decimal places.



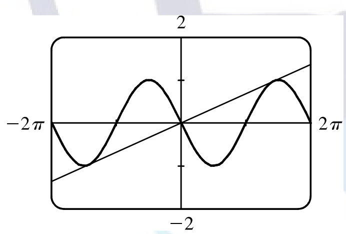
37.



$$y = x^2\sin x \quad \Rightarrow \quad y' = x^2\cos x + (\sin x)(2x) \quad \Rightarrow$$
$$y'' = x^2(-\sin x) + (\cos x)(2x) + (\sin x)(2) + 2x\cos x$$
$$= -x^2\sin x + 4x\cos x + 2\sin x \quad \Rightarrow$$
$$y''' = -x^2\cos x + (\sin x)(-2x) + 4x(-\sin x) + (\cos x)(4) + 2\cos x$$
$$= -x^2\cos x - 6x\sin x + 6\cos x.$$

From the graph of $y = x^2\sin x$, we see that $x = 1.5$ is a reasonable guess for the x-coordinate of the inflection point. Using Newton's method with $g(x) = y''$ and $g'(x) = y'''$, we get $x_1 = 1.5 \quad \Rightarrow \quad x_2 \approx 1.520092$, $x_3 \approx 1.519855 \approx x_4$. The inflection point is about $(1.519855, 2.306964)$.

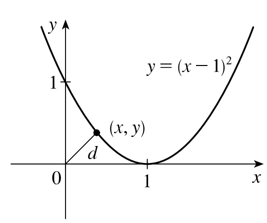
38.



$f(x) = -\sin x \quad \Rightarrow \quad f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \quad \Rightarrow \quad \tan a = a$.

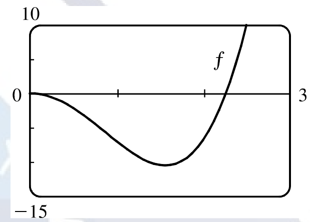
To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $\quad x_3 \approx 4.493410$, $\quad x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

39. We need to minimize the distance from $(0,0)$ to an arbitrary point (x,y) on the curve $y = (x-1)^2$. $d = \sqrt{x^2 + y^2} \quad \Rightarrow$ $d(x) = \sqrt{x^2 + [(x-1)^2]^2} = \sqrt{x^2 + (x-1)^4}$. When $d' = 0$, d will be minimized and equivalently, $s = d^2$ will be minimized, so we will use Newton's method with $f = s'$ and $f' = s''$.



$f(x) = 2x + 4(x-1)^3 \quad \Rightarrow \quad f'(x) = 2 + 12(x-1)^2$, so $x_{n+1} = x_n - \dfrac{2x_n + 4(x_n - 1)^3}{2 + 12(x_n - 1)^2}$. Try $x_1 = 0.5 \quad \Rightarrow$ $x_2 = 0.4$, $x_3 \approx 0.410127$, $x_4 \approx 0.410245 \approx x_5$. Now $d(0.410245) \approx 0.537841$ is the minimum distance and the point on the parabola is $(0.410245, 0.347810)$, correct to six decimal places.

40. Let the radius of the circle be r. Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get $4^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos\theta \quad \Leftrightarrow \quad 16 = 2r^2(1 - \cos\theta) = 2(5/\theta)^2\,(1 - \cos\theta)$. Multiplying by θ^2 gives $16\theta^2 = 50(1 - \cos\theta)$, so we take $f(\theta) = 16\theta^2 + 50\cos\theta - 50$ and $f'(\theta) = 32\theta - 50\sin\theta$. The formula for Newton's method is $\theta_{n+1} = \theta_n - \dfrac{16\theta_n^2 + 50\cos\theta_n - 50}{32\theta_n - 50\sin\theta_n}$. From the graph of f, we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$, $\theta_3 \approx 2.2622 \approx \theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130°$.



41. In this case, $A = 18{,}000$, $R = 375$, and $n = 5(12) = 60$. So the formula $A = \dfrac{R}{i}[1 - (1+i)^{-n}]$ becomes

$18{,}000 = \dfrac{375}{x}[1 - (1+x)^{-60}] \quad \Leftrightarrow \quad 48x = 1 - (1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$ $48x(1+x)^{60} - (1+x)^{60} + 1 = 0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59} + 48(1+x)^{60} - 60(1+x)^{59} \\ &= 12(1+x)^{59}[4x(60) + 4(1+x) - 5] = 12(1+x)^{59}(244x - 1) \end{aligned}$$

$x_{n+1} = x_n - \dfrac{48x_n(1+x_n)^{60} - (1+x_n)^{60} + 1}{12(1+x_n)^{59}(244x_n - 1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 \approx 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

42. (a) $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$p'(x) = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r)$. So we use

$$x_{n+1} = x_n - \frac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}.$$

We substitute in the value $r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the sun, so we take $x_1 = 0.95$ as our first approximation, and get $x_2 \approx 0.96682$, $x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$. So, to five decimal places, L_1 is located 0.98999 AU from the sun (or 0.01001 AU from the earth).

(b) In this case we use Newton's method with the function

$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute

$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the sun and, judging from the result of part (a), probably less than 0.02 AU from earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the sun (or 0.01008 AU from the earth).

4.9 Antiderivatives

1. $f(x) = 4x + 7 = 4x^1 + 7 \quad\Rightarrow\quad F(x) = 4\dfrac{x^{1+1}}{1+1} + 7x + C = 2x^2 + 7x + C$

   Check: $F'(x) = 2(2x) + 7 + 0 = 4x + 7 = f(x)$

2. $f(x) = x^2 - 3x + 2 \quad\Rightarrow\quad F(x) = \dfrac{x^3}{3} - 3\dfrac{x^2}{2} + 2x + C = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

   Check: $F'(x) = \frac{1}{3}(3x^2) - \frac{3}{2}(2x) + 2 + 0 = x^2 - 3x + 2 = f(x)$

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x \quad\Rightarrow\quad F(x) = 2\dfrac{x^{3+1}}{3+1} - \dfrac{2}{3}\dfrac{x^{2+1}}{2+1} + 5\dfrac{x^{1+1}}{1+1} = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$

   Check: $F'(x) = \frac{1}{2}(4x^3) - \frac{2}{9}(3x^2) + \frac{5}{2}(2x) + 0 = 2x^3 - \frac{2}{3}x^2 + 5x = f(x)$

4. $f(x) = 6x^5 - 8x^4 - 9x^2 \quad\Rightarrow\quad F(x) = 6\dfrac{x^6}{6} - 8\dfrac{x^5}{5} - 9\dfrac{x^3}{3} + C = x^6 - \frac{8}{5}x^5 - 3x^3 + C$

5. $f(x) = x(12x + 8) = 12x^2 + 8x \quad\Rightarrow\quad F(x) = 12\dfrac{x^3}{3} + 8\dfrac{x^2}{2} + C = 4x^3 + 4x^2 + C$

6. $f(x) = (x-5)^2 = x^2 - 10x + 25 \quad\Rightarrow\quad F(x) = \dfrac{x^3}{3} - 10\dfrac{x^2}{2} + 25x + C = \frac{1}{3}x^3 - 5x^2 + 25x + C$

7. $f(x) = 7x^{2/5} + 8x^{-4/5} \quad\Rightarrow\quad F(x) = 7\left(\frac{5}{7}x^{7/5}\right) + 8(5x^{1/5}) + C = 5x^{7/5} + 40x^{1/5} + C$

8. $f(x) = x^{3.4} - 2x^{\sqrt{2}-1} \quad\Rightarrow\quad F(x) = \dfrac{x^{4.4}}{4.4} - 2\left(\dfrac{x^{\sqrt{2}}}{\sqrt{2}}\right) + C = \dfrac{5}{22}x^{4.4} - \sqrt{2}\,x^{\sqrt{2}} + C$

9. $f(x) = \sqrt{2}$ is a constant function, so $F(x) = \sqrt{2}\,x + C$.

10. $f(x) = e^2$ is a constant function, so $F(x) = e^2x + C$.

11. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x} = 3x^{1/2} - 2x^{1/3} \quad\Rightarrow\quad F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

12. $f(x) = \sqrt[3]{x^2} + x\sqrt{x} = x^{2/3} + x^{3/2} \quad\Rightarrow\quad F(x) = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$

13. $f(x) = \dfrac{1}{5} - \dfrac{2}{x} = \dfrac{1}{5} - 2\left(\dfrac{1}{x}\right)$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \frac{1}{5}x - 2\ln|x| + C_1 & \text{if } x < 0 \\ \frac{1}{5}x - 2\ln|x| + C_2 & \text{if } x > 0 \end{cases}$

    See Example 1(b) for a similar problem.

14. $f(t) = \dfrac{3t^4 - t^3 + 6t^2}{t^4} = 3 - \dfrac{1}{t} + \dfrac{6}{t^2}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(t) = \begin{cases} 3t - \ln|t| - \dfrac{6}{t} + C_1 & \text{if } t < 0 \\ 3t - \ln|t| - \dfrac{6}{t} + C_2 & \text{if } t > 0 \end{cases}$

See Example 1(b) for a similar problem.

15. $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}} = t^{-1/2} + t^{1/2} + t^{3/2} \quad \Rightarrow \quad G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$

16. $r(\theta) = \sec\theta \, \tan\theta - 2e^\theta \quad \Rightarrow \quad R(\theta) = \sec\theta - 2e^\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

17. $h(\theta) = 2\sin\theta - \sec^2\theta \quad \Rightarrow \quad H(\theta) = -2\cos\theta - \tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

18. $g(v) = 2\cos v - \dfrac{3}{\sqrt{1 - v^2}} \quad \Rightarrow \quad G(v) = 2\sin v - 3\sin^{-1} v + C$

19. $f(x) = 2^x + 4\sinh x \quad \Rightarrow \quad F(x) = \dfrac{2^x}{\ln 2} + 4\cosh x + C$

20. $f(x) = 1 + 2\sin x + 3/\sqrt{x} = 1 + 2\sin x + 3x^{-1/2} \quad \Rightarrow \quad F(x) = x - 2\cos x + 3\,\dfrac{x^{1/2}}{1/2} + C = x - 2\cos x + 6\sqrt{x} + C$

21. $f(x) = \dfrac{2x^4 + 4x^3 - x}{x^3}, x > 0; \; f(x) = 2x + 4 - x^{-2} \quad \Rightarrow$

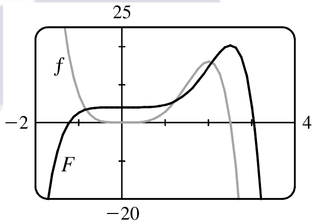
$F(x) = 2\,\dfrac{x^2}{2} + 4x - \dfrac{x^{-2+1}}{-2+1} + C = x^2 + 4x + \dfrac{1}{x} + C, \; x > 0$

22. $f(x) = \dfrac{2x^2 + 5}{x^2 + 1} = \dfrac{2(x^2 + 1) + 3}{x^2 + 1} = 2 + \dfrac{3}{x^2 + 1} \quad \Rightarrow \quad F(x) = 2x + 3\tan^{-1} x + C$

23. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C.$

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so $F(x) = x^5 - \frac{1}{3}x^6 + 4.$

The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.

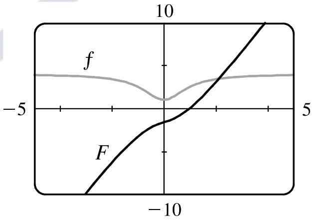


24. $f(x) = 4 - 3(1 + x^2)^{-1} = 4 - \dfrac{3}{1 + x^2} \quad \Rightarrow \quad F(x) = 4x - 3\tan^{-1} x + C.$

$F(1) = 0 \quad \Rightarrow \quad 4 - 3\left(\frac{\pi}{4}\right) + C = 0 \quad \Rightarrow \quad C = \frac{3\pi}{4} - 4$, so

$F(x) = 4x - 3\tan^{-1} x + \frac{3\pi}{4} - 4$. Note that f is positive and F is increasing on $\mathbb{R}$.

Also, f has smaller values where the slopes of the tangent lines of F are smaller.



25. $f''(x) = 20x^3 - 12x^2 + 6x \quad \Rightarrow \quad f'(x) = 20\left(\dfrac{x^4}{4}\right) - 12\left(\dfrac{x^3}{3}\right) + 6\left(\dfrac{x^2}{2}\right) + C = 5x^4 - 4x^3 + 3x^2 + C \quad \Rightarrow$

$f(x) = 5\left(\dfrac{x^5}{5}\right) - 4\left(\dfrac{x^4}{4}\right) + 3\left(\dfrac{x^3}{3}\right) + Cx + D = x^5 - x^4 + x^3 + Cx + D$

26. $f''(x) = x^6 - 4x^4 + x + 1 \quad\Rightarrow\quad f'(x) = \frac{1}{7}x^7 - \frac{4}{5}x^5 + \frac{1}{2}x^2 + x + C \quad\Rightarrow$

$f(x) = \frac{1}{56}x^8 - \frac{2}{15}x^6 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + Cx + D$

27. $f''(x) = 2x + 3e^x \quad\Rightarrow\quad f'(x) = x^2 + 3e^x + C \quad\Rightarrow\quad f(x) = \frac{1}{3}x^3 + 3e^x + Cx + D$

28. $f''(x) = 1/x^2 = x^{-2} \quad\Rightarrow\quad f'(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases} \quad\Rightarrow\quad f(x) = \begin{cases} -\ln(-x) + C_1x + D_1 & \text{if } x < 0 \\ -\ln x + C_2x + D_2 & \text{if } x > 0 \end{cases}$

29. $f'''(t) = 12 + \sin t \quad\Rightarrow\quad f''(t) = 12t - \cos t + C_1 \quad\Rightarrow\quad f'(t) = 6t^2 - \sin t + C_1t + D \quad\Rightarrow$

$f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

30. $f'''(t) = \sqrt{t} - 2\cos t = t^{1/2} - 2\cos t \quad\Rightarrow\quad f''(t) = \frac{2}{3}t^{3/2} - 2\sin t + C_1 \quad\Rightarrow\quad f'(t) = \frac{4}{15}t^{5/2} + 2\cos t + C_1t + D \quad\Rightarrow$

$f(t) = \frac{8}{105}t^{7/2} + 2\sin t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

31. $f'(x) = 1 + 3\sqrt{x} \quad\Rightarrow\quad f(x) = x + 3\left(\frac{2}{3}x^{3/2}\right) + C = x + 2x^{3/2} + C$. $f(4) = 4 + 2(8) + C$ and $f(4) = 25 \quad\Rightarrow$

$20 + C = 25 \quad\Rightarrow\quad C = 5$, so $f(x) = x + 2x^{3/2} + 5$.

32. $f'(x) = 5x^4 - 3x^2 + 4 \quad\Rightarrow\quad f(x) = x^5 - x^3 + 4x + C$. $f(-1) = -1 + 1 - 4 + C$ and $f(-1) = 2 \quad\Rightarrow$

$-4 + C = 2 \quad\Rightarrow\quad C = 6$, so $f(x) = x^5 - x^3 + 4x + 6$.

33. $f'(t) = \dfrac{4}{1+t^2} \quad\Rightarrow\quad f(t) = 4\arctan t + C$. $f(1) = 4\left(\dfrac{\pi}{4}\right) + C$ and $f(1) = 0 \quad\Rightarrow\quad \pi + C = 0 \quad\Rightarrow\quad C = -\pi$,

so $f(t) = 4\arctan t - \pi$.

34. $f'(t) = t + \dfrac{1}{t^3},\ t > 0 \quad\Rightarrow\quad f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + C$. $f(1) = \dfrac{1}{2} - \dfrac{1}{2} + C$ and $f(1) = 6 \quad\Rightarrow\quad C = 6$, so

$f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + 6$.

35. $f'(x) = 5x^{2/3} \quad\Rightarrow\quad f(x) = 5\left(\frac{3}{5}x^{5/3}\right) + C = 3x^{5/3} + C$.

$f(8) = 3 \cdot 32 + C$ and $f(8) = 21 \quad\Rightarrow\quad 96 + C = 21 \quad\Rightarrow\quad C = -75$, so $f(x) = 3x^{5/3} - 75$.

36. $f'(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2} \quad\Rightarrow\quad f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$. $f(1) = \frac{2}{3} + 2 + C = \frac{8}{3} + C$ and $f(1) = 5 \quad\Rightarrow$

$C = 5 - \frac{8}{3} = \frac{7}{3}$, so $f(x) = \frac{2}{3}x^{3/2} + 2\sqrt{x} + \frac{7}{3}$.

37. $f'(t) = \sec t(\sec t + \tan t) = \sec^2 t + \sec t\,\tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2} \quad\Rightarrow\quad f(t) = \tan t + \sec t + C$. $f\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} + C$

and $f\left(\frac{\pi}{4}\right) = -1 \quad\Rightarrow\quad 1 + \sqrt{2} + C = -1 \quad\Rightarrow\quad C = -2 - \sqrt{2}$, so $f(t) = \tan t + \sec t - 2 - \sqrt{2}$.

Note: The fact that f is defined and continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ means that we have only one constant of integration.

38. $f'(t) = 3^t - \dfrac{3}{t} \quad\Rightarrow\quad f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + C & \text{if } t < 0 \\ 3^t/\ln 3 - 3\ln t + D & \text{if } t > 0 \end{cases}$

$f(-1) = \dfrac{1}{3\ln 3} - 3\ln 1 + C$ and $f(-1) = 1 \quad\Rightarrow\quad C = 1 - \dfrac{1}{3\ln 3}$.

$f(1) = \dfrac{3}{\ln 3} - 3\ln 1 + D$ and $f(1) = 2 \quad\Rightarrow\quad D = 2 - \dfrac{3}{\ln 3}$.

Thus, $f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + 1 - 1/(3\ln 3) & \text{if } t < 0 \\ 3^t/\ln 3 - 3\ln t + 2 - 3/\ln 3 & \text{if } t > 0 \end{cases}$

39. $f''(x) = -2 + 12x - 12x^2 \quad\Rightarrow\quad f'(x) = -2x + 6x^2 - 4x^3 + C$. $f'(0) = C$ and $f'(0) = 12 \quad\Rightarrow\quad C = 12$, so $f'(x) = -2x + 6x^2 - 4x^3 + 12$ and hence, $f(x) = -x^2 + 2x^3 - x^4 + 12x + D$. $f(0) = D$ and $f(0) = 4 \quad\Rightarrow\quad D = 4$, so $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$.

40. $f''(x) = 8x^3 + 5 \quad\Rightarrow\quad f'(x) = 2x^4 + 5x + C$. $f'(1) = 2 + 5 + C$ and $f'(1) = 8 \quad\Rightarrow\quad C = 1$, so $f'(x) = 2x^4 + 5x + 1$. $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + D$. $f(1) = \frac{2}{5} + \frac{5}{2} + 1 + D = D + \frac{39}{10}$ and $f(1) = 0 \quad\Rightarrow\quad D = -\frac{39}{10}$, so $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$.

41. $f''(\theta) = \sin\theta + \cos\theta \quad\Rightarrow\quad f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \quad\Rightarrow\quad C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \quad\Rightarrow\quad D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

42. $f''(t) = t^2 + \dfrac{1}{t^2} = t^2 + t^{-2}$, $t > 0 \quad\Rightarrow\quad f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + C$. $f'(1) = \frac{1}{3} - 1 + C$ and $f'(1) = 2 \quad\Rightarrow\quad C - \frac{2}{3} = 2 \quad\Rightarrow\quad C = \frac{8}{3}$, so $f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + \frac{8}{3}$ and hence, $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + D$. $f(2) = \frac{4}{3} - \ln 2 + \frac{16}{3} + D$ and $f(2) = 3 \quad\Rightarrow\quad \frac{20}{3} - \ln 2 + D = 3 \quad\Rightarrow\quad D = \ln 2 - \frac{11}{3}$, so $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + \ln 2 - \frac{11}{3}$.

43. $f''(x) = 4 + 6x + 24x^2 \quad\Rightarrow\quad f'(x) = 4x + 3x^2 + 8x^3 + C \quad\Rightarrow\quad f(x) = 2x^2 + x^3 + 2x^4 + Cx + D$. $f(0) = D$ and $f(0) = 3 \quad\Rightarrow\quad D = 3$, so $f(x) = 2x^2 + x^3 + 2x^4 + Cx + 3$. $f(1) = 8 + C$ and $f(1) = 10 \quad\Rightarrow\quad C = 2$, so $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$.

44. $f''(x) = x^3 + \sinh x \quad\Rightarrow\quad f'(x) = \frac{1}{4}x^4 + \cosh x + C \quad\Rightarrow\quad f(x) = \frac{1}{20}x^5 + \sinh x + Cx + D$. $f(0) = D$ and $f(0) = 1 \quad\Rightarrow\quad D = 1$, so $f(x) = \frac{1}{20}x^5 + \sinh x + Cx + 1$. $f(2) = \frac{32}{20} + \sinh 2 + 2C + 1$ and $f(2) = 2.6 \quad\Rightarrow\quad \sinh 2 + 2C = 0 \quad\Rightarrow\quad C = -\frac{1}{2}\sinh 2$, so $f(x) = \frac{1}{20}x^5 + \sinh x - \frac{1}{2}(\sinh 2)x + 1$.

45. $f''(x) = e^x - 2\sin x \quad\Rightarrow\quad f'(x) = e^x + 2\cos x + C \quad\Rightarrow\quad f(x) = e^x + 2\sin x + Cx + D$.
$f(0) = 1 + 0 + D$ and $f(0) = 3 \quad\Rightarrow\quad D = 2$, so $f(x) = e^x + 2\sin x + Cx + 2$. $f\left(\frac{\pi}{2}\right) = e^{\pi/2} + 2 + \frac{\pi}{2}C + 2$ and $f\left(\frac{\pi}{2}\right) = 0 \quad\Rightarrow\quad e^{\pi/2} + 4 + \frac{\pi}{2}C = 0 \quad\Rightarrow\quad \frac{\pi}{2}C = -e^{\pi/2} - 4 \quad\Rightarrow\quad C = -\frac{2}{\pi}(e^{\pi/2} + 4)$, so
$f(x) = e^x + 2\sin x + -\frac{2}{\pi}(e^{\pi/2} + 4)x + 2$.

46. $f''(t) = \sqrt[3]{t} - \cos t = t^{1/3} - \cos t \quad \Rightarrow \quad f'(t) = \frac{3}{4}t^{4/3} - \sin t + C \quad \Rightarrow \quad f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + D.$

$f(0) = 0 + 1 + 0 + D$ and $f(0) = 2 \quad \Rightarrow \quad D = 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + 1$. $\quad f(1) = \frac{9}{28} + \cos 1 + C + 1$ and $f(1) = 2 \quad \Rightarrow \quad C = 2 - \frac{9}{28} - \cos 1 - 1 = \frac{19}{28} - \cos 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$.

47. $f''(x) = x^{-2}, x > 0 \quad \Rightarrow \quad f'(x) = -1/x + C \quad \Rightarrow \quad f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ [since $x > 0$].
$f(1) = 0 \quad \Rightarrow \quad C + D = 0$ and $f(2) = 0 \quad \Rightarrow \quad -\ln 2 + 2C + D = 0 \quad \Rightarrow \quad -\ln 2 + 2C - C = 0$ [since $D = -C$] $\Rightarrow$ $-\ln 2 + C = 0 \quad \Rightarrow \quad C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

48. $f'''(x) = \cos x \quad \Rightarrow \quad f''(x) = \sin x + C$. $f''(0) = C$ and $f''(0) = 3 \quad \Rightarrow \quad C = 3$. $f''(x) = \sin x + 3 \quad \Rightarrow$ $f'(x) = -\cos x + 3x + D$. $f'(0) = -1 + D$ and $f'(0) = 2 \quad \Rightarrow \quad D = 3$. $f'(x) = -\cos x + 3x + 3 \quad \Rightarrow$ $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + E$. $f(0) = E$ and $f(0) = 1 \quad \Rightarrow \quad E = 1$. Thus, $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

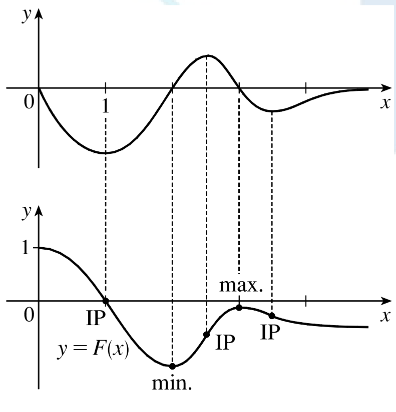
49. "The slope of its tangent line at $(x, f(x))$ is $3 - 4x$" means that $f'(x) = 3 - 4x$, so $f(x) = 3x - 2x^2 + C$.
"The graph of f passes through the point $(2, 5)$" means that $f(2) = 5$, but $f(2) = 3(2) - 2(2)^2 + C$, so $5 = 6 - 8 + C \quad \Rightarrow$ $C = 7$. Thus, $f(x) = 3x - 2x^2 + 7$ and $f(1) = 3 - 2 + 7 = 8$.

50. $f'(x) = x^3 \quad \Rightarrow \quad f(x) = \frac{1}{4}x^4 + C$. $\quad x + y = 0 \quad \Rightarrow \quad y = -x \quad \Rightarrow \quad m = -1$. Now $m = f'(x) \quad \Rightarrow \quad -1 = x^3 \quad \Rightarrow$ $x = -1 \quad \Rightarrow \quad y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f, $1 = \frac{1}{4}(-1)^4 + C \quad \Rightarrow \quad C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

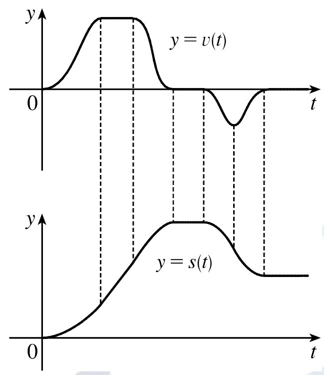
51. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

52. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

53.



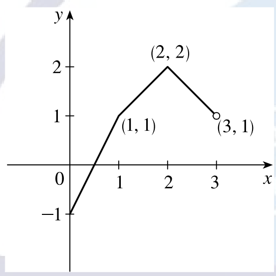
The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.
Where f is negative (positive), F is decreasing (increasing).
Where f changes from negative to positive, F will have a minimum.
Where f changes from positive to negative, F will have a maximum.
Where f is decreasing (increasing), F is concave downward (upward).

54. 

Where v is positive (negative), s is increasing (decreasing).

Where v is increasing (decreasing), s is concave upward (downward).

Where v is horizontal (a steady velocity), s is linear.

55. 

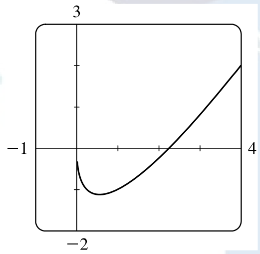
$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x < 3 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x < 3 \end{cases}$$

$f(0) = -1 \quad \Rightarrow \quad 2(0) + C = -1 \quad \Rightarrow \quad C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$.

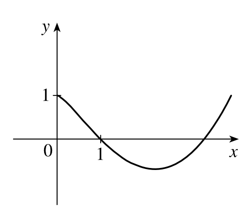
The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x < 3$ is -1, so we get to $(3, 1)$. $\quad f(2) = 2 \quad \Rightarrow \quad -2 + E = 2 \quad \Rightarrow \quad E = 4$. Thus,

$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x < 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$, 2, or 3.

56. (a) 

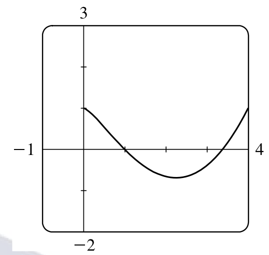
(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.



(c) $f(x) = 2x - 3\sqrt{x} \;\Rightarrow\; F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$.

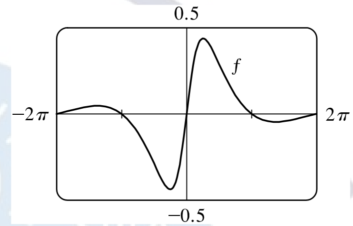
$F(0) = C$ and $F(0) = 1 \;\Rightarrow\; C = 1$, so $F(x) = x^2 - 2x^{3/2} + 1$.

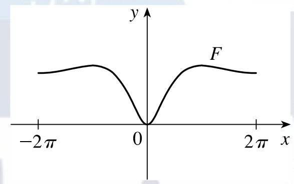
(d)



57. $f(x) = \dfrac{\sin x}{1 + x^2}$, $-2\pi \le x \le 2\pi$

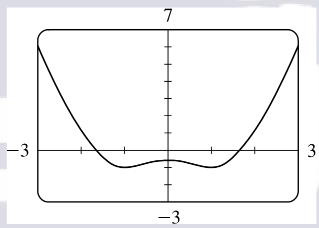
Note that the graph of f is one of an odd function, so the graph of F will be one of an even function.

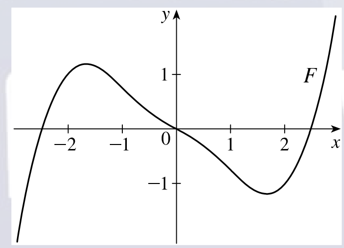




58. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2$, $-3 \le x \le 3$

Note that the graph of f is one of an even function, so the graph of F will be one of an odd function.





59. $v(t) = s'(t) = \sin t - \cos t \;\Rightarrow\; s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \;\Rightarrow\; C = 1$, so $s(t) = -\cos t - \sin t + 1$.

60. $v(t) = s'(t) = t^2 - 3\sqrt{t} = t^2 - 3t^{1/2} \;\Rightarrow\; s(t) = \frac{1}{3}t^3 - 2t^{3/2} + C$. $s(4) = \frac{64}{3} - 16 + C$ and $s(4) = 8 \;\Rightarrow$ $C = 8 - \frac{64}{3} + 16 = \frac{8}{3}$, so $s(t) = \frac{1}{3}t^3 - 2t^{3/2} + \frac{8}{3}$.

61. $a(t) = v'(t) = 2t + 1 \;\Rightarrow\; v(t) = t^2 + t + C$. $v(0) = C$ and $v(0) = -2 \;\Rightarrow\; C = -2$, so $v(t) = t^2 + t - 2$ and $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + D$. $s(0) = D$ and $s(0) = 3 \;\Rightarrow\; D = 3$, so $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$.

62. $a(t) = v'(t) = 3\cos t - 2\sin t \;\Rightarrow\; v(t) = 3\sin t + 2\cos t + C$. $v(0) = 2 + C$ and $v(0) = 4 \;\Rightarrow\; C = 2$, so $v(t) = 3\sin t + 2\cos t + 2$ and $s(t) = -3\cos t + 2\sin t + 2t + D$. $s(0) = -3 + D$ and $s(0) = 0 \;\Rightarrow\; D = 3$, so $s(t) = -3\cos t + 2\sin t + 2t + 3$.

63. $a(t) = v'(t) = 10\sin t + 3\cos t \;\Rightarrow\; v(t) = -10\cos t + 3\sin t + C \;\Rightarrow\; s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \;\Rightarrow\; D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

64. $a(t) = t^2 - 4t + 6 \quad\Rightarrow\quad v(t) = \frac{1}{3}t^3 - 2t^2 + 6t + C \quad\Rightarrow\quad s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + Ct + D$. $s(0) = D$ and $s(0) = 0 \quad\Rightarrow\quad D = 0$. $s(1) = \frac{29}{12} + C$ and $s(1) = 20 \quad\Rightarrow\quad C = \frac{211}{12}$. Thus, $s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t$.

65. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \quad\Rightarrow\quad v(t) = -9.8t + C$. Now $v(0) = 0 \quad\Rightarrow\quad C = 0$, so $v(t) = -9.8t \quad\Rightarrow$ $s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \quad\Rightarrow\quad D = 450 \quad\Rightarrow\quad s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \quad\Rightarrow\quad t^2 = 450/4.9 \quad\Rightarrow\quad t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \quad\Rightarrow\quad 0 + C = -5 \quad\Rightarrow$ $v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \quad\Rightarrow\quad D = 450 \quad\Rightarrow\quad s(t) = -4.9t^2 - 5t + 450$.
Solving $s(t) = 0$ by using the quadratic formula gives us $t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \quad\Rightarrow\quad t_1 \approx 9.09$ s.

66. $v'(t) = a(t) = a \quad\Rightarrow\quad v(t) = at + C$ and $v_0 = v(0) = C \quad\Rightarrow\quad v(t) = at + v_0 \quad\Rightarrow$ $s(t) = \frac{1}{2}at^2 + v_0t + D \quad\Rightarrow\quad s_0 = s(0) = D \quad\Rightarrow\quad s(t) = \frac{1}{2}at^2 + v_0t + s_0$

67. By Exercise 66 with $a = -9.8$, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2t^2 - 19.6v_0t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0t = v_0^2 - 19.6\left(-4.9t^2 + v_0t\right)$.
But $-4.9t^2 + v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

68. For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 7. For the second ball, $a(t) = -32 \quad\Rightarrow\quad v(t) = -32t + C$, but $v(1) = -32(1) + C = 24 \quad\Rightarrow\quad C = 56$, so $v(t) = -32t + 56 \quad\Rightarrow\quad s(t) = -16t^2 + 56t + D$, but $s(1) = -16(1)^2 + 56(1) + D = 432 \quad\Rightarrow\quad D = 392$, and $s_2(t) = -16t^2 + 56t + 392$. The balls pass each other when $s_1(t) = s_2(t) \quad\Rightarrow\quad -16t^2 + 48t + 432 = -16t^2 + 56t + 392 \quad\Leftrightarrow\quad 8t = 40 \quad\Leftrightarrow\quad t = 5$ s.

Another solution: From Exercise 66, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$.
We now want to solve $s_1(t) = s_2(t-1) \quad\Rightarrow\quad -16t^2 + 48t + 432 = -16(t-1)^2 + 24(t-1) + 432 \quad\Rightarrow$ $48t = 32t - 16 + 24t - 24 \quad\Rightarrow\quad 40 = 8t \quad\Rightarrow\quad t = 5$ s.

69. Using Exercise 66 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \quad\Rightarrow\quad -32t = -120 \quad\Rightarrow\quad t = 3.75$, so $0 = s(3.75) = -16(3.75)^2 + h \quad\Rightarrow\quad h = 16(3.75)^2 = 225$ ft.

70. (a) $EIy'' = mg(L-x) + \frac{1}{2}\rho g(L-x)^2 \quad\Rightarrow\quad EIy' = -\frac{1}{2}mg(L-x)^2 - \frac{1}{6}\rho g(L-x)^3 + C \quad\Rightarrow$
$EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + Cx + D$. Since the left end of the board is fixed, we must have $y = y' = 0$ when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that
$EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right)$ and

$$f(x) = y = \frac{1}{EI}\left[\tfrac{1}{6}mg(L-x)^3 + \tfrac{1}{24}\rho g(L-x)^4 + \left(\tfrac{1}{2}mgL^2 + \tfrac{1}{6}\rho gL^3\right)x - \left(\tfrac{1}{6}mgL^3 + \tfrac{1}{24}\rho gL^4\right)\right]$$

(b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$-f(L) = \frac{-1}{EI}\left[\tfrac{1}{2}mgL^3 + \tfrac{1}{6}\rho gL^4 - \tfrac{1}{6}mgL^3 - \tfrac{1}{24}\rho gL^4\right] = \frac{-1}{EI}\left(\tfrac{1}{3}mgL^3 + \tfrac{1}{8}\rho gL^4\right) = -\frac{gL^3}{EI}\left(\frac{m}{3} + \frac{\rho L}{8}\right)$$

Note: This is positive because g is negative.

71. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \Rightarrow K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \Rightarrow C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

72. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \Rightarrow m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the 100-centimeter rod is $m(100) = 2\sqrt{100} = 20$ g.

73. Taking the upward direction to be positive we have that for $0 \leq t \leq 10$ (using the subscript 1 to refer to $0 \leq t \leq 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \Rightarrow$ $v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \Rightarrow$ $s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow$ $v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow v(t) = -55$.
At 55 m/s, it will take $100/55 \approx 1.8$ s to fall the last 100 m. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

74. $v'(t) = a(t) = -22$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -22t + \frac{220}{3}$.
The car stops when $v(t) = 0 \Leftrightarrow t = \frac{220}{3 \cdot 22} = \frac{10}{3}$. Since $s(t) = -11t^2 + \frac{220}{3}t$, the distance covered is $s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3} \cdot \frac{10}{3} = \frac{1100}{9} = 122.\overline{2}$ ft.

75. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is 50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \Rightarrow C = 44$. Thus, $v(t) = kt + 44 \Rightarrow$ $v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \Rightarrow 5k = \frac{88}{3} \Rightarrow k = \frac{88}{15} \approx 5.87$ ft/s^2.

76. $a(t) = -16 \Rightarrow v(t) = -16t + v_0$ where v_0 is the car's speed (in ft/s) when the brakes were applied. The car stops when $-16t + v_0 = 0 \Leftrightarrow t = \frac{1}{16}v_0$. Now $s(t) = \frac{1}{2}(-16)t^2 + v_0t = -8t^2 + v_0t$. The car travels 200 ft in the time that it takes to stop, so $s\left(\frac{1}{16}v_0\right) = 200 \Rightarrow 200 = -8\left(\frac{1}{16}v_0\right)^2 + v_0\left(\frac{1}{16}v_0\right) = \frac{1}{32}v_0^2 \Rightarrow v_0^2 = 32 \cdot 200 = 6400 \Rightarrow$ $v_0 = 80$ ft/s [$54.\overline{54}$ mi/h].

77. Let the acceleration be $a(t) = k$ km/h^2. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0. We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$.

Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$s(t_f) = \frac{1}{2}k\left(-\frac{100}{k}\right)^2 + 100\left(-\frac{100}{k}\right) = 10{,}000\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is

$-\frac{5{,}000}{k} < 0.08 \quad\Rightarrow\quad -\frac{5{,}000}{0.08} > k \quad [k \text{ is negative}] \quad\Rightarrow\quad k < -62{,}500 \text{ km/h}^2$, or equivalently,

$k < -\frac{3125}{648} \approx -4.82 \text{ m/s}^2$.

78. (a) For $0 \le t \le 3$ we have $a(t) = 60t \quad\Rightarrow\quad v(t) = 30t^2 + C \quad\Rightarrow\quad v(0) = 0 = C \quad\Rightarrow\quad v(t) = 30t^2$, so

$s(t) = 10t^3 + C \quad\Rightarrow\quad s(0) = 0 = C \quad\Rightarrow\quad s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$.

For $3 < t \le 17$: $a(t) = -g = -32$ ft/s $\quad\Rightarrow\quad v(t) = -32(t-3) + C \quad\Rightarrow\quad v(3) = 270 = C \quad\Rightarrow$

$v(t) = -32(t-3) + 270 \quad\Rightarrow\quad s(t) = -16(t-3)^2 + 270(t-3) + C \quad\Rightarrow\quad s(3) = 270 = C \quad\Rightarrow$

$s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$.

For $17 < t \le 22$: The velocity increases linearly from -178 ft/s to -18 ft/s during this period, so

$\frac{\Delta v}{\Delta t} = \frac{-18 - (-178)}{22 - 17} = \frac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \quad\Rightarrow$

$s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424$ ft.

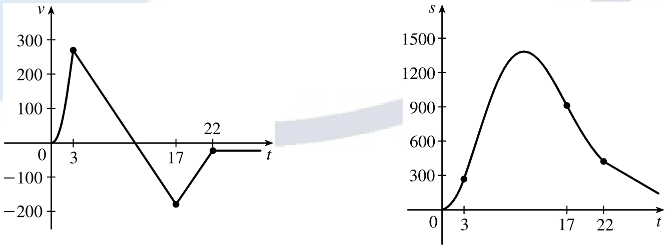
For $t > 22$: $v(t) = -18 \quad\Rightarrow\quad s(t) = -18(t-22) + C$. But $s(22) = 424 = C \quad\Rightarrow\quad s(t) = -18(t-22) + 424$.

Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32\,(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178\,(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$



(b) To find the maximum height, set $v(t)$ on $3 < t \le 17$ equal to 0. $\quad -32(t-3) + 270 = 0 \quad\Rightarrow\quad t_1 = 11.4375$ s and the maximum height is $s(t_1) = -16(t_1 - 3)^2 + 270(t_1 - 3) + 270 = 1409.0625$ ft.

(c) To find the time to land, set $s(t) = -18(t-22) + 424 = 0$. Then $t - 22 = \frac{424}{18} = 23.\overline{5}$, so $t \approx 45.6$ s.

79. (a) First note that $90\text{ mi/h} = 90 \times \frac{5280}{3600}\text{ ft/s} = 132\text{ ft/s}$. Then $a(t) = 4\text{ ft/s}^2 \quad\Rightarrow\quad v(t) = 4t + C$, but $v(0) = 0 \quad\Rightarrow$ $C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \le t \le 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 < t \le 933$ we have $v(t) = 132\text{ ft/s} \quad\Rightarrow\quad s(933) = 132(900) + 2178 = 120{,}978\text{ ft} = 22.9125$ mi.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is $2178 + 110{,}088 + 2178 = 114{,}444\text{ ft} = 21.675$ mi.

(c) $45\text{ mi} = 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is $1767 + 2(33) = 1833\text{ s} = 30\text{ min }33\text{ s} = 30.55$ min.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

4 Review

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad\Leftrightarrow\quad c \in (-1, 1)$.

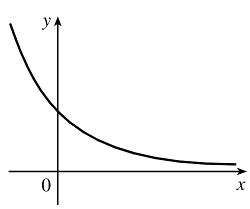
5. True. This is an example of part (b) of the I/D Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad\Rightarrow\quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \ne g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis—at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ [since f and g are increasing on I], so $(f+g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f+g)(x_2)$.

12. False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)\,g(x) = x(x-1)$ is not increasing on $(0, 1)$.

14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ [since f and g are both positive and increasing]. Hence, $f(x_1)\,g(x_1) < f(x_2)\,g(x_1) < f(x_2)\,g(x_2)$. So fg is increasing on I.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ [f is increasing] $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ [f is positive] $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

16. False. If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

17. True. If f is periodic, then there is a number p such that $f(x+p) = f(p)$ for all x. Differentiating gives $f'(x) = f'(x+p) \cdot (x+p)' = f'(x+p) \cdot 1 = f'(x+p)$, so f' is periodic.

18. False. The most general antiderivative of $f(x) = x^{-2}$ is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ [see Example 4.9.1(b)].

19. True. By the Mean Value Theorem, there exists a number c in $(0, 1)$ such that $f(1) - f(0) = f'(c)(1-0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

20. False. Let $f(x) = 1 + \dfrac{1}{x}$ and $g(x) = x$. Then $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} g(x) = \infty$, but

$$\lim_{x\to\infty} [f(x)]^{g(x)} = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e, \text{ not } 1.$$

21. False. $\displaystyle\lim_{x\to 0}\frac{x}{e^x} = \frac{\lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} e^x} = \frac{0}{1} = 0$, not 1.

EXERCISES

1. $f(x) = x^3 - 9x^2 + 24x - 2$, $[0, 5]$. $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$. $f'(x) = 0$ $\Leftrightarrow$ $x = 2$ or $x = 4$. $f'(x) > 0$ for $0 < x < 2$, $f'(x) < 0$ for $2 < x < 4$, and $f'(x) > 0$ for $4 < x < 5$, so $f(2) = 18$ is a local maximum value and $f(4) = 14$ is a local minimum value. Checking the endpoints, we find $f(0) = -2$ and $f(5) = 18$. Thus, $f(0) = -2$ is the absolute minimum value and $f(2) = f(5) = 18$ is the absolute maximum value.

2. $f(x) = x\sqrt{1-x}$, $[-1,1]$. $f'(x) = x \cdot \frac{1}{2}(1-x)^{-1/2}(-1) + (1-x)^{1/2}(1) = (1-x)^{-1/2}\left[-\frac{1}{2}x + (1-x)\right] = \dfrac{1-\frac{3}{2}x}{\sqrt{1-x}}$.

$f'(x) = 0 \;\Rightarrow\; x = \frac{2}{3}$. $f'(x)$ does not exist $\Leftrightarrow$ $x = 1$. $f'(x) > 0$ for $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ for $\frac{2}{3} < x < 1$, so $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{3}$ [≈ 0.38] is a local maximum value. Checking the endpoints, we find $f(-1) = -\sqrt{2}$ and $f(1) = 0$. Thus, $f(-1) = -\sqrt{2}$ is the absolute minimum value and $f\left(\frac{2}{3}\right) = \frac{2}{9}\sqrt{3}$ is the absolute maximum value.

3. $f(x) = \dfrac{3x-4}{x^2+1}$, $[-2,2]$. $f'(x) = \dfrac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2} = \dfrac{-(3x^2-8x-3)}{(x^2+1)^2} = \dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$.

$f'(x) = 0 \;\Rightarrow\; x = -\frac{1}{3}$ or $x = 3$, but 3 is not in the interval. $f'(x) > 0$ for $-\frac{1}{3} < x < 2$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{3}$, so $f\left(-\frac{1}{3}\right) = \frac{-5}{10/9} = -\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2) = -2$ and $f(2) = \frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$ is the absolute minimum value and $f(2) = \frac{2}{5}$ is the absolute maximum value.

4. $f(x) = \sqrt{x^2+x+1}$, $[-2,1]$. $f'(x) = \frac{1}{2}(x^2+x+1)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$. $f'(x) = 0 \;\Rightarrow\; x = -\dfrac{1}{2}$.

$f'(x) > 0$ for $-\frac{1}{2} < x < 1$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{2}$, so $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is a local minimum value. Checking the endpoints, we find $f(-2) = f(1) = \sqrt{3}$. Thus, $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is the absolute minimum value and $f(-2) = f(1) = \sqrt{3}$ is the absolute maximum value.

5. $f(x) = x + 2\cos x$, $[-\pi,\pi]$. $f'(x) = 1 - 2\sin x$. $f'(x) = 0 \;\Rightarrow\; \sin x = \frac{1}{2} \;\Rightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}$. $f'(x) > 0$ for $\left(-\pi, \frac{\pi}{6}\right)$ and $\left(\frac{5\pi}{6}, \pi\right)$, and $f'(x) < 0$ for $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, so $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \approx 2.26$ is a local maximum value and $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.89$ is a local minimum value. Checking the endpoints, we find $f(-\pi) = -\pi - 2 \approx -5.14$ and $f(\pi) = \pi - 2 \approx 1.14$. Thus, $f(-\pi) = -\pi - 2$ is the absolute minimum value and $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ is the absolute maximum value.

6. $f(x) = x^2 e^{-x}$, $[-1,3]$. $f'(x) = x^2(-e^{-x}) + e^{-x}(2x) = xe^{-x}(-x+2)$. $f'(x) = 0 \;\Rightarrow\; x = 0$ or $x = 2$. $f'(x) > 0$ for $0 < x < 2$ and $f'(x) < 0$ for $-1 < x < 0$ and $2 < x < 3$, so $f(0) = 0$ is a local minimum value and $f(2) = 4e^{-2} \approx 0.54$ is a local maximum value. Checking the endpoints, we find $f(-1) = e \approx 2.72$ and $f(3) = 9e^{-3} \approx 0.45$. Thus, $f(0) = 0$ is the absolute minimum value and $f(-1) = e$ is the absolute maximum value.

7. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{\sec^2 x} = \frac{1}{1} = 1$

8. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{x + \sin 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4\sec^2 4x}{1 + 2\cos 2x} = \frac{4(1)}{1 + 2(1)} = \frac{4}{3}$

9. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \frac{2+2}{1} = 4$

10. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to \infty} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \lim_{x\to\infty} 2(x+1)(e^{2x} + e^{-2x}) = \infty$

since $2(x+1) \to \infty$ and $(e^{2x} + e^{-2x}) \to \infty$ as $x \to \infty$.

11. This limit has the form $\infty \cdot 0$.

$$\lim_{x\to-\infty}(x^2-x^3)e^{2x} = \lim_{x\to-\infty}\frac{x^2-x^3}{e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to-\infty}\frac{2x-3x^2}{-2e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right]$$

$$\overset{\text{H}}{=} \lim_{x\to-\infty}\frac{2-6x}{4e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to-\infty}\frac{-6}{-8e^{-2x}} = 0$$

12. This limit has the form $0 \cdot \infty$. $\displaystyle \lim_{x\to\pi^-}(x-\pi)\csc x = \lim_{x\to\pi^-}\frac{x-\pi}{\sin x} \quad \left[\tfrac{0}{0} \text{ form}\right] \overset{\text{H}}{=} \lim_{x\to\pi^-}\frac{1}{\cos x} = \frac{1}{-1} = -1$

13. This limit has the form $\infty - \infty$.

$$\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) = \lim_{x\to1^+}\left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to1^+}\frac{x\cdot(1/x)+\ln x - 1}{(x-1)\cdot(1/x)+\ln x} = \lim_{x\to1^+}\frac{\ln x}{1-1/x+\ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to1^+}\frac{1/x}{1/x^2+1/x} = \frac{1}{1+1} = \frac{1}{2}$$

14. $y = (\tan x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \, \ln \tan x$, so

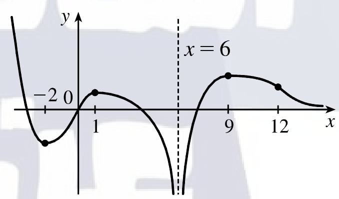
$$\lim_{x\to(\pi/2)^-}\ln y = \lim_{x\to(\pi/2)^-}\frac{\ln\tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-}\frac{(1/\tan x)\sec^2 x}{\sec x\,\tan x} = \lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan^2 x} = \lim_{x\to(\pi/2)^-}\frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0,$$

so $\displaystyle \lim_{x\to(\pi/2)^-}(\tan x)^{\cos x} = \lim_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1.$

15. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $\displaystyle\lim_{x\to\infty} f(x) = 0$, $\displaystyle\lim_{x\to6} f(x) = -\infty$,

$f'(x) < 0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$, $f'(x) > 0$ on $(-2,1)$ and $(6,9)$,

$f''(x) > 0$ on $(-\infty,0)$ and $(12,\infty)$, $f''(x) < 0$ on $(0,6)$ and $(6,12)$



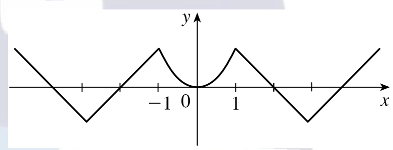
16. For $0 < x < 1$, $f'(x) = 2x$, so $f(x) = x^2 + C$. Since $f(0) = 0$,

$f(x) = x^2$ on $[0,1]$. For $1 < x < 3$, $f'(x) = -1$, so $f(x) = -x + D$.

$1 = f(1) = -1 + D \quad \Rightarrow \quad D = 2$, so $f(x) = 2 - x$. For $x > 3$, $f'(x) = 1$,

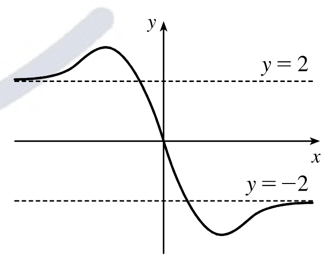
so $f(x) = x + E$. $-1 = f(3) = 3 + E \quad \Rightarrow \quad E = -4$, so $f(x) = x - 4$.

Since f is even, its graph is symmetric about the y-axis.



17. f is odd, $f'(x) < 0$ for $0 < x < 2$, $\quad f'(x) > 0$ for $x > 2$,

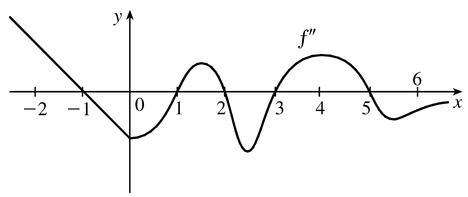
$f''(x) > 0$ for $0 < x < 3$, $\quad f''(x) < 0$ for $x > 3$, $\lim_{x\to\infty} f(x) = -2$



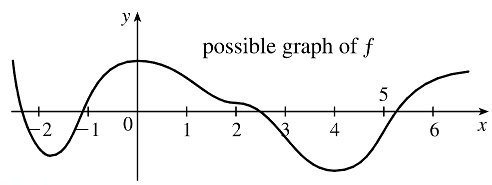
18. (a) Using the Test for Monotonic Functions we know that f is increasing on $(-2,0)$ and $(4,\infty)$ because $f' > 0$ on $(-2,0)$ and $(4,\infty)$, and that f is decreasing on $(-\infty,-2)$ and $(0,4)$ because $f' < 0$ on $(-\infty,-2)$ and $(0,4)$.

(b) Using the First Derivative Test, we know that f has a local maximum at $x = 0$ because f' changes from positive to negative at $x = 0$, and that f has a local minimum at $x = 4$ because f' changes from negative to positive at $x = 4$.

(c)



(d)

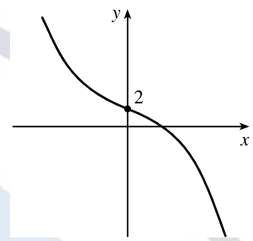


19. $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$. The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.

F. No extreme value **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

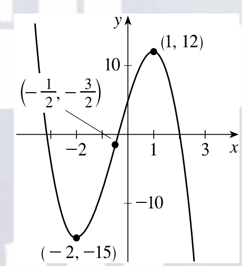
H.



20. $y = f(x) = -2x^3 - 3x^2 + 12x + 5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 5$; x-intercept: $f(x) = 0 \Leftrightarrow x \approx -3.15, -0.39, 2.04$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x + 2)(x - 1)$. $f'(x) > 0$ for $-2 < x < 1$, so f is increasing on $(-2, 1)$ and decreasing on $(-\infty, -2)$ and $(1, \infty)$. **F.** Local minimum value $f(-2) = -15$, local maximum value $f(1) = 12$ **G.** $f''(x) = -12x - 6 = -12\left(x + \frac{1}{2}\right)$. $f''(x) > 0$ for $x < -\frac{1}{2}$, so f is CU on $\left(-\infty, -\frac{1}{2}\right)$ and CD on $\left(-\frac{1}{2}, \infty\right)$. There is an IP at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

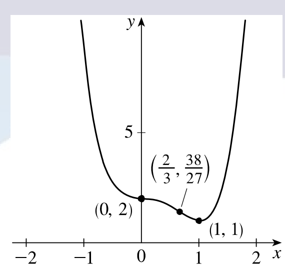
H.



21. $y = f(x) = 3x^4 - 4x^3 + 2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptote

E. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** $f'(x)$ does not change sign at $x = 0$, so there is no local extremum there. $f(1) = 1$ is a local minimum value. **G.** $f''(x) = 36x^2 - 24x = 12x(3x - 2)$. $f''(x) < 0$ for $0 < x < \frac{2}{3}$, so f is CD on $\left(0, \frac{2}{3}\right)$ and f is CU on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$. There are inflection points at $(0, 2)$ and $\left(\frac{2}{3}, \frac{38}{27}\right)$.

H.



22. $y = f(x) = \dfrac{x}{1 - x^2}$ **A.** $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: 0

C. $f(-x) = -f(x)$, so f is odd and the graph is symmetric about the origin. **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{x}{1 - x^2} = 0$, so $y = 0$ is a HA.

$\displaystyle\lim_{x \to -1^-} \frac{x}{1 - x^2} = \infty$ and $\displaystyle\lim_{x \to -1^+} \frac{x}{1 - x^2} = -\infty$, so $x = -1$ is a VA. Similarly, $\displaystyle\lim_{x \to 1^-} \frac{x}{1 - x^2} = \infty$ and

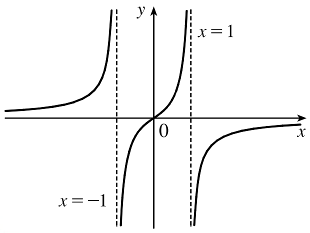
$\displaystyle\lim_{x \to 1^+} \frac{x}{1 - x^2} = -\infty$, so $x = 1$ is a VA. **E.** $f'(x) = \dfrac{(1 - x^2)(1) - x(-2x)}{(1 - x^2)^2} = \dfrac{1 + x^2}{(1 - x^2)^2} > 0$ for $x \neq \pm 1$, so f is increasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. **F.** No local extrema

G. $f''(x) = \dfrac{(1-x^2)^2(2x) - (1+x^2)2(1-x^2)(-2x)}{[(1-x^2)^2]^2}$

$= \dfrac{2x(1-x^2)[(1-x^2)+2(1+x^2)]}{(1-x^2)^4} = \dfrac{2x(3+x^2)}{(1-x^2)^3}$

$f''(x) > 0$ for $x < -1$ and $0 < x < 1$, and $f''(x) < 0$ for $-1 < x < 0$ and $x > 1$, so f is CU on $(-\infty, -1)$ and $(0, 1)$, and f is CD on $(-1, 0)$ and $(1, \infty)$. $(0, 0)$ is an IP.

H.



23. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts. **C.** No symmetry.

D. $\lim\limits_{x \to \pm\infty} \dfrac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 0^+} \dfrac{1}{x(x-3)^2} = \infty$, $\lim\limits_{x \to 0^-} \dfrac{1}{x(x-3)^2} = -\infty$, $\lim\limits_{x \to 3} \dfrac{1}{x(x-3)^2} = \infty$,

so $x = 0$ and $x = 3$ are VA. **E.** $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3} \Rightarrow f'(x) > 0 \Leftrightarrow 1 < x < 3$,

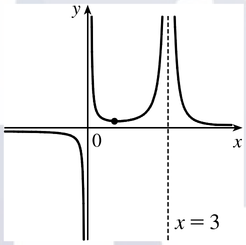
so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$.

F. Local minimum value $f(1) = \frac{1}{4}$ **G.** $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$.

Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant.

So $f''(x) > 0 \Leftrightarrow x > 0 \Rightarrow f$ is CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



24. $y = f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x-2)^2}$ **A.** $D = \{x \mid x \neq 0, 2\}$ **B.** y-intercept: none; x-intercept: $f(x) = 0 \Rightarrow$

$\dfrac{1}{x^2} = \dfrac{1}{(x-2)^2} \Leftrightarrow (x-2)^2 = x^2 \Leftrightarrow x^2 - 4x + 4 = x^2 \Leftrightarrow 4x = 4 \Leftrightarrow x = 1$ **C.** No symmetry

D. $\lim\limits_{x \to 0} f(x) = \infty$ and $\lim\limits_{x \to 2} f(x) = -\infty$, so $x = 0$ and $x = 2$ are VA; $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a HA

E. $f'(x) = -\dfrac{2}{x^3} + \dfrac{2}{(x-2)^3} > 0 \Rightarrow \dfrac{-(x-2)^3 + x^3}{x^3(x-2)^3} > 0 \Leftrightarrow \dfrac{-x^3 + 6x^2 - 12x + 8 + x^3}{x^3(x-2)^3} > 0 \Leftrightarrow$

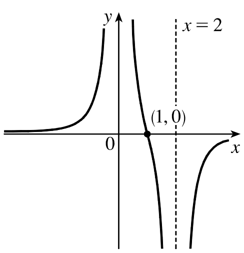
$\dfrac{2(3x^2 - 6x + 4)}{x^3(x-2)^3} > 0$. The numerator is positive (the discriminant of the quadratic is negative), so $f'(x) > 0$ if $x < 0$ or $x > 2$, and hence, f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

F. No local extreme values **G.** $f''(x) = \dfrac{6}{x^4} - \dfrac{6}{(x-2)^4} > 0 \Rightarrow$

$\dfrac{(x-2)^4 - x^4}{x^4(x-2)^4} > 0 \Leftrightarrow \dfrac{x^4 - 8x^3 + 24x^2 - 32x + 16 - x^4}{x^4(x-2)^4} > 0 \Leftrightarrow$

$\dfrac{-8(x^3 - 3x^2 + 4x - 2)}{x^4(x-2)^4} > 0 \Leftrightarrow \dfrac{-8(x-1)(x^2 - 2x + 2)}{x^4(x-2)^4} > 0$. So f'' is positive for $x < 1$ $[x \neq 0]$ and negative for $x > 1$ $[x \neq 2]$. Thus, f is CU on $(-\infty, 0)$ and $(0, 1)$ and f is CD on $(1, 2)$ and $(2, \infty)$. IP at $(1, 0)$

H.



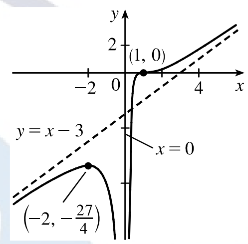
25. $y = f(x) = \dfrac{(x-1)^3}{x^2} = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2} = x - 3 + \dfrac{3x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$

B. y-intercept: none; x-intercept: $f(x) = 0 \Leftrightarrow x = 1$ **C.** No symmetry **D.** $\displaystyle\lim_{x\to 0^-} \frac{(x-1)^3}{x^2} = -\infty$ and

$\displaystyle\lim_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA. $f(x) - (x-3) = \dfrac{3x-1}{x^2} \to 0$ as $x \to \pm\infty$, so $y = x - 3$ is a SA.

E. $f'(x) = \dfrac{x^2 \cdot 3(x-1)^2 - (x-1)^3(2x)}{(x^2)^2} = \dfrac{x(x-1)^2[3x - 2(x-1)]}{x^4} = \dfrac{(x-1)^2(x+2)}{x^3}$. $f'(x) < 0$ for $-2 < x < 0$,

so f is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, \infty)$. **H.**



F. Local maximum value $f(-2) = -\frac{27}{4}$ **G.** $f(x) = x - 3 + \dfrac{3}{x} - \dfrac{1}{x^2} \Rightarrow$

$f'(x) = 1 - \dfrac{3}{x^2} + \dfrac{2}{x^3} \Rightarrow f''(x) = \dfrac{6}{x^3} - \dfrac{6}{x^4} = \dfrac{6x-6}{x^4} = \dfrac{6(x-1)}{x^4}$.

$f''(x) > 0$ for $x > 1$, so f is CD on $(-\infty, 0)$ and $(0, 1)$, and f is CU on $(1, \infty)$.

There is an inflection point at $(1, 0)$.

26. $y = f(x) = \sqrt{1-x} + \sqrt{1+x}$ **A.** $1 - x \geq 0$ and $1 + x \geq 0 \Rightarrow x \leq 1$ and $x \geq -1$, so $D = [-1, 1]$.

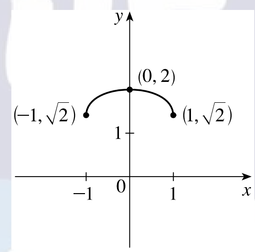
B. y-intercept: $f(0) = 1 + 1 = 2$; no x-intercept because $f(x) > 0$ for all x.

C. $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1) + \frac{1}{2}(1+x)^{-1/2} = \dfrac{-1}{2\sqrt{1-x}} + \dfrac{1}{2\sqrt{1+x}} = \dfrac{-\sqrt{1+x} + \sqrt{1-x}}{2\sqrt{1-x}\sqrt{1+x}} > 0 \Rightarrow$

$-\sqrt{1+x} + \sqrt{1-x} > 0 \Rightarrow \sqrt{1-x} > \sqrt{1+x} \Rightarrow 1 - x > 1 + x \Rightarrow -2x > 0 \Rightarrow x < 0$, so $f'(x) > 0$ for

$-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 1$. Thus, f is increasing on $(-1, 0)$ **H.**



and decreasing on $(0, 1)$. **F.** Local maximum value $f(0) = 2$

G. $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) + \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$

$$= \frac{-1}{4(1-x)^{3/2}} + \frac{-1}{4(1+x)^{3/2}} < 0$$

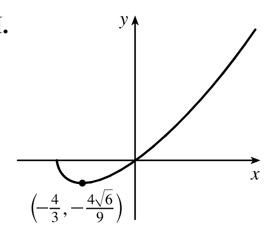
for all x in the domain, so f is CD on $(-1, 1)$. No IP

27. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is

decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$,

no local maximum **H.**



G. $f''(x) = \dfrac{2\sqrt{2+x} \cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \dfrac{6(2+x) - (3x+4)}{4(2+x)^{3/2}}$

$$= \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

28. $y = f(x) = x^{2/3}(x-3)^2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0, 3$

C. No symmetry **D.** No asymptote

E. $f'(x) = x^{2/3} \cdot 2(x-3) + (x-3)^2 \cdot \frac{2}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}(x-3)[3x + (x-3)] = \frac{2}{3}x^{-1/3}(x-3)(4x-3)$.

$f'(x) > 0 \Leftrightarrow 0 < x < \frac{3}{4}$ or $x > 3$, so f is decreasing on $(-\infty, 0)$, increasing on $\left(0, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 3\right)$, and increasing on $(3, \infty)$. **F.** Local minimum value $f(0) = f(3) = 0$; local maximum value

$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{2/3}\left(-\frac{9}{4}\right)^2 = \frac{81}{16}\sqrt[3]{\frac{9}{16}} = \frac{81}{32}\sqrt[3]{\frac{9}{2}}$ $[\approx 4.18]$

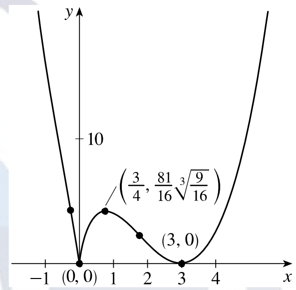
G. $f'(x) = \left(\frac{2}{3}x^{-1/3}\right)(4x^2 - 15x + 9) \Rightarrow$

$$f''(x) = \left(\frac{2}{3}x^{-1/3}\right)(8x - 15) + (4x^2 - 15x + 9)\left(-\frac{2}{9}x^{-4/3}\right)$$
$$= \frac{2}{9}x^{-4/3}[3x(8x-15) - (4x^2 - 15x + 9)]$$
$$= \frac{2}{9}x^{-4/3}(20x^2 - 30x - 9)$$

$f''(x) = 0 \Leftrightarrow x \approx -0.26$ or 1.76. $f''(x)$ does not exist at $x = 0$.

f is CU on $(-\infty, -0.26)$, CD on $(-0.26, 0)$, CD on $(0, 1.76)$, and CU on $(1.76, \infty)$. There are inflection points at $(-0.26, 4.28)$ and $(1.76, 2.25)$.

H.



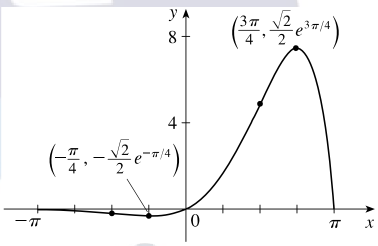
29. $y = f(x) = e^x \sin x$, $-\pi \le x \le \pi$ **A.** $D = [-\pi, \pi]$ **B.** y-intercept: $f(0) = 0$; $f(x) = 0 \Leftrightarrow \sin x = 0 \Rightarrow x = -\pi, 0, \pi$. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^x \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$.

$f'(x) = 0 \Leftrightarrow -\cos x = \sin x \Leftrightarrow -1 = \tan x \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}$. $f'(x) > 0$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $f'(x) < 0$ for $-\pi < x < -\frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$, so f is increasing on $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and f is decreasing on $\left(-\pi, -\frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

F. Local minimum value $f\left(-\frac{\pi}{4}\right) = (-\sqrt{2}/2)e^{-\pi/4} \approx -0.32$ and local maximum value $f\left(\frac{3\pi}{4}\right) = \left(\sqrt{2}/2\right)e^{3\pi/4} \approx 7.46$

G. $f''(x) = e^x(-\sin x + \cos x) + (\cos x + \sin x)e^x = e^x(2\cos x) > 0 \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f''(x) < 0 \Rightarrow -\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and f is CD on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. There are inflection points at $\left(-\frac{\pi}{2}, -e^{-\pi/2}\right)$ and $\left(\frac{\pi}{2}, e^{\pi/2}\right)$.

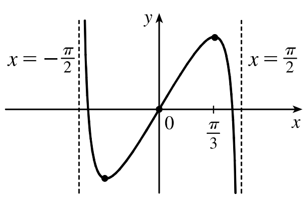
H.



30. $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x \to \pi/2^-}(4x - \tan x) = -\infty$, $\lim\limits_{x \to -\pi/2^+}(4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \Leftrightarrow \sec x < 2 \Leftrightarrow \cos x > \frac{1}{2} \Leftrightarrow -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum value, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value.

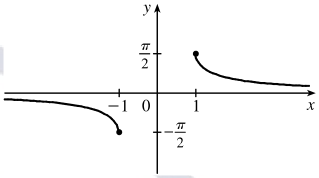
G. $f''(x) = -2\sec^2 x \tan x > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, so f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.



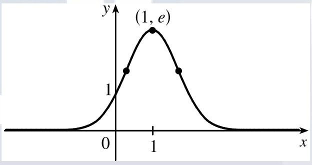
31. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept
C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.
E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.
F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.
G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2-1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

H.



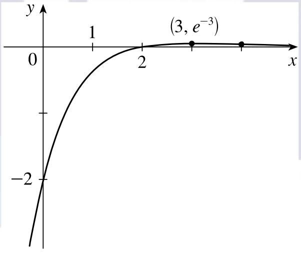
32. $y = f(x) = e^{2x-x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.
G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.
$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.



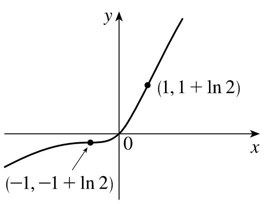
33. $y = f(x) = (x-2)e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$; x-intercept: $f(x) = 0 \Leftrightarrow x = 2$
C. No symmetry **D.** $\lim_{x\to\infty} \dfrac{x-2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. No VA
E. $f'(x) = (x-2)(-e^{-x}) + e^{-x}(1) = e^{-x}[-(x-2)+1] = (3-x)e^{-x}$.
$f'(x) > 0$ for $x < 3$, so f is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.
F. Local maximum value $f(3) = e^{-3}$, no local minimum value
G. $f''(x) = (3-x)(-e^{-x}) + e^{-x}(-1) = e^{-x}[-(3-x)+(-1)]$
$= (x-4)e^{-x} > 0$
for $x > 4$, so f is CU on $(4, \infty)$ and CD on $(-\infty, 4)$. IP at $(4, 2e^{-4})$

H.



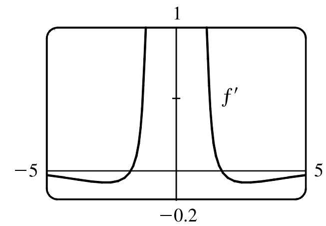
34. $y = f(x) = x + \ln(x^2+1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0 + \ln 1 = 0$; x-intercept: $f(x) = 0 \Leftrightarrow \ln(x^2+1) = -x \Leftrightarrow x^2 + 1 = e^{-x} \Rightarrow x = 0$ since the graphs of $y = x^2 + 1$ and $y = e^{-x}$ intersect only at $x = 0$.
C. No symmetry **D.** No asymptote **E.** $f'(x) = 1 + \dfrac{2x}{x^2+1} = \dfrac{x^2+2x+1}{x^2+1} = \dfrac{(x+1)^2}{x^2+1}$. $f'(x) > 0$ if $x \neq -1$ and f is increasing on $\mathbb{R}$. **F.** No local extreme values
G. $f''(x) = \dfrac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2} = \dfrac{2[(x^2+1) - 2x^2]}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.
$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(-1, -1 + \ln 2)$ and $(1, 1 + \ln 2)$

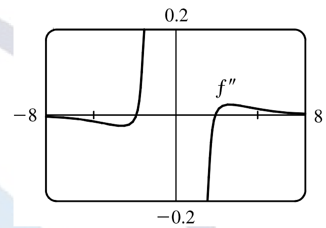
H.

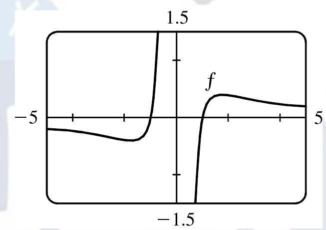


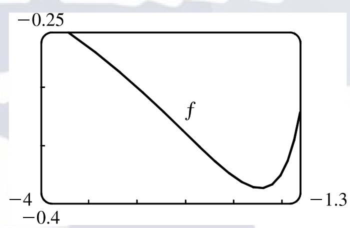
35. $f(x)=\dfrac{x^2-1}{x^3} \quad\Rightarrow\quad f'(x)=\dfrac{x^3(2x)-(x^2-1)3x^2}{x^6}=\dfrac{3-x^2}{x^4} \quad\Rightarrow$

$f''(x)=\dfrac{x^4(-2x)-(3-x^2)4x^3}{x^8}=\dfrac{2x^2-12}{x^5}$









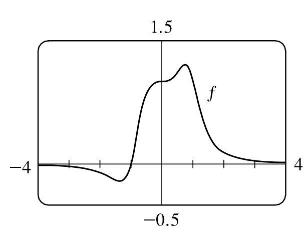
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73,0)$ and $(0,1.73)$ and decreasing on $(-\infty,-1.73)$ and $(1.73,\infty)$; f has a local maximum of about $f(1.73)=0.38$ and a local minimum of about $f(-1.7)=-0.38$; f is CU on $(-2.45,0)$ and $(2.45,\infty)$, and CD on $(-\infty,-2.45)$ and $(0,2.45)$; and f has inflection points at about $(-2.45,-0.34)$ and $(2.45,0.34)$.

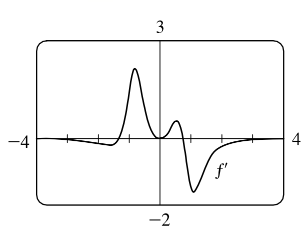
Exact: Now $f'(x)=\dfrac{3-x^2}{x^4}$ is positive for $0<x^2<3$, that is, f is increasing on $(-\sqrt{3},0)$ and $(0,\sqrt{3})$; and $f'(x)$ is negative (and so f is decreasing) on $(-\infty,-\sqrt{3})$ and $(\sqrt{3},\infty)$. $f'(x)=0$ when $x=\pm\sqrt{3}$.

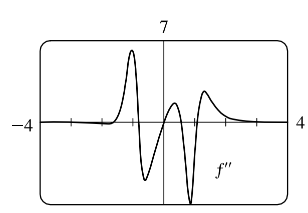
f' goes from positive to negative at $x=\sqrt{3}$, so f has a local maximum of $f(\sqrt{3})=\frac{(\sqrt{3})^2-1}{(\sqrt{3})^3}=\frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0,\infty)$ correspond to minima on $(-\infty,0)$, so f has a local minimum of $f(-\sqrt{3})=-\frac{2\sqrt{3}}{9}$. Also, $f''(x)=\dfrac{2x^2-12}{x^5}$ is positive (so f is CU) on $(-\sqrt{6},0)$ and $(\sqrt{6},\infty)$, and negative (so f is CD) on $(-\infty,-\sqrt{6})$ and $(0,\sqrt{6})$. There are IP at $\left(\sqrt{6},\frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6},-\frac{5\sqrt{6}}{36}\right)$.

36. $f(x)=\dfrac{x^3+1}{x^6+1} \quad\Rightarrow\quad f'(x)=-\dfrac{3x^2(x^6+2x^3-1)}{(x^6+1)^2} \quad\Rightarrow\quad f''(x)=\dfrac{6x(2x^{12}+7x^9-9x^6-5x^3+1)}{(x^6+1)^3}.$

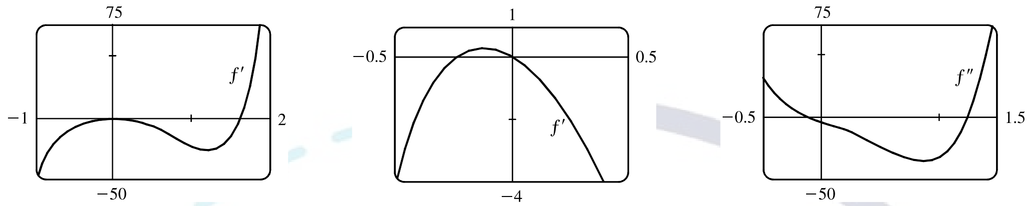
$f(x)=0 \;\Leftrightarrow\; x=-1$. $f'(x)=0 \;\Leftrightarrow\; x=0$ or $x\approx-1.34, 0.75$. $f''(x)=0 \;\Leftrightarrow\; x=0$ or $x\approx-1.64,-0.82,0.54, 1.09$. From the graphs of f and f', it appears that f is decreasing on $(-\infty,-1.34)$, increasing on $(-1.34,0.75)$, and decreasing on $(0.75,\infty)$. f has a local minimum value of $f(-1.34)\approx-0.21$ and a local maximum value of $f(0.75)\approx1.21$. From the graphs of f and f'', it appears that f is CD on $(-\infty,-1.64)$, CU on $(-1.64,-0.82)$, CD on $(-0.82,0)$, CU on $(0,0.54)$, CD on $(0.54,1.09)$ and CU on $(1.09,\infty)$. There are inflection points at about $(-1.64,-0.17)$, $(-0.82,0.34)$, $(0.54,1.13)$, $(1.09,0.86)$, and at $(0,1)$.



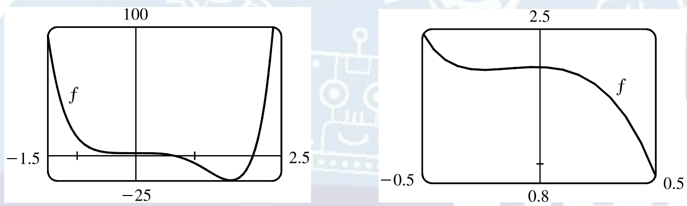




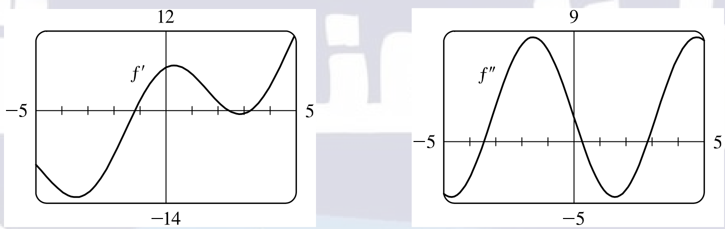
37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$



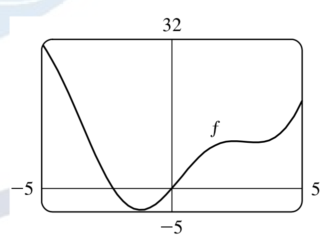
From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.



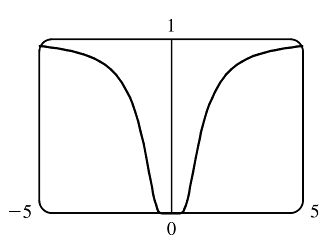
38. $f(x) = x^2 + 6.5\sin x,\ -5 \le x \le 5 \quad \Rightarrow \quad f'(x) = 2x + 6.5\cos x \quad \Rightarrow \quad f''(x) = 2 - 6.5\sin x.\quad f(x) = 0 \quad \Leftrightarrow$
$x \approx -2.25$ and $x = 0$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.19, 2.40, 3.24$; $\quad f''(x) = 0 \quad \Leftrightarrow \quad x \approx -3.45, 0.31, 2.83$.



From the graphs of f' and f'', it appears that f is decreasing on $(-5, -1.19)$ and $(2.40, 3.24)$ and increasing on $(-1.19, 2.40)$ and $(3.24, 5)$; f has a local maximum of about $f(2.40) = 10.15$ and local minima of about $f(-1.19) = -4.62$ and $f(3.24) = 9.86$; f is CU on $(-3.45, 0.31)$ and $(2.83, 5)$ and CD on $(-5, -3.45)$ and $(0.31, 2.83)$; and f has inflection points at about $(-3.45, 13.93)$, $(0.31, 2.10)$, and $(2.83, 10.00)$.



39.



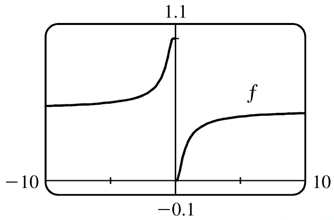
From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$.

$f(x) = e^{-1/x^2} \quad \Rightarrow \quad f'(x) = 2x^{-3}e^{-1/x^2} \quad \Rightarrow$

$f''(x) = 2[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right)$.

This is 0 when $2 - 3x^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

40. (a)



(b) $f(x) = \dfrac{1}{1+e^{1/x}}$.

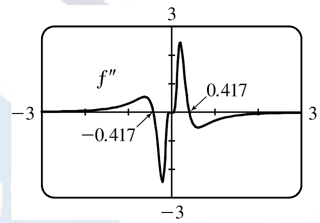
$\lim\limits_{x\to\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$, $\lim\limits_{x\to-\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$,

as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty \;\Rightarrow\; \lim\limits_{x\to 0^+} f(x) = 0$,

as $x \to 0^-$, $1/x \to -\infty$, so $e^{1/x} \to 0 \;\Rightarrow\; \lim\limits_{x\to 0^-} f(x) = \dfrac{1}{1+0} = 1$

(c) From the graph of f, estimates for the IP are $(-0.4, 0.9)$ and $(0.4, 0.08)$.

(d) $f''(x) = -\dfrac{e^{1/x}[e^{1/x}(2x-1)+2x+1]}{x^4(e^{1/x}+1)^3}$



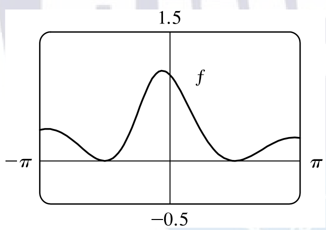
(e) From the graph, we see that f'' changes sign at $x = \pm 0.417$ ($x = 0$ is not in the domain of f). IP are approximately $(0.417, 0.083)$ and $(-0.417, 0.917)$.

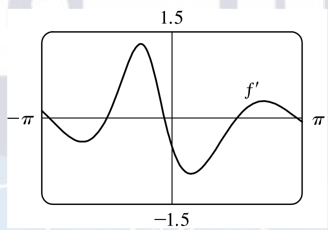
41. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2+x+1}}$, $-\pi \le x \le \pi \;\Rightarrow\; f'(x) = -\dfrac{\cos x\,[(2x+1)\cos x + 4(x^2+x+1)\sin x]}{2(x^2+x+1)^{3/2}} \;\Rightarrow$

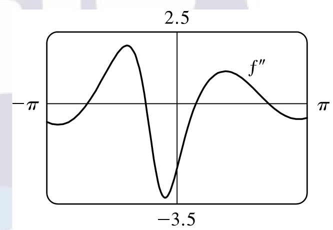
$$f''(x) = -\frac{(8x^4+16x^3+16x^2+8x+9)\cos^2 x - 8(x^2+x+1)(2x+1)\sin x\,\cos x - 8(x^2+x+1)^2\sin^2 x}{4(x^2+x+1)^{5/2}}$$

$f(x) = 0 \;\Leftrightarrow\; x = \pm\frac{\pi}{2}$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -2.96, -1.57, -0.18, 1.57, 3.01$;

$f''(x) = 0 \;\Leftrightarrow\; x \approx -2.16, -0.75, 0.46$, and 2.21.







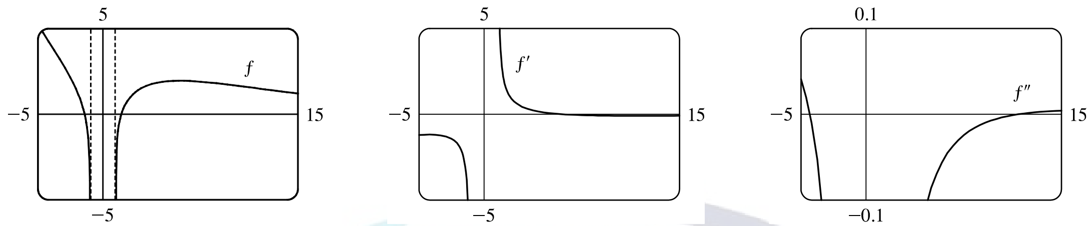
The x-coordinates of the maximum points are the values at which f' changes from positive to negative, that is, -2.96, -0.18, and 3.01. The x-coordinates of the minimum points are the values at which f' changes from negative to positive, that is, -1.57 and 1.57. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -2.16, -0.75, 0.46, and 2.21.

42. $f(x) = e^{-0.1x}\ln(x^2-1) \;\Rightarrow\; f'(x) = \dfrac{e^{-0.1x}\left[(x^2-1)\ln(x^2-1) - 20x\right]}{10(1-x^2)} \;\Rightarrow$

$f''(x) = \dfrac{e^{-0.1x}\left[(x^2-1)^2\ln(x^2-1) - 40(x^3+5x^2-x+5)\right]}{100(x^2-1)^2}$.

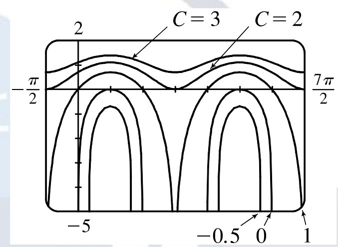
The domain of f is $(-\infty,-1)\cup(1,\infty)$. $f(x) = 0 \;\Leftrightarrow\; x = \pm\sqrt{2}$; $f'(x) = 0 \;\Leftrightarrow\; x \approx 5.87$;

$f''(x) = 0 \;\Leftrightarrow\; x \approx -4.31$ and 11.74.



f' changes from positive to negative at $x \approx 5.87$, so 5.87 is the x-coordinate of the maximum point. There is no minimum point. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -4.31 and 11.74.

43. The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \leq \sin x \leq 1$, this happens if $C > -1$, that is, f has no graph if $C \leq -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and f is continuous on $(-\infty, \infty)$. As C increases, the graph of f is shifted vertically upward and flattens out. If $-1 < C \leq 1$, f is defined where $\sin x + C > 0 \;\Leftrightarrow\; \sin x > -C \;\Leftrightarrow\; \sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is 2π, the domain of f is $\left(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\right)$, n an integer.

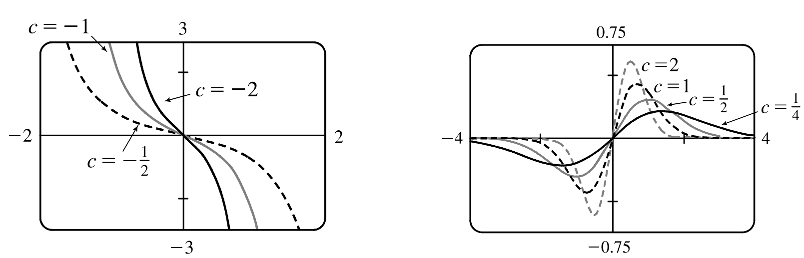


44. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \quad\Rightarrow\quad f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}(-2cx^2 + 1)$$

This is 0 where $-2cx^2 + 1 = 0 \;\Leftrightarrow\; x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right) e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x\to\pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x\to\pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2 + 1\right) \;\Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + (-2cx^2 + 1)(-2cxe^{-cx^2})\right] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \;\Leftrightarrow\; x = \pm\sqrt{3/(2c)} \;\Rightarrow\;$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.



45. Let $f(x) = 3x + 2\cos x + 5$. Then $f(0) = 7 > 0$ and $f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 = -3(\pi - 1) < 0$, and since f is continuous on $\mathbb{R}$ (hence on $[-\pi, 0]$), the Intermediate Value Theorem assures us that there is at least one zero of f in $[-\pi, 0]$. Now $f'(x) = 3 - 2\sin x > 0$ implies that f is increasing on $\mathbb{R}$, so there is exactly one zero of f, and hence, exactly one real root of the equation $3x + 2\cos x + 5 = 0$.

46. By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad \Leftrightarrow \quad 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since $2 \le f'(c) \le 5$, we have $4(2) \le 4f'(c) \le 4(5) \quad \Leftrightarrow \quad 4(2) \le f(4) - 1 \le 4(5) \quad \Leftrightarrow \quad 8 \le f(4) - 1 \le 20 \quad \Leftrightarrow$ $9 \le f(4) \le 21$.

47. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad \Rightarrow \quad \sqrt[5]{33} - 2 > 0 \quad \Rightarrow \quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad \Rightarrow \quad 0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad \Rightarrow \quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

48. Since the point $(1, 3)$ is on the curve $y = ax^3 + bx^2$, we have $3 = a(1)^3 + b(1)^2 \quad \Rightarrow \quad 3 = a + b$ **(1)**. $y' = 3ax^2 + 2bx \quad \Rightarrow \quad y'' = 6ax + 2b$. $\;y'' = 0$ [for inflection points] $\Leftrightarrow \quad x = \dfrac{-2b}{6a} = -\dfrac{b}{3a}$. Since we want $x = 1$, $1 = -\dfrac{b}{3a} \quad \Rightarrow \quad b = -3a$. Combining with **(1)** gives us $3 = a - 3a \quad \Leftrightarrow \quad 3 = -2a \quad \Leftrightarrow \quad a = -\frac{3}{2}$. Hence, $b = -3\left(-\frac{3}{2}\right) = \frac{9}{2}$ and the curve is $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

49. (a) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \ne 0$, we must have $f'(x^2) > 0$ for $x \ne 0$, so $g'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad \Rightarrow \quad g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2 f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \ne 0$, $f''(x^2) > 0$ [since f is CU for $x > 0$], and $f'(x^2) > 0$ for all $x \ne 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \ne 0$. Whether $g''(0)$ is positive or 0 doesn't matter [since the sign of g'' does not change there]; g is concave upward on $\mathbb{R}$.

50. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad \Leftrightarrow \quad y = 125$. $\;P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

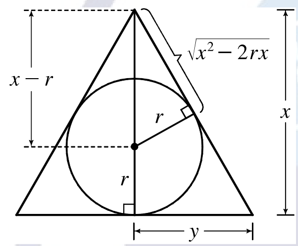
51. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \ne 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so

we minimize $f(x) = (x - x_1)^2 + \left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)^2 \Rightarrow f'(x) = 2(x - x_1) + 2\left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)\left(-\frac{A}{B}\right)$.

$f'(x) = 0 \Rightarrow x = \dfrac{B^2 x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \frac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

52. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x-3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x-3) - 128/x^3 = 0 \Rightarrow x^4 - 3x^3 - 64 = 0 \Rightarrow (x-4)(x^3 + x^2 + 4x + 16) = 0 \Rightarrow x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

53.



By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

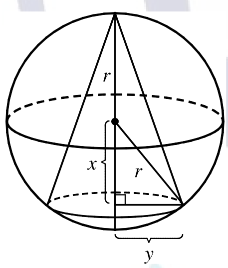
$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0$$

when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = \dfrac{r(9r^2)}{\sqrt{3}\,r} = 3\sqrt{3}\,r^2$.

54.



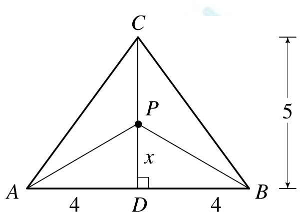
The volume of the cone is $V = \frac{1}{3}\pi y^2(r + x) = \frac{1}{3}\pi(r^2 - x^2)(r + x)$, $-r \le x \le r$.

$$V'(x) = \tfrac{\pi}{3}[(r^2 - x^2)(1) + (r + x)(-2x)] = \tfrac{\pi}{3}[(r + x)(r - x - 2x)]$$
$$= \tfrac{\pi}{3}(r + x)(r - 3x) = 0 \text{ when } x = -r \text{ or } x = r/3.$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is

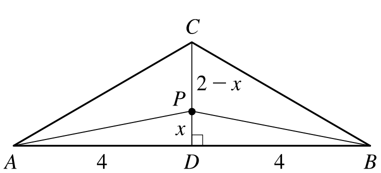
$$V\left(\frac{r}{3}\right) = \frac{\pi}{3}\left(r^2 - \frac{r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}.$$

55.



We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2 + 16} + (5 - x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2 + 16} - 1 = 0 \Leftrightarrow 2x = \sqrt{x^2 + 16} \Leftrightarrow 4x^2 = x^2 + 16 \Leftrightarrow x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

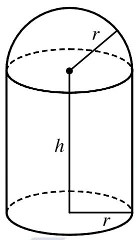
56.



If $|CD| = 2$, the last part of $L(x)$ changes from $(5 - x)$ to $(2 - x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \Leftrightarrow x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

57. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \;\Rightarrow\; \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C) + (C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \;\Leftrightarrow\; \dfrac{1}{C} = \dfrac{C}{L^2} \;\Leftrightarrow\; L^2 = C^2 \;\Leftrightarrow\; L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

58. 

We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \;\Leftrightarrow\; \frac{10}{3}\pi r^3 = 2V \;\Leftrightarrow\; r^3 = \dfrac{3V}{5\pi} \;\Leftrightarrow\; r = \sqrt[3]{\dfrac{3V}{5\pi}}$.

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus,

$$h = \frac{V - \frac{2}{3}\pi \cdot \dfrac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \frac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \frac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\frac{3V}{5\pi}} = r$$

59. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

for $0 \le x \le 4$ [since the seating capacity is 15,000] $\Rightarrow$ $R'(x) = -2000x + 1000 = 0 \;\Leftrightarrow\; x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R.

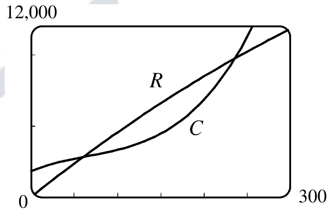
$R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

60. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and

$R(x) = xp(x) = 48.2x - 0.03x^2$.

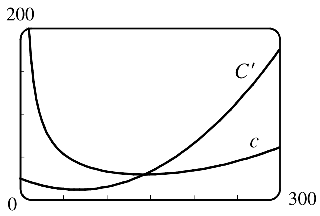
The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.



(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \;\Rightarrow\; 0.003x^2 - 0.34x - 23.2 = 0 \;\Rightarrow$ $x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \;\Rightarrow$ profit is maximized by producing 161 units.

(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \quad \Leftrightarrow \quad x \approx 144$.

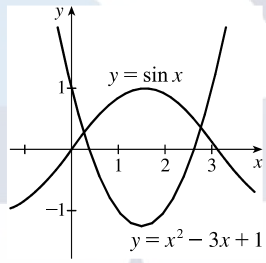


61. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 4x^3 + 6x - 3$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$.

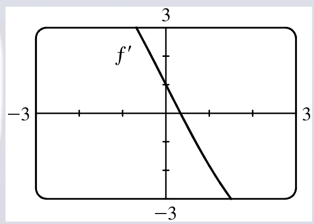
Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1.5 \quad \Rightarrow \quad x_3 \approx 1.343860 \quad \Rightarrow \quad x_4 \approx 1.300320 \quad \Rightarrow \quad x_5 \approx 1.297396 \quad \Rightarrow$ $x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is 1.297383, to six decimal places.

62. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two roots, one about 0.3 and the other about 2.8. $f(x) = \sin x - x^2 + 3x - 1 \quad \Rightarrow$ $f'(x) = \cos x - 2x + 3 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$.

Now $x_1 = 0.3 \quad \Rightarrow \quad x_2 \approx 0.268552 \quad \Rightarrow \quad x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \quad \Rightarrow \quad x_2 \approx 2.770354 \quad \Rightarrow \quad x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the roots are 0.268881 and 2.770058.



63. $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



64. $y = f(x) = x \sin x, \ \ 0 \le x \le 2\pi$. **A.** $D = [0, 2\pi]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, or 2π. **C.** There is no symmetry on D, but if f is defined for all real numbers x, then f is an even function. **D.** No asymptote **E.** $f'(x) = x \cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph f' and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting $g(x) = f'(x) = x \cos x + \sin x$, so that $g'(x) = f''(x) = 2 \cos x - x \sin x$ and $x_{n+1} = x_n - \dfrac{x_n \cos x_n + \sin x_n}{2 \cos x_n - x_n \sin x_n}$.

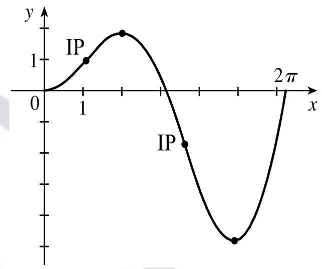
$x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.029048$, $x_3 \approx 2.028758 \approx x_4$ and $x_1 = 4.9 \quad \Rightarrow \quad x_2 \approx 4.913214$, $x_3 \approx 4.913180 \approx x_4$, so the critical numbers, to six decimal places, are $r_1 = 2.028758$ and $r_2 = 4.913180$. By checking sample values of f' in $(0, r_1)$, (r_1, r_2), and $(r_2, 2\pi)$, we see that f is increasing on $(0, r_1)$, decreasing on (r_1, r_2), and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2 \cos x - x \sin x$. To find points where $f''(x) = 0$, we graph f'' and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set $h(x) = f''(x) = 2\cos x - x\sin x$. Then $h'(x) = -3\sin x - x\cos x$, so

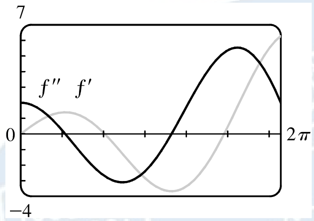
$x_{n+1} = x_n - \dfrac{2\cos x_n - x_n \sin x_n}{-3\sin x_n - x_n \cos x_n}$. $x_1 = 1 \quad \Rightarrow \quad x_2 \approx 1.078028$, $x_3 \approx 1.076874 \approx x_4$ and $x_1 = 3.6 \quad \Rightarrow$

$x_2 \approx 3.643996$, $x_3 \approx 3.643597 \approx x_4$, so the zeros of f'', to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$.

By checking sample values of f'' in $(0, r_3)$, (r_3, r_4), and $(r_4, 2\pi)$, we see that f is CU on $(0, r_3)$, CD on (r_3, r_4), and CU on $(r_4, 2\pi)$. f has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.

H.





65. $f(x) = 4\sqrt{x} - 6x^2 + 3 = 4x^{1/2} - 6x^2 + 3 \quad \Rightarrow \quad F(x) = 4\left(\frac{2}{3}x^{3/2}\right) - 6\left(\frac{1}{3}x^3\right) + 3x + C = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$

66. $g(x) = \dfrac{1}{x} + \dfrac{1}{x^2 + 1} \quad \Rightarrow \quad G(x) = \begin{cases} \ln x + \tan^{-1} x + C_1 & \text{if } x > 0 \\ \ln(-x) + \tan^{-1} x + C_2 & \text{if } x < 0 \end{cases}$

67. $f(t) = 2\sin t - 3e^t \quad \Rightarrow \quad F(t) = -2\cos t - 3e^t + C$

68. $f(x) = x^{-3} + \cosh x \quad \Rightarrow \quad F(x) = \begin{cases} -1/(2x^2) + \sinh x + C_1 & \text{if } x > 0 \\ -1/(2x^2) + \sinh x + C_2 & \text{if } x < 0 \end{cases}$

69. $f'(t) = 2t - 3\sin t \quad \Rightarrow \quad f(t) = t^2 + 3\cos t + C$.

$f(0) = 3 + C$ and $f(0) = 5 \quad \Rightarrow \quad C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

70. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \quad \Rightarrow \quad f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C$.

$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \quad \Rightarrow \quad C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

71. $f''(x) = 1 - 6x + 48x^2 \quad \Rightarrow \quad f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \quad \Rightarrow \quad C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.

$f(0) = D$ and $f(0) = 1 \quad \Rightarrow \quad D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

72. $f''(x) = 5x^3 + 6x^2 + 2 \quad \Rightarrow \quad f'(x) = \frac{5}{4}x^4 + 2x^3 + 2x + C \quad \Rightarrow \quad f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + Cx + D$. Now $f(0) = D$ and $f(0) = 3$, so $D = 3$. Also, $f(1) = \frac{1}{4} + \frac{1}{2} + 1 + C + 3 = C + \frac{19}{4}$ and $f(1) = -2$, so $C + \frac{19}{4} = -2 \quad \Rightarrow \quad C = -\frac{27}{4}$.

Thus, $f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3$.

73. $v(t) = s'(t) = 2t - \dfrac{1}{1 + t^2} \quad \Rightarrow \quad s(t) = t^2 - \tan^{-1} t + C$.

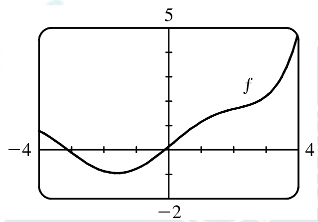
$s(0) = 0 - 0 + C = C$ and $s(0) = 1 \quad \Rightarrow \quad C = 1$, so $s(t) = t^2 - \tan^{-1} t + 1$.

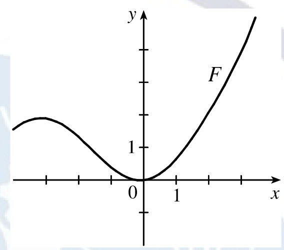
74. $a(t) = v'(t) = \sin t + 3\cos t \quad \Rightarrow \quad v(t) = -\cos t + 3\sin t + C$.

$v(0) = -1 + 0 + C$ and $v(0) = 2 \quad \Rightarrow \quad C = 3$, so $v(t) = -\cos t + 3\sin t + 3$ and $s(t) = -\sin t - 3\cos t + 3t + D$.

$s(0) = -3 + D$ and $s(0) = 0 \quad \Rightarrow \quad D = 3$, and $s(t) = -\sin t - 3\cos t + 3t + 3$.

75. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.





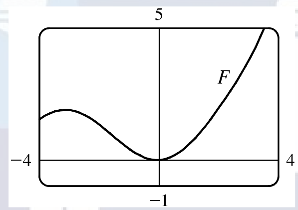
(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow$

$F(x) = 0.1e^x - \cos x + C$. $F(0) = 0 \quad \Rightarrow$

$0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so

$F(x) = 0.1e^x - \cos x + 0.9$.

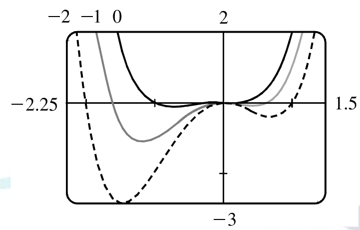
(c)

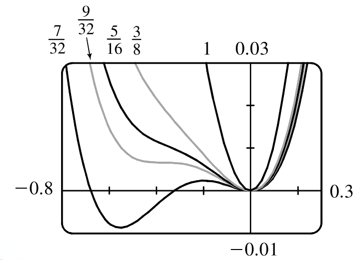


76. $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x(4x^2 + 3x + 2c) = 0 \quad \Leftrightarrow \quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \le 0 \quad \Leftrightarrow \quad c \ge \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

[continued]

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \geq \frac{3}{8}$	1	0

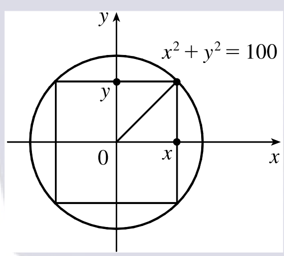




77. Choosing the positive direction to be upward, we have $a(t) = -9.8 \Rightarrow v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \Rightarrow v(t) = -9.8t = s'(t) \Rightarrow s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \Rightarrow s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \Rightarrow t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \Rightarrow v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

78. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.22), there is a number c such that $f''(c) = 0$. So $s''_A(c) = s''_B(c)$; that is, A and B had equal accelerations at $t = c$. We assume that f is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where T is the total time of the race.

79. (a)



The cross-sectional area of the rectangular beam is

$A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \leq x \leq 10$, so

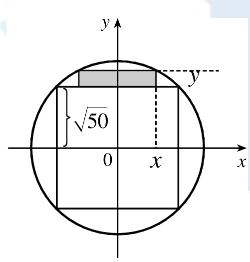
$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x) + (100 - x^2)^{1/2} \cdot 4$$

$$= \frac{-4x^2}{(100 - x^2)^{1/2}} + 4(100 - x^2)^{1/2} = \frac{4[-x^2 + (100 - x^2)]}{(100 - x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + (100 - x^2) = 0 \Rightarrow x^2 = 50 \Rightarrow x = \sqrt{50} \approx 7.07 \Rightarrow y = \sqrt{100 - (\sqrt{50})^2} = \sqrt{50}$.

Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x(y - \sqrt{50}) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \leq x \leq \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x)$$

$$= 2(100 - x^2)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $(100 - x^2) - \sqrt{50}\,(100 - x^2)^{1/2} - x^2 = 0 \Rightarrow 100 - 2x^2 = \sqrt{50}\,(100 - x^2)^{1/2} \Rightarrow$

$10{,}000 - 400x^2 + 4x^4 = 50(100 - x^2) \Rightarrow 4x^4 - 350x^2 + 5000 = 0 \Rightarrow 2x^4 - 175x^2 + 2500 = 0 \Rightarrow$

$x^2 = \frac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \Rightarrow x \approx 8.34$ or 4.24. But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \Rightarrow$

$y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

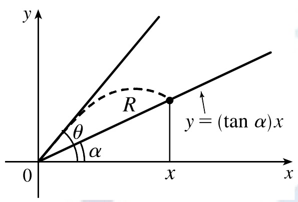
(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is

$S = k(2x)(2y)^2 = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3,\ 0 \le x \le 10.$ $dS/dx = 800k - 24kx^2 = 0$ when

$24kx^2 = 800k \Rightarrow x^2 = \frac{100}{3} \Rightarrow x = \frac{10}{\sqrt{3}} \Rightarrow y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the

maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

80. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$(\tan\alpha)x = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2 \Rightarrow$

$x = \dfrac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \Rightarrow$

$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\,\cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

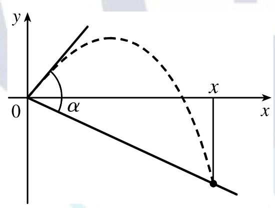
$$= (\sin\theta\,\cos\alpha - \sin\alpha\,\cos\theta)\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta - \alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $$R'(\theta) = \frac{2v^2}{g\cos^2\alpha}[\cos\theta\cdot\cos(\theta-\alpha) + \sin(\theta-\alpha)(-\sin\theta)] = \frac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta-\alpha)]$$

$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta - \alpha) = 0$$

when $\cos(2\theta - \alpha) = 0 \Rightarrow 2\theta - \alpha = \frac{\pi}{2} \Rightarrow \theta = \dfrac{\pi/2 + \alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\,\cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\,\sin(\theta+\alpha)}{g\cos^2\alpha}$.

Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

81. $$\lim_{E\to0^+} P(E) = \lim_{E\to0^+}\left(\frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}\right)$$

$$= \lim_{E\to0^+}\frac{E\left(e^E + e^{-E}\right) - 1\left(e^E - e^{-E}\right)}{\left(e^E - e^{-E}\right)E} = \lim_{E\to0^+}\frac{Ee^E + Ee^{-E} - e^E + e^{-E}}{Ee^E - Ee^{-E}} \qquad \left[\text{form is } \tfrac{0}{0}\right]$$

$$\overset{\text{H}}{=} \lim_{E\to0^+}\frac{Ee^E + e^E\cdot1 + E\left(-e^{-E}\right) + e^{-E}\cdot1 - e^E + \left(-e^{-E}\right)}{Ee^E + e^E\cdot1 - \left[E(-e^{-E}) + e^{-E}\cdot1\right]}$$

$$= \lim_{E\to0^+}\frac{Ee^E - Ee^{-E}}{Ee^E + e^E + Ee^{-E} - e^{-E}} = \lim_{E\to0^+}\frac{e^E - e^{-E}}{e^E + \dfrac{e^E}{E} + e^{-E} - \dfrac{e^{-E}}{E}} \qquad [\text{divide by } E]$$

$$= \frac{0}{2+L}, \quad \text{where } L = \lim_{E\to0^+}\frac{e^E - e^{-E}}{E} \quad \left[\text{form is } \tfrac{0}{0}\right] \quad \overset{\text{H}}{=} \lim_{E\to0^+}\frac{e^E + e^{-E}}{1} = \frac{1+1}{1} = 2$$

Thus, $\displaystyle\lim_{E\to0^+} P(E) = \frac{0}{2+2} = 0.$

82. $\lim\limits_{c\to 0^+} s(t) = \lim\limits_{c\to 0^+}\left(\dfrac{m}{c}\ln\cosh\sqrt{\dfrac{gc}{mt}}\right) = m\lim\limits_{c\to 0^+}\dfrac{\ln\cosh\sqrt{ac}}{c}$ [let $a = g/(mt)$]

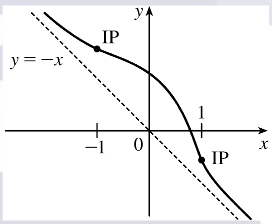
$$\overset{\text{H}}{=} m\lim_{c\to 0^+}\frac{\dfrac{1}{\cosh\sqrt{ac}}\left(\sinh\sqrt{ac}\right)\left(\dfrac{\sqrt{a}}{2\sqrt{c}}\right)}{1} = \frac{m\sqrt{a}}{2}\lim_{c\to 0^+}\frac{\tanh\sqrt{ac}}{\sqrt{c}}$$

$$\overset{\text{H}}{=} \frac{m\sqrt{a}}{2}\lim_{c\to 0^+}\frac{\operatorname{sech}^2\sqrt{ac}\left[\sqrt{a}/\left(2\sqrt{c}\right)\right]}{1/\left(2\sqrt{c}\right)} = \frac{ma}{2}\lim_{c\to 0^+}\operatorname{sech}^2\sqrt{ac} = \frac{ma}{2}(1)^2 = \frac{mg}{2mt} = \frac{g}{2t}$$

83. We first show that $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $x > 0$. Let $f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. Then

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \dfrac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $0 < x$. We next show that $\tan^{-1}x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1}x$. Then $h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1}x$. Hence, $\tan^{-1}x < x$ for $x > 0$, and we conclude that $\dfrac{x}{1+x^2} < \tan^{-1}x < x$ for $x > 0$.

84. If $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim\limits_{x\to\pm\infty}[f(x)+x] = 0$, then f is decreasing everywhere, concave up on $(-\infty,-1)$ and $(1,\infty)$, concave down on $(-1,1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.



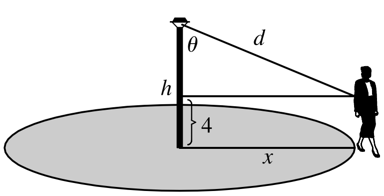
85. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \quad\Rightarrow$

$$\frac{dI}{dh} = k\,\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{[(1600+h^2)^{3/2}]^2} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^3}$$

$$= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}} \qquad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \;\Rightarrow\; h^2 = 800 \;\Rightarrow\; h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$\dfrac{dx}{dt} = 4$ ft/s

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3}$$
$$= \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

[continued]

$$\frac{dI}{dt} = \frac{dI}{dx} \cdot \frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2 + x^2\right]^{-5/2} \cdot 2x \cdot \frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2 + x^2\right]^{-5/2} \cdot 4 = \frac{-12xk(h-4)}{\left[(h-4)^2 + x^2\right]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{\left[(h-4)^2 + 1600\right]^{5/2}}$$

86. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \quad \Leftrightarrow \quad \cos x = -\sin x \quad \Leftrightarrow$ $\tan x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \quad \Rightarrow \quad |\sin x - \cos x| \leq \sqrt{2}$.

3. $f(x)$ has the form $e^{g(x)}$, so it will have an absolute maximum (minimum) where g has an absolute maximum (minimum).

$$g(x) = 10|x-2| - x^2 = \begin{cases} 10(x-2) - x^2 & \text{if } x - 2 > 0 \\ 10[-(x-2)] - x^2 & \text{if } x - 2 < 0 \end{cases} = \begin{cases} -x^2 + 10x - 20 & \text{if } x > 2 \\ -x^2 - 10x + 20 & \text{if } x < 2 \end{cases} \quad \Rightarrow$$

$$g'(x) = \begin{cases} -2x + 10 & \text{if } x > 2 \\ -2x - 10 & \text{if } x < 2 \end{cases}$$

$g'(x) = 0$ if $x = -5$ or $x = 5$, and $g'(2)$ does not exist, so the critical numbers of g are -5, 2, and 5. Since $g''(x) = -2$ for all $x \neq 2$, g is concave downward on $(-\infty, 2)$ and $(2, \infty)$, and g will attain its absolute maximum at one of the critical numbers. Since $g(-5) = 45$, $g(2) = -4$, and $g(5) = 5$, we see that $f(-5) = e^{45}$ is the absolute maximum value of f. Also, $\lim\limits_{x \to \infty} g(x) = -\infty$, so $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{g(x)} = 0$. But $f(x) > 0$ for all x, so there is no absolute minimum value of f.

4. $x^2y^2\left(4 - x^2\right)\left(4 - y^2\right) = x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) = f(x)f(y)$, where $f(t) = t^2\left(4 - t^2\right)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.

$f(t) = 4t^2 - t^4 \quad \Rightarrow \quad f'(t) = 8t - 4t^3 = 4t\left(2 - t^2\right) = 0 \quad \Rightarrow \quad t = 0$ or $\pm\sqrt{2}$.

$f(0) = 0$, $f\left(\pm\sqrt{2}\right) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence, $0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) \leq 16$.

5. $y = \dfrac{\sin x}{x} \;\Rightarrow\; y' = \dfrac{x\cos x - \sin x}{x^2} \;\Rightarrow\; y'' = \dfrac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$. If (x, y) is an inflection point, then $y'' = 0 \;\Rightarrow\; (2 - x^2)\sin x = 2x\cos x \;\Rightarrow\; (2 - x^2)^2\sin^2 x = 4x^2\cos^2 x \;\Rightarrow$ $(2 - x^2)^2\sin^2 x = 4x^2(1 - \sin^2 x) \;\Rightarrow\; (4 - 4x^2 + x^4)\sin^2 x = 4x^2 - 4x^2\sin^2 x \;\Rightarrow$ $(4 + x^4)\sin^2 x = 4x^2 \;\Rightarrow\; (x^4 + 4)\dfrac{\sin^2 x}{x^2} = 4 \;\Rightarrow\; y^2(x^4 + 4) = 4$ since $y = \dfrac{\sin x}{x}$.

6. Let $P(a, 1 - a^2)$ be the point of contact. The equation of the tangent line at P is $y - (1 - a^2) = (-2a)(x - a) \;\Rightarrow$ $y - 1 + a^2 = -2ax + 2a^2 \;\Rightarrow\; y = -2ax + a^2 + 1$. To find the x-intercept, put $y = 0$: $2ax = a^2 + 1 \;\Rightarrow$ $x = \dfrac{a^2 + 1}{2a}$. To find the y-intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is $\dfrac{1}{2}\left(\dfrac{a^2 + 1}{2a}\right)(a^2 + 1) = \dfrac{(a^2 + 1)^2}{4a}$. Therefore, we minimize the function $A(a) = \dfrac{(a^2 + 1)^2}{4a}$, $a > 0$.

$$A'(a) = \frac{(4a)2(a^2 + 1)(2a) - (a^2 + 1)^2(4)}{16a^2} = \frac{(a^2 + 1)[4a^2 - (a^2 + 1)]}{4a^2} = \frac{(a^2 + 1)(3a^2 - 1)}{4a^2}.$$

$A'(a) = 0$ when $3a^2 - 1 = 0 \;\Rightarrow\; a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ and the corresponding minimum area is $A\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$.

7. Let $L = \lim\limits_{x\to 0} \dfrac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6}$. Now L has the indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule. $L = \lim\limits_{x\to 0} \dfrac{2ax + b\cos bx + c\cos cx + d\cos dx}{6x + 20x^3 + 42x^5}$. The denominator approaches 0 as $x \to 0$, so the numerator must also approach 0 (because the limit exists). But the numerator approaches $0 + b + c + d$, so $b + c + d = 0$. Apply l'Hospital's Rule again. $L = \lim\limits_{x\to 0} \dfrac{2a - b^2\sin bx - c^2\sin cx - d^2\sin dx}{6 + 60x^2 + 210x^4} = \dfrac{2a - 0}{6 + 0} = \dfrac{2a}{6}$, which must equal 8.

$\dfrac{2a}{6} = 8 \;\Rightarrow\; a = 24$. Thus, $a + b + c + d = a + (b + c + d) = 24 + 0 = 24$.

8. We first present some preliminary results that we will invoke when calculating the limit.

(1) If $y = (1 + ax)^x$, then $\ln y = x\ln(1 + ax)$, and $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x\ln(1 + ax) = 0$. Thus, $\lim\limits_{x\to 0^+} (1 + ax)^x = e^0 = 1$.

(2) If $y = (1 + ax)^x$, then $\ln y = x\ln(1 + ax)$, and implicitly differentiating gives us $\dfrac{y'}{y} = x \cdot \dfrac{a}{1 + ax} + \ln(1 + ax) \;\Rightarrow$ $y' = y\left[\dfrac{ax}{1 + ax} + \ln(1 + ax)\right]$. Thus, $y = (1 + ax)^x \;\Rightarrow\; y' = (1 + ax)^x\left[\dfrac{ax}{1 + ax} + \ln(1 + ax)\right]$.

(3) If $y = \dfrac{ax}{1 + ax}$, then $y' = \dfrac{(1 + ax)a - ax(a)}{(1 + ax)^2} = \dfrac{a + a^2x - a^2x}{(1 + ax)^2} = \dfrac{a}{(1 + ax)^2}$.

$$\lim_{x\to\infty}\frac{(x+2)^{1/x}-x^{1/x}}{(x+3)^{1/x}-x^{1/x}}=\lim_{x\to\infty}\frac{x^{1/x}[(1+2/x)^{1/x}-1]}{x^{1/x}[(1+3/x)^{1/x}-1]} \qquad \text{[factor out } x^{1/x}\text{]}$$

$$=\lim_{x\to\infty}\frac{(1+2/x)^{1/x}-1}{(1+3/x)^{1/x}-1}$$

$$=\lim_{t\to0^+}\frac{(1+2t)^t-1}{(1+3t)^t-1} \qquad \text{[let } t=1/x\text{, form } 0/0 \text{ by } \mathbf{(1)}\text{]}$$

$$\overset{\text{H}}{=}\lim_{t\to0^+}\frac{(1+2t)^t\left[\dfrac{2t}{1+2t}+\ln(1+2t)\right]}{(1+3t)^t\left[\dfrac{3t}{1+3t}+\ln(1+3t)\right]} \qquad \text{[by } \mathbf{(2)}\text{]}$$

$$=\lim_{t\to0^+}\frac{(1+2t)^t}{(1+3t)^t}\cdot\lim_{t\to0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)}$$

$$=\frac{1}{1}\cdot\lim_{t\to0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)} \qquad \text{[by } \mathbf{(1)}\text{, now form } 0/0\text{]}$$

$$\overset{\text{H}}{=}\lim_{t\to0^+}\frac{\dfrac{2}{(1+2t)^2}+\dfrac{2}{1+2t}}{\dfrac{3}{(1+3t)^2}+\dfrac{3}{1+3t}} \qquad \text{[by } \mathbf{(3)}\text{]}$$

$$=\frac{2+2}{3+3}=\frac{4}{6}=\frac{2}{3}$$

9. Differentiating $x^2+xy+y^2=12$ implicitly with respect to x gives $2x+y+x\,\dfrac{dy}{dx}+2y\,\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{2x+y}{x+2y}$.

At a highest or lowest point, $\dfrac{dy}{dx}=0 \quad\Leftrightarrow\quad y=-2x$. Substituting $-2x$ for y in the original equation gives $x^2+x(-2x)+(-2x)^2=12$, so $3x^2=12$ and $x=\pm2$. If $x=2$, then $y=-2x=-4$, and if $x=-2$ then $y=4$. Thus, the highest and lowest points are $(-2,4)$ and $(2,-4)$.

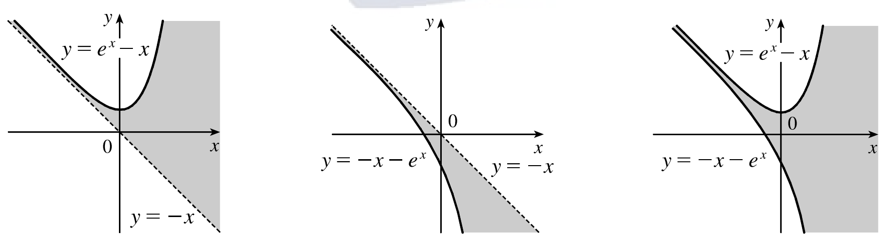
10. *Case (i) (first graph):* For $x+y\geq0$, that is, $y\geq-x$, $|x+y|=x+y\leq e^x \quad\Rightarrow\quad y\leq e^x-x$.

Note that $y=e^x-x$ is always above the line $y=-x$ and that $y=-x$ is a slant asymptote.

Case (ii) (second graph): For $x+y<0$, that is, $y<-x$, $|x+y|=-x-y\leq e^x \quad\Rightarrow\quad y\geq-x-e^x$.

Note that $-x-e^x$ is always below the line $y=-x$ and $y=-x$ is a slant asymptote.

Putting the two pieces together gives the third graph.



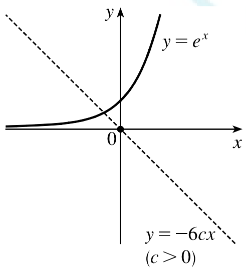
11. (a) $y = x^2 \Rightarrow y' = 2x$, so the slope of the tangent line at $P(a, a^2)$ is $2a$ and the slope of the normal line is $-\frac{1}{2a}$ for $a \neq 0$. An equation of the normal line is $y - a^2 = -\frac{1}{2a}(x - a)$. Substitute x^2 for y to find the x-coordinates of the two points of intersection of the parabola and the normal line. $x^2 - a^2 = -\frac{x}{2a} + \frac{1}{2} \Leftrightarrow x^2 + \left(\frac{1}{2a}\right)x - \frac{1}{2} - a^2 = 0$. We know that a is a root of this quadratic equation, so $x - a$ is a factor, and we have $(x - a)\left(x + \frac{1}{2a} + a\right) = 0$, and hence, $x = -a - \frac{1}{2a}$ is the x-coordinate of the point Q. We want to minimize the y-coordinate of Q, which is $\left(-a - \frac{1}{2a}\right)^2 = a^2 + 1 + \frac{1}{4a^2} = y(a)$. Now $y'(a) = 2a - \frac{1}{2a^3} = \frac{4a^4 - 1}{2a^3} = \frac{(2a^2+1)(2a^2-1)}{2a^3} = 0 \Rightarrow$ $a = \frac{1}{\sqrt{2}}$ for $a > 0$. Since $y''(a) = 2 + \frac{3}{2a^4} > 0$, we see that $a = \frac{1}{\sqrt{2}}$ gives us the minimum value of the y-coordinate of Q.

(b) The square S of the distance from $P(a, a^2)$ to $Q\left(-a - \frac{1}{2a}, \left(-a - \frac{1}{2a}\right)^2\right)$ is given by

$$\begin{aligned} S &= \left(-a - \frac{1}{2a} - a\right)^2 + \left[\left(-a - \frac{1}{2a}\right)^2 - a^2\right]^2 = \left(-2a - \frac{1}{2a}\right)^2 + \left[\left(a^2 + 1 + \frac{1}{4a^2}\right) - a^2\right]^2 \\ &= \left(4a^2 + 2 + \frac{1}{4a^2}\right) + \left(1 + \frac{1}{4a^2}\right)^2 = \left(4a^2 + 2 + \frac{1}{4a^2}\right) + 1 + \frac{2}{4a^2} + \frac{1}{16a^4} \\ &= 4a^2 + 3 + \frac{3}{4a^2} + \frac{1}{16a^4} \end{aligned}$$

$S' = 8a - \frac{6}{4a^3} - \frac{4}{16a^5} = 8a - \frac{3}{2a^3} - \frac{1}{4a^5} = \frac{32a^6 - 6a^2 - 1}{4a^5} = \frac{(2a^2-1)(4a^2+1)^2}{4a^5}$. The only real positive zero of the equation $S' = 0$ is $a = \frac{1}{\sqrt{2}}$. Since $S'' = 8 + \frac{9}{2a^4} + \frac{5}{4a^6} > 0$, $a = \frac{1}{\sqrt{2}}$ corresponds to the shortest possible length of the line segment PQ.

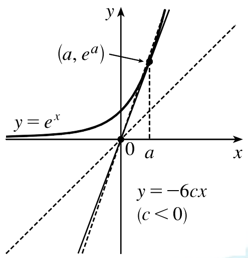
12.



$y = cx^3 + e^x \Rightarrow y' = 3cx^2 + e^x \Rightarrow y'' = 6cx + e^x$. The curve will have inflection points when y'' changes sign. $y'' = 0 \Rightarrow -6cx = e^x$, so y'' will change sign when the line $y = -6cx$ intersects the curve $y = e^x$ (but is not tangent to it). Note that if $c = 0$, the curve is just $y = e^x$, which has no inflection point.

The first figure shows that for $c > 0$, $y = -6cx$ will intersect $y = e^x$ once, so $y = cx^3 + e^x$ will have one inflection point.

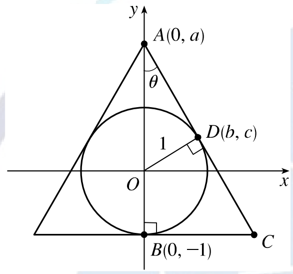
[continued]



The second figure shows that for $c < 0$, the line $y = -6cx$ can intersect the curve $y = e^x$ in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). The tangent line at (a, e^a) has slope e^a, but from the diagram we see that the slope is $\dfrac{e^a}{a}$. So $\dfrac{e^a}{a} = e^a \quad \Rightarrow \quad a = 1$. Thus, the slope is e. The line $y = -6cx$ must have slope greater than e, so $-6c > e \quad \Rightarrow \quad c < -e/6$.

Therefore, the curve $y = cx^3 + e^x$ will have one inflection point if $c > 0$ and two inflection points if $c < -e/6$.

13.



$\overline{AC}$ is tangent to the unit circle at D. To find the slope of $\overline{AC}$ at D, use implicit differentiation. $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\,y' = 0 \quad \Rightarrow \quad y\,y' = -x \quad \Rightarrow \quad y' = -\dfrac{x}{y}$.

Thus, the tangent line at $D(b, c)$ has equation $y = -\dfrac{b}{c}\,x + a$. At D, $x = b$ and $y = c$, so $c = -\dfrac{b}{c}(b) + a \quad \Rightarrow \quad a = c + \dfrac{b^2}{c} = \dfrac{c^2 + b^2}{c} = \dfrac{1}{c}$, and hence $c = \dfrac{1}{a}$.

Since $b^2 + c^2 = 1$, $b = \sqrt{1 - c^2} = \sqrt{1 - 1/a^2} = \sqrt{\dfrac{a^2 - 1}{a^2}} = \dfrac{\sqrt{a^2 - 1}}{a}$, and now we have both b and c in terms of a. At C, $y = -1$, so $-1 = -\dfrac{b}{c}\,x + a \quad \Rightarrow \quad \dfrac{b}{c}\,x = a + 1 \quad \Rightarrow$

$x = \dfrac{c}{b}(a + 1) = \dfrac{1/a}{\sqrt{a^2 - 1}/a}(a + 1) = \dfrac{a + 1}{\sqrt{(a + 1)(a - 1)}} = \sqrt{\dfrac{a + 1}{a - 1}}$, and C has coordinates $\left(\sqrt{\dfrac{a + 1}{a - 1}}, -1\right)$. Let S be the square of the distance from A to C. Then $S(a) = \left(0 - \sqrt{\dfrac{a + 1}{a - 1}}\right)^2 + (a + 1)^2 = \dfrac{a + 1}{a - 1} + (a + 1)^2 \quad \Rightarrow$

$$S'(a) = \frac{(a - 1)(1) - (a + 1)(1)}{(a - 1)^2} + 2(a + 1) = \frac{-2 + 2(a + 1)(a - 1)^2}{(a - 1)^2}$$

$$= \frac{-2 + 2(a^3 - a^2 - a + 1)}{(a - 1)^2} = \frac{2a^3 - 2a^2 - 2a}{(a - 1)^2} = \frac{2a(a^2 - a - 1)}{(a - 1)^2}$$

Using the quadratic formula, we find that the solutions of $a^2 - a - 1 = 0$ are $a = \dfrac{1 \pm \sqrt{5}}{2}$, so $a_1 = \dfrac{1 + \sqrt{5}}{2}$ (the "golden mean") since $a > 0$. For $1 < a < a_1$, $S'(a) < 0$, and for $a > a_1$, $S'(a) > 0$, so a_1 minimizes S.

Note: The minimum length of the equal sides is $\sqrt{S(a_1)} = \cdots = \sqrt{\dfrac{11 + 5\sqrt{5}}{2}} \approx 3.33$ and the corresponding length of the third side is $2\sqrt{\dfrac{a_1 + 1}{a_1 - 1}} = \cdots = 2\sqrt{2 + \sqrt{5}} \approx 4.12$, so the triangle is *not* equilateral.

Another method: In ΔABC, $\cos\theta = \dfrac{a + 1}{\overline{AC}}$, so $\overline{AC} = \dfrac{a + 1}{\cos\theta}$. In ΔADO, $\sin\theta = \dfrac{1}{a}$, so $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - 1/a^2} = \dfrac{1}{a}\sqrt{a^2 - 1}$. Thus $\overline{AC} = \dfrac{a + 1}{(1/a)\sqrt{a^2 - 1}} = \dfrac{a(a + 1)}{\sqrt{a^2 - 1}} = f(a)$. Now find the minimum of f.

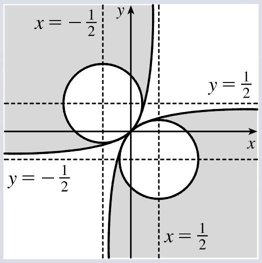
4.PP.13: changed "minimmum" to "corresponding" as circled here.

14. To sketch the region $\{(x,y) \mid 2xy \le |x-y| \le x^2+y^2\}$, we consider two cases.

Case 1: $x \ge y$ This is the case in which (x,y) lies on or below the line $y = x$. The double inequality becomes $2xy \le x - y \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 - x + y^2 + y \ge 0 \Leftrightarrow$ $\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 \ge \frac{1}{2} \Leftrightarrow (x,y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

The left-hand inequality holds if and only if $2xy - x + y \le 0 \Leftrightarrow xy - \frac{1}{2}x + \frac{1}{2}y \le 0 \Leftrightarrow$ $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) \le -\frac{1}{4} \Leftrightarrow (x,y)$ lies on or below the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ asymptotically.

Case 2: $y \ge x$ This is the case in which (x,y) lies on or above the line $y = x$. The double inequality becomes $2xy \le y - x \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 + x + y^2 - y \ge 0 \Leftrightarrow$ $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \ge \frac{1}{2} \Leftrightarrow (x,y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy + x - y \le 0 \Leftrightarrow xy + \frac{1}{2}x - \frac{1}{2}y \le 0 \Leftrightarrow \left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \le -\frac{1}{4} \Leftrightarrow (x,y)$ lies on or above the left-hand branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = -\frac{1}{2}$ and $x = \frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y = x$. So we need only have analyzed case 1 and then reflected that region about the line $y = x$, instead of considering case 2.



15. $A = \left(x_1, x_1^2\right)$ and $B = \left(x_2, x_2^2\right)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = \left(x, x^2\right)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2 + x_2^2\right)(x_2 - x_1) - \tfrac{1}{2}\left(x_1^2 + x^2\right)(x - x_1) - \tfrac{1}{2}\left(x^2 + x_2^2\right)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

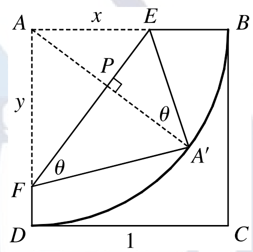
$f'(x) = 0 \Rightarrow 2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x_P = \frac{1}{2}(x_1 + x_2)$.

$$f(x_P) = \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right)$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] = \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right]$$
$$= \tfrac{1}{8}(x_2 - x_1)\left(x_1^2 - 2x_1x_2 + x_2^2\right) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 = \tfrac{1}{8}(x_2 - x_1)^3$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad \Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

16. Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the ΔAEF is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between x and y, so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF. Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A', and that $|AP| = |PA'|$ for the same reason.



But $|AA'| = 1$, since AA' is a radius of the circle. Since $|AP| + |PA'| = |AA'|$, we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is $\mathcal{A} = \frac{1}{2}\,|EF|\,|AP| = \frac{1}{2}\sqrt{x^2 + y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2 + y^2}$. Equating the two expressions for $\mathcal{A}$, we get $\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2 + y^2} \quad \Rightarrow \quad 4x^2y^2 = x^2 + y^2 \quad \Rightarrow \quad y^2\left(4x^2 - 1\right) = x^2 \quad \Rightarrow \quad y = x/\sqrt{4x^2 - 1}$.

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,

$\dfrac{|A'P|}{|PE|} = \dfrac{|A'F|}{|A'E|} \quad \Rightarrow \quad \dfrac{\frac{1}{2}}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}} = \dfrac{y}{x} \quad \Rightarrow \quad y = \dfrac{\frac{1}{2}x}{\sqrt{4x^2 - 1}\,/2} = \dfrac{x}{\sqrt{4x^2 - 1}}$.) Now we can substitute for y and calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2 - 1}} \quad \Rightarrow \quad \dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2 - 1}\,(2x) - x^2\left(\frac{1}{2}\right)\left(4x^2 - 1\right)^{-1/2}(8x)}{4x^2 - 1}\right]$. This is 0 when

$2x\sqrt{4x^2 - 1} - 4x^3\left(4x^2 - 1\right)^{-1/2} = 0 \quad \Leftrightarrow \quad 2x\left(4x^2 - 1\right)^{-1/2}\left[\left(4x^2 - 1\right) - 2x^2\right] = 0 \quad \Rightarrow \quad \left(4x^2 - 1\right) - 2x^2 = 0$ $(x > 0) \quad \Leftrightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \quad \Leftrightarrow$ $1 = x/\sqrt{4x^2 - 1} \quad \Leftrightarrow \quad x^2 = 4x^2 - 1 \quad \Leftrightarrow \quad 3x^2 = 1 \quad \Leftrightarrow \quad x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate

$\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{\left(1/\sqrt{2}\right)^2}{\sqrt{4\left(1/\sqrt{2}\right)^2 - 1}} = \dfrac{1}{4} = 0.25$, and $\mathcal{A}(1) = \dfrac{1}{2}\dfrac{1^2}{\sqrt{4(1)^2 - 1}} = \dfrac{1}{2\sqrt{3}} \approx 0.29$. So the maximum area is $\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

[continued]

Another method: Use the angle θ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\,\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin\,2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \;\Rightarrow\; \sec\theta \le 2 \;\Rightarrow$ $\cos\theta \ge \frac{1}{2} \;\Rightarrow\; \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \;\Rightarrow\; \csc\theta \le 2 \;\Rightarrow\; \sin\theta \le \frac{1}{2} \;\Rightarrow\; \theta \ge \frac{\pi}{6}$.

As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$;

and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

17. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, $x > 0$, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

18. If $L = \displaystyle\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

$$\ln L = \lim_{x\to\infty}\ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty}\frac{\ln(x+a)-\ln(x-a)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+a}-\dfrac{1}{x-a}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{(x-a)-(x+a)}{(x+a)(x-a)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = \lim_{x\to\infty}\frac{2a}{1-a^2/x^2} = 2a$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \;\Rightarrow\; 2a = 1 \;\Rightarrow\; a = \frac{1}{2}$.

19. Note that $f(0) = 0$, so for $x \ne 0$, $\left|\dfrac{f(x)-f(0)}{x-0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\right| = \lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0}\right| \le \lim_{x\to 0}\frac{\sin x}{x} = 1$.

But $f(x) = a_1\sin x + a_2\sin 2x + \cdots + a_n\sin nx \;\Rightarrow\; f'(x) = a_1\cos x + 2a_2\cos 2x + \cdots + na_n\cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \le 1$.

Another solution: We are given that $\left|\sum_{k=1}^n a_k\sin kx\right| \le |\sin x|$. So for x close to 0, and $x \ne 0$, we have

$\left|\displaystyle\sum_{k=1}^n a_k\frac{\sin kx}{\sin x}\right| \le 1 \;\Rightarrow\; \lim_{x\to 0}\left|\sum_{k=1}^n a_k\frac{\sin kx}{\sin x}\right| \le 1 \;\Rightarrow\; \left|\sum_{k=1}^n a_k\lim_{x\to 0}\frac{\sin kx}{\sin x}\right| \le 1$. But by l'Hospital's Rule,

$\displaystyle\lim_{x\to 0}\frac{\sin kx}{\sin x} = \lim_{x\to 0}\frac{k\cos kx}{\cos x} = k$, so $\left|\displaystyle\sum_{k=1}^n ka_k\right| \le 1$.

20. Let the circle have radius r, so $|OP| = |OQ| = r$, where O is the center of the circle. Now $\angle POR$ has measure $\frac{1}{2}\theta$, and $\angle OPR$ is a right angle, so $\tan\frac{1}{2}\theta = \dfrac{|PR|}{r}$ and the area of $\triangle OPR$ is $\frac{1}{2}\,|OP|\,|PR| = \frac{1}{2}r^2\tan\frac{1}{2}\theta$. The area of the sector cut by OP and OR is $\frac{1}{2}r^2\left(\frac{1}{2}\theta\right) = \frac{1}{4}r^2\theta$. Let S be the intersection of PQ and OR. Then $\sin\frac{1}{2}\theta = \dfrac{|PS|}{r}$ and $\cos\frac{1}{2}\theta = \dfrac{|OS|}{r}$, and the area of $\triangle OSP$ is $\frac{1}{2}\,|OS|\,|PS| = \frac{1}{2}\left(r\cos\frac{1}{2}\theta\right)\left(r\sin\frac{1}{2}\theta\right) = \frac{1}{2}r^2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta = \frac{1}{4}r^2\sin\theta$.

So $B(\theta) = 2\left(\frac{1}{2}r^2\tan\frac{1}{2}\theta - \frac{1}{4}r^2\theta\right) = r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)$ and $A(\theta) = 2\left(\frac{1}{4}r^2\theta - \frac{1}{4}r^2\sin\theta\right) = \frac{1}{2}r^2(\theta - \sin\theta)$. Thus,

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{r^2\left(\tan\frac{1}{2}\theta-\frac{1}{2}\theta\right)} = \lim_{\theta\to 0^+}\frac{\theta-\sin\theta}{2\left(\tan\frac{1}{2}\theta-\frac{1}{2}\theta\right)} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{2\left(\frac{1}{2}\sec^2\frac{1}{2}\theta-\frac{1}{2}\right)}\\ &= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\sec^2\frac{1}{2}\theta-1} = \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\tan^2\frac{1}{2}\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{2\left(\tan\frac{1}{2}\theta\right)\left(\sec^2\frac{1}{2}\theta\right)\frac{1}{2}}\\ &= \lim_{\theta\to 0^+}\frac{\sin\theta\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \lim_{\theta\to 0^+}\frac{\left(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta\right)\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 2\lim_{\theta\to 0^+}\cos^4\left(\tfrac{1}{2}\theta\right) = 2(1)^4 = 2\end{aligned}$$

21. (a) Distance = rate × time, so time = distance/rate. $T_1 = \dfrac{D}{c_1}$, $T_2 = \dfrac{2\,|PR|}{c_1} + \dfrac{|RS|}{c_2} = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$,

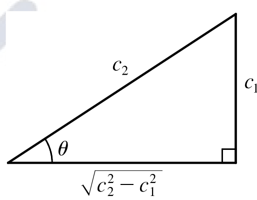
$$T_3 = \frac{2\sqrt{h^2 + D^2/4}}{c_1} = \frac{\sqrt{4h^2 + D^2}}{c_1}.$$

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \quad\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \quad\Rightarrow\quad \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \quad\Rightarrow\quad \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \quad\Rightarrow\quad c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2 + D^2}}{c_1} \quad\Rightarrow$

$4h^2 + D^2 = T_3^2c_1^2 \quad\Rightarrow\quad h = \frac{1}{2}\sqrt{T_3^2c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,

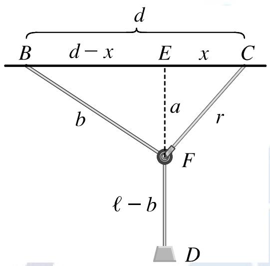


$\sin\theta = \dfrac{c_1}{c_2} \quad\Rightarrow\quad \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2 - c_1^2}}$, so

$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$. Using the values for T_2 [given as 0.32], h, c_1, and D, we can graph $\text{Y}_1 = T_2$ and $\text{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$ and find their intersection points. Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6°$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

22. A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of f has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}$. There are two distinct roots for $f''(x) = 0$ (and hence two inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \quad \Leftrightarrow \quad c^2 > 32 \quad \Leftrightarrow \quad |c| > \sqrt{32}$. Thus, the desired values of c are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

23. 

Let $a = |EF|$ and $b = |BF|$ as shown in the figure.

Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$|ED| = |EF| + |FD| = a + \ell - b$$
$$\sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2}$$
$$= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2}$$
$$= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$$f'(x) = \tfrac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \tfrac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \frac{-x}{\sqrt{r^2 - x^2}} + \frac{d}{\sqrt{d^2 + r^2 - 2dx}}.$$

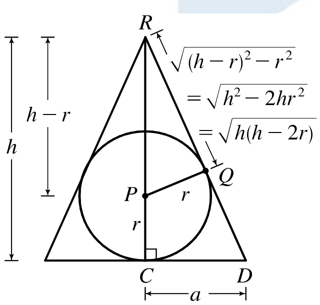
$$f'(x) = 0 \quad \Rightarrow \quad \frac{x}{\sqrt{r^2 - x^2}} = \frac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \frac{x^2}{r^2 - x^2} = \frac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$$

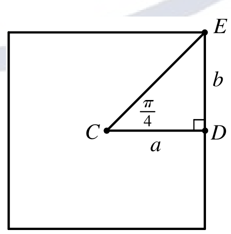
$$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$$

$$0 = 2dx^2(x - d) - r^2\left(x^2 - d^2\right) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow \quad 0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

$x = \dfrac{-\left(-r^2\right) \pm \sqrt{\left(-r^2\right)^2 - 4(2d)\left(-dr^2\right)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus, $x = \dfrac{r^2 + \sqrt{r^2}\sqrt{r^2 + 8d^2}}{4d} = \dfrac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad [r > 0] \quad = \dfrac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$. The maximum value of $|ED|$ occurs at this value of x.

24. 



Let $a = \overline{CD}$ denote the distance from the center C of the base to the midpoint D of a side of the base.

Since ΔPQR is similar to ΔDCR, $\dfrac{a}{h} = \dfrac{r}{\sqrt{h(h-2r)}} \quad \Rightarrow \quad a = \dfrac{rh}{\sqrt{h(h-2r)}} = r\dfrac{\sqrt{h}}{\sqrt{h-2r}}$.

Let b denote one-half the length of a side of the base. The area A of the base is

$A = 8(\text{area of } \Delta CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a\tan\frac{\pi}{4}\right) = 4a^2$.

The volume of the pyramid is $V = \frac{1}{3}Ah = \frac{1}{3}\left(4a^2\right)h = \dfrac{4}{3}\left(r\dfrac{\sqrt{h}}{\sqrt{h-2r}}\right)^2 h = \frac{4}{3}r^2\dfrac{h^2}{h-2r}$, with domain $h > 2r$.

Now $\dfrac{dV}{dh} = \dfrac{4}{3}r^2 \cdot \dfrac{(h-2r)(2h) - h^2(1)}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h^2-4hr}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h(h-4r)}{(h-2r)^2}$

and
$$\begin{aligned}\frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h-2r)^2(2h-4r) - \left(h^2-4hr\right)(2)(h-2r)(1)}{\left[(h-2r)^2\right]^2} \\ &= \frac{4}{3}r^2 \cdot \frac{2(h-2r)\left[\left(h^2-4hr+4r^2\right) - \left(h^2-4hr\right)\right]}{(h-2r)^2} \\ &= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h-2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h-2r)^3}.\end{aligned}$$

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular n-gon, we make the following changes:

(1) the number of sides of the base is n

(2) the number of triangles in the base is $2n$

(3) $\angle DCE = \dfrac{\pi}{n}$

(4) $b = a\tan\dfrac{\pi}{n}$

We then obtain the following results: $A = na^2\tan\dfrac{\pi}{n}$, $V = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h^2}{h-2r}$, $\dfrac{dV}{dh} = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h(h-4r)}{(h-2r)^2}$,

and $\dfrac{d^2V}{dh^2} = \dfrac{8nr^4}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the base of the polygon!

25. $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k\cdot 4\pi r^2$ for some constant k.

Therefore, $4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t is kt, so $r = kt + C$.

When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k+r_0)^3 = \frac{1}{2}\cdot\frac{4}{3}\pi r_0^3 \quad \Rightarrow \quad (3k+r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow$

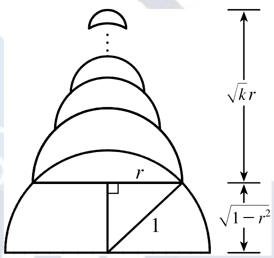
$3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball

has melted completely we have $r = 0 \quad \Rightarrow \quad \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2}-1}$. Hence, it takes

$\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2}-1} - 3 = \dfrac{3}{\sqrt[3]{2}-1} \approx 11$ h 33 min longer.

26. By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of n hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius r. Then by our induction hypothesis (scaled to the setting of a bottom hemisphere of radius r), the height of the stack formed by the top k bubbles is $\sqrt{k}\,r$. (If it were shorter, then the total stack of $k + 1$ bubbles wouldn't have maximum height.)

The height of the whole stack is $H(r) = \sqrt{k}\,r + \sqrt{1 - r^2}$. (See the figure.) We want to choose r so as to maximize $H(r)$. Note that $0 < r < 1$.



We calculate $H'(r) = \sqrt{k} - \dfrac{r}{\sqrt{1 - r^2}}$ and $H''(r) = \dfrac{-1}{(1 - r^2)^{3/2}}$.

$$H'(r) = 0 \quad \Leftrightarrow \quad r^2 = k(1 - r^2) \quad \Leftrightarrow \quad (k + 1)r^2 = k \quad \Leftrightarrow \quad r = \sqrt{\frac{k}{k + 1}}.$$

This is the only critical number in $(0, 1)$ and it represents a local maximum (hence an absolute maximum) since $H''(r) < 0$ on $(0, 1)$. When $r = \sqrt{\dfrac{k}{k + 1}}$,

$H(r) = \sqrt{k}\dfrac{\sqrt{k}}{\sqrt{k + 1}} + \sqrt{1 - \dfrac{k}{k + 1}} = \dfrac{k}{\sqrt{k + 1}} + \dfrac{1}{\sqrt{k + 1}} = \sqrt{k + 1}$. Thus, the assertion is true for $n = k + 1$ when it is true for $n = k$. By induction, it is true for all positive integers n.

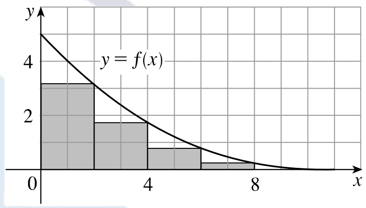
Note: In general, a maximally tall stack of n hemispherical bubbles consists of bubbles with radii

$$1, \sqrt{\frac{n - 1}{n}}, \sqrt{\frac{n - 2}{n}}, \ldots, \sqrt{\frac{2}{n}}, \sqrt{\frac{1}{n}}.$$

5 □ INTEGRALS

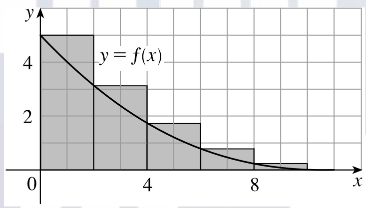
5.1 Areas and Distances

1. (a) Since f is *decreasing* , we can obtain a *lower* estimate by using *right* endpoints. We are instructed to use five rectangles, so $n = 5$.



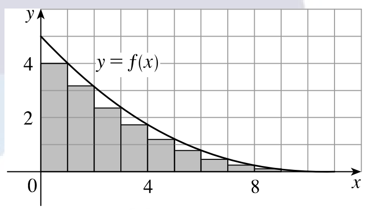
$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x \qquad \left[\Delta x = \frac{b-a}{n} = \frac{10-0}{5} = 2\right]$$
$$= f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2 + f(x_5)\cdot 2$$
$$= 2[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3.2 + 1.8 + 0.8 + 0.2 + 0)$$
$$= 2(6) = 12$$

Since f is *decreasing* , we can obtain an *upper* estimate by using *left* endpoints.

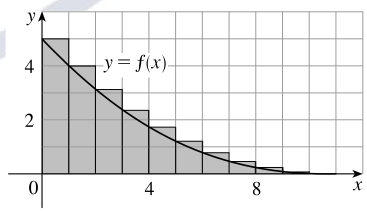


$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(5 + 3.2 + 1.8 + 0.8 + 0.2)$$
$$= 2(11) = 22$$

(b) $R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x \qquad \left[\Delta x = \frac{10-0}{10} = 1\right]$



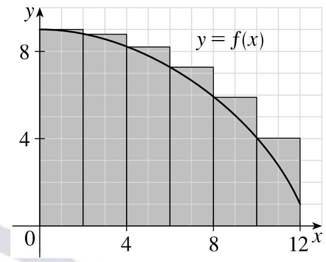
$$= 1[f(x_1) + f(x_2) + \cdots + f(x_{10})]$$
$$= f(1) + f(2) + \cdots + f(10)$$
$$\approx 4 + 3.2 + 2.5 + 1.8 + 1.3 + 0.8 + 0.5 + 0.2 + 0.1 + 0$$
$$= 14.4$$



$$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$= R_{10} + 1\cdot f(0) - 1\cdot f(10) \qquad \left[\begin{array}{c}\text{add leftmost upper rectangle,}\\ \text{subtract rightmost lower rectangle}\end{array}\right]$$
$$= 14.4 + 5 - 0$$
$$= 19.4$$

2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \qquad [\Delta x = \frac{12-0}{6} = 2]$

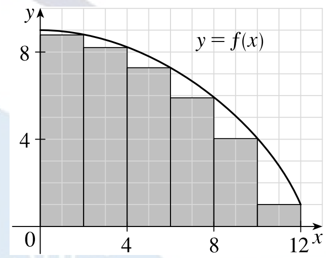
$$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$$
$$= 2(43.3) = 86.6$$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

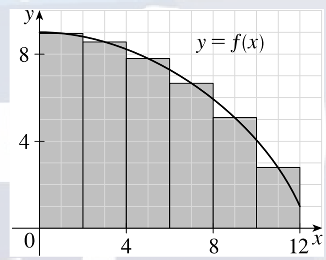
$$\approx 86.6 + 2(1) - 2(9) = 70.6$$

[Add area of rightmost lower rectangle and subtract area of leftmost upper rectangle.]



(iii) $M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x$

$$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$$
$$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$$
$$= 2(39.7) = 79.4$$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

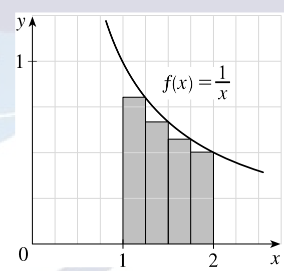
(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \qquad \left[\Delta x = \frac{2-1}{4} = \frac{1}{4}\right] = \left[\sum_{i=1}^{4} f(x_i)\right]\Delta x$

$$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$$
$$= \left[\frac{1}{5/4} + \frac{1}{6/4} + \frac{1}{7/4} + \frac{1}{8/4}\right]\frac{1}{4} = \left[\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7} + \tfrac{1}{2}\right]\tfrac{1}{4} \approx 0.6345$$

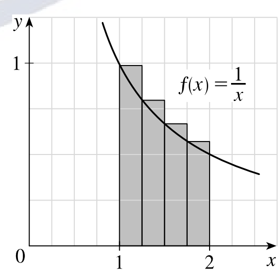
Since f is *decreasing* on $[1, 2]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.



(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x = \left[\sum_{i=1}^{4} f(x_{i-1})\right]\Delta x$

$$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\,\Delta x$$
$$= \left[\frac{1}{1} + \frac{1}{5/4} + \frac{1}{6/4} + \frac{1}{7/4}\right]\frac{1}{4} = \left[1 + \tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7}\right]\tfrac{1}{4} \approx 0.7595$$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot \frac{1}{4} - f(2) \cdot \frac{1}{4}$.



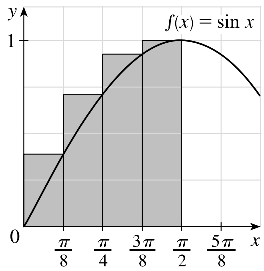
4. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\ \Delta x \qquad \left[\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}\right] = \left[\sum_{i=1}^{4} f(x_i)\right] \Delta x$

$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\ \Delta x$

$= \left[\sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8}\right]\frac{\pi}{8}$

≈ 1.1835

Since f is *increasing* on $\left[0, \frac{\pi}{2}\right]$, R_4 is an *overestimate*.



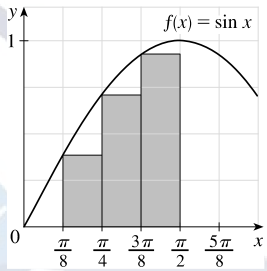
(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\ \Delta x = \left[\sum_{i=1}^{4} f(x_{i-1})\right] \Delta x$

$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\ \Delta x$

$= \left[\sin 0 + \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8}\right]\frac{\pi}{8}$

≈ 0.7908

Since f is *increasing* on $\left[0, \frac{\pi}{2}\right]$, L_4 is an *underestimate*.



5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

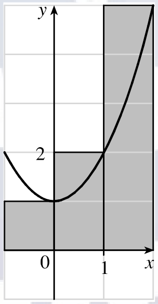
$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8.$

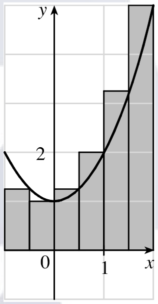
$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad \Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$



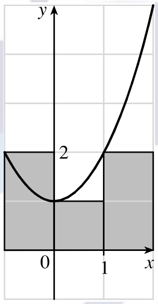


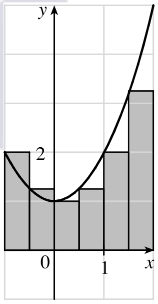
(b) $L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$





(c) $M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5)$

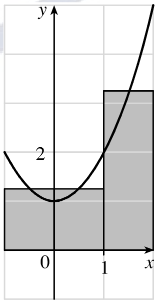
$= 1 \cdot 1.25 + 1 \cdot 1.25 + 1 \cdot 3.25 = 5.75$

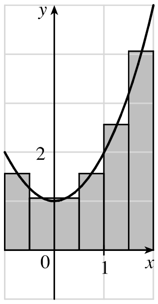
$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$

$+\ f(0.75) + f(1.25) + f(1.75)]$

$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

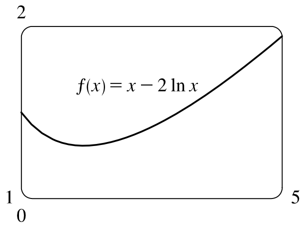
$= 0.5(11.875) = 5.9375$





(d) M_6 appears to be the best estimate.

6. (a)

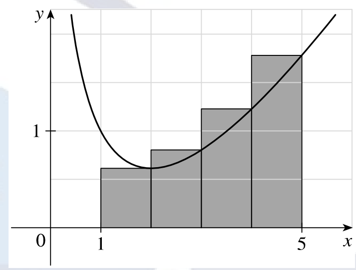


(b) $f(x) = x - 2\ln x$ and $\Delta x = \dfrac{5-1}{4} = 1 \quad \Rightarrow$

(i) $R_4 = 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) + 1 \cdot f(5)$

$= (2 - 2\ln 2) + (3 - 2\ln 3) + (4 - 2\ln 4) + (5 - 2\ln 5)$

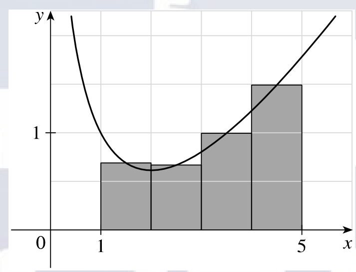
≈ 4.425



(ii) $M_4 = 1 \cdot f(1.5) + 1 \cdot f(2.5) + 1 \cdot f(3.5) + 1 \cdot f(4.5)$

$= (1.5 - 2\ln 1.5) + (2.5 - 2\ln 2.5)$
$+ (3.5 - 2\ln 3.5) + (4.5 - 2\ln 4.5)$

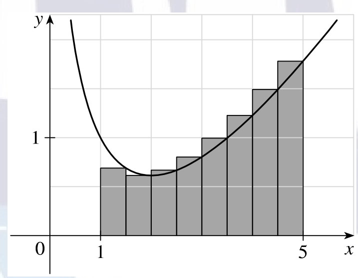
≈ 3.843



(c) (i) $R_8 = \frac{1}{2}[f(1.5) + f(2) + \cdots + f(5)]$

$= \frac{1}{2}[(1.5 - 2\ln 1.5) + (2 - 2\ln 2) + \cdots + (5 - 2\ln 5)]$

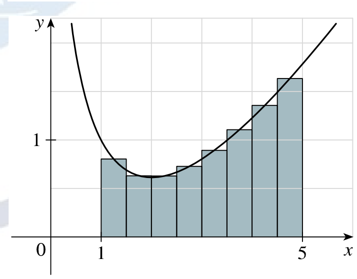
≈ 4.134



(ii) $M_8 = \frac{1}{2}[f(1.25) + f(1.75) + \cdots + f(4.75)]$

$= \frac{1}{2}[(1.25 - 2\ln 1.25) + (1.75 - 2\ln 1.75) + \cdots$
$+ (4.75 - 2\ln 4.75)]$

≈ 3.889



7. $f(x) = 2 + \sin x$, $0 \le x \le \pi$, $\Delta x = \pi/n$.

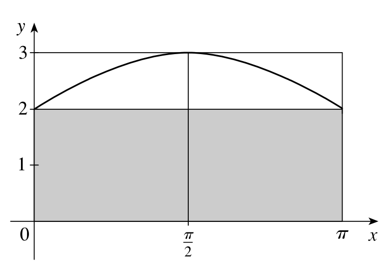
$\boldsymbol{n = 2}$**:** The maximum values of f on both subintervals occur at $x = \frac{\pi}{2}$, so

upper sum $= f\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} + f\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} = 3 \cdot \frac{\pi}{2} + 3 \cdot \frac{\pi}{2}$

$= 3\pi \approx 9.42.2$

The minimum values of f on the subintervals occur at $x = 0$ and $x = \pi$, so

lower sum $= f(0) \cdot \frac{\pi}{2} + f(\pi) \cdot \frac{\pi}{2} = 2 \cdot \frac{\pi}{2} + 2 \cdot \frac{\pi}{2} = 2\pi \approx 6.28.$



$\boldsymbol{n=4}$: upper sum $= \left[f\left(\frac{\pi}{4}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{3\pi}{4}\right)\right]\left(\frac{\pi}{4}\right)$

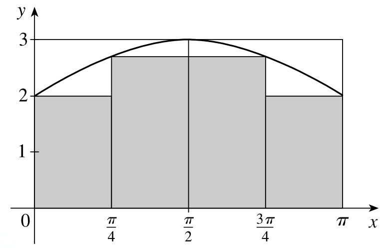
$= \left[\left(2+\frac{1}{2}\sqrt{2}\right)+(2+1)+(2+1)+\left(2+\frac{1}{2}\sqrt{2}\right)\right]\left(\frac{\pi}{4}\right)$

$= \left(10+\sqrt{2}\right)\left(\frac{\pi}{4}\right)\approx 8.96$

lower sum $= \left[f(0)+f\left(\frac{\pi}{4}\right)+f\left(\frac{3\pi}{4}\right)+f(\pi)\right]\left(\frac{\pi}{4}\right)$

$= \left[(2+0)+\left(2+\frac{1}{2}\sqrt{2}\right)+\left(2+\frac{1}{2}\sqrt{2}\right)+(2+0)\right]\left(\frac{\pi}{4}\right)$

$= \left(8+\sqrt{2}\right)\left(\frac{\pi}{4}\right)\approx 7.39$

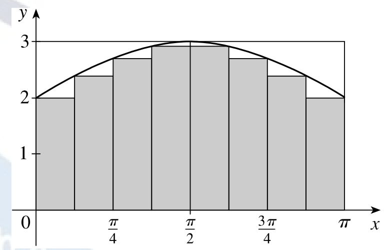


$\boldsymbol{n=8}$: upper sum $= \left[f\left(\frac{\pi}{8}\right)+f\left(\frac{\pi}{4}\right)+f\left(\frac{3\pi}{8}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{5\pi}{8}\right)+f\left(\frac{3\pi}{4}\right)+f\left(\frac{7\pi}{8}\right)\right]\left(\frac{\pi}{8}\right)$

≈ 8.65

lower sum $= \left[f(0)+f\left(\frac{\pi}{8}\right)+f\left(\frac{\pi}{4}\right)+f\left(\frac{3\pi}{8}\right)+f\left(\frac{5\pi}{8}\right)+f\left(\frac{3\pi}{4}\right)+f\left(\frac{7\pi}{8}\right)+f(\pi)\right]\left(\frac{\pi}{8}\right)$

≈ 7.86



8. $f(x)=1+x^2$, $-1\le x\le 1$, $\Delta x=2/n$.

$\boldsymbol{n=3}$: upper sum $= \left[f(-1)+f\left(\frac{1}{3}\right)+f(1)\right]\left(\frac{2}{3}\right)$

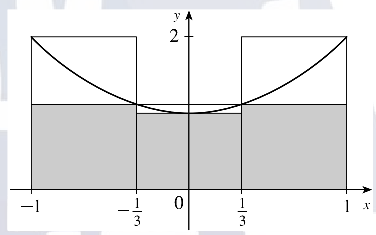
$= \left(2+\frac{10}{9}+2\right)\left(\frac{2}{3}\right)$

$= \frac{92}{27}\approx 3.41$

lower sum $= \left[f\left(-\frac{1}{3}\right)+f(0)+f\left(\frac{1}{3}\right)\right]\left(\frac{2}{3}\right)$

$= \left(\frac{10}{9}+1+\frac{10}{9}\right)\left(\frac{2}{3}\right)$

$= \frac{58}{27}\approx 2.15$



$\boldsymbol{n=4}$: upper sum $= \left[f(-1)+f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)+f(1)\right]\left(\frac{2}{4}\right)$

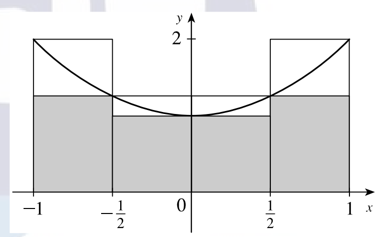
$= \left(2+\frac{5}{4}+\frac{5}{4}+2\right)\left(\frac{1}{2}\right)$

$= \frac{13}{4}=3.25$

lower sum $= \left[f\left(-\frac{1}{2}\right)+f(0)+f(0)+f\left(\frac{1}{2}\right)\right]\left(\frac{2}{4}\right)$

$= \left(\frac{5}{4}+1+1+\frac{5}{4}\right)\left(\frac{1}{2}\right)$

$= \frac{9}{4}=2.25$



9. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = 1, N = 10 (depending on which sum we are calculating),

DELTA_X = (X_MAX - X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT $>$ X_MAX.

2a Add (RIGHT_ENDPOINT)^4 to SUM.

Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X)·(SUM) is equal to the answer we are looking for. We find that

$R_{10}=\frac{1}{10}\sum_{i=1}^{10}\left(\frac{i}{10}\right)^4\approx 0.2533$, $R_{30}=\frac{1}{30}\sum_{i=1}^{30}\left(\frac{i}{30}\right)^4\approx 0.2170$, $R_{50}=\frac{1}{50}\sum_{i=1}^{50}\left(\frac{i}{50}\right)^4\approx 0.2101$, and

$R_{100} = \frac{1}{100} \sum_{i=1}^{100} \left(\frac{i}{100} \right)^4 \approx 0.2050$. It appears that the exact area is 0.2. The following display shows the program SUMRIGHT and its output from a TI-83/4 Plus calculator. To generalize the program, we have input (rather than assign) values for Xmin, Xmax, and N. Also, the function, x^4, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y₁(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

```
prgmSUMRIGHT
Xmin=?0
Xmax=?1
N=?10
                .25333
                  Done
```

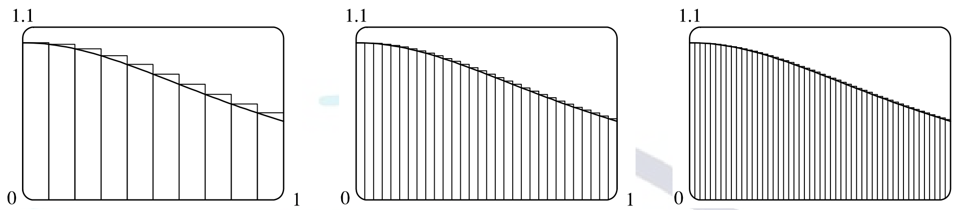
10. We can use the algorithm from Exercise 9 with X_MIN = 0, X_MAX = $\pi/2$, and cos(RIGHT_ENDPOINT) instead of (RIGHT_ENDPOINT)^4 in step 2a. We find that $R_{10} = \frac{\pi/2}{10} \sum_{i=1}^{10} \cos\left(\frac{i\pi}{20}\right) \approx 0.9194$, $R_{30} = \frac{\pi/2}{30} \sum_{i=1}^{30} \cos\left(\frac{i\pi}{60}\right) \approx 0.9736$, and $R_{50} = \frac{\pi/2}{50} \sum_{i=1}^{50} \cos\left(\frac{i\pi}{100}\right) \approx 0.9842$, and $R_{100} = \frac{\pi/2}{100} \sum_{i=1}^{100} \cos\left(\frac{i\pi}{200}\right) \approx 0.9921$. It appears that the exact area is 1.

11. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(1/(x^2+1),x=0..1,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(1/10)*Sum[1/(((i-1)/10)^2+1)],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

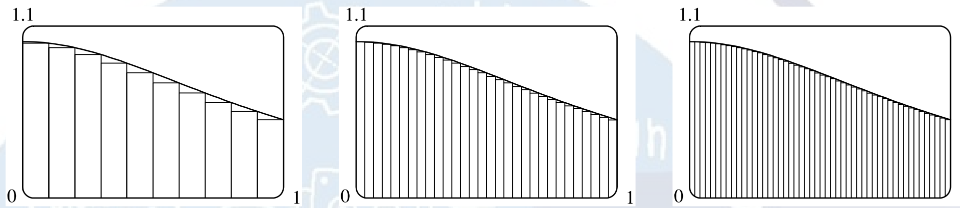
In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \frac{1}{x^2+1}$, $0 \le x \le 1$, the left sums are of the form $L_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{i-1}{n}\right)^2 + 1}$. Specifically, $L_{10} \approx 0.8100$, $L_{30} \approx 0.7937$, and $L_{50} \approx 0.7904$. The right sums are of the form $R_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n}\right)^2 + 1}$. Specifically, $R_{10} \approx 0.7600$, $R_{30} \approx 0.7770$, and $R_{50} \approx 0.7804$.

(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.



left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$



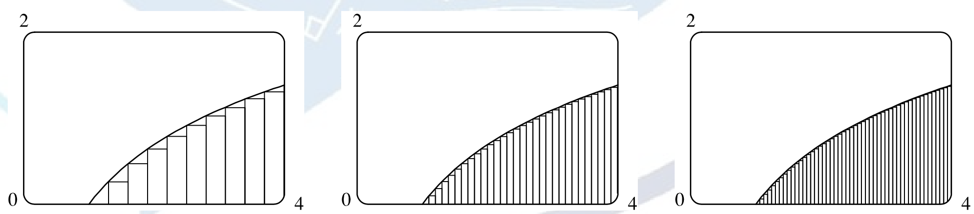
right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $y = 1/(x^2 + 1)$ is a decreasing function on $(0, 1)$, all of the left sums are larger than the actual area, and all of the right sums are smaller than the actual area. Since the left sum with $n = 50$ is about $0.7904 < 0.791$ and the right sum with $n = 50$ is about $0.7804 > 0.780$, we conclude that $0.780 < R_{50} <$ exact area $< L_{50} < 0.791$, so the exact area is between 0.780 and 0.791.

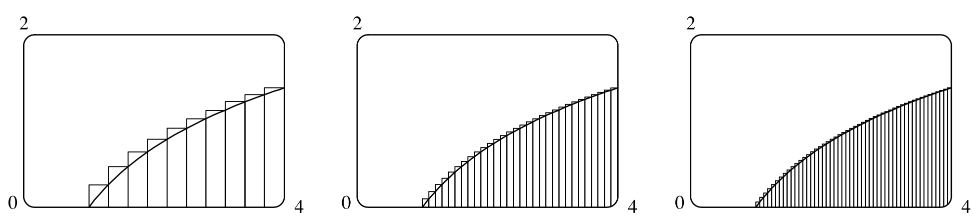
12. See the solution to Exercise 11 for the CAS commands for evaluating the sums.

(a) With $f(x) = \ln x$, $1 \leq x \leq 4$, the left sums are of the form $L_n = \dfrac{3}{n} \sum\limits_{i=1}^{n} \ln\left(1 + \dfrac{3(i-1)}{n}\right)$. In particular, $L_{10} \approx 2.3316$, $L_{30} \approx 2.4752$, and $L_{50} \approx 2.5034$. The right sums are of the form $R_n = \dfrac{3}{n} \sum\limits_{i=1}^{n} \ln\left(1 + \dfrac{3i}{n}\right)$. In particular, $R_{10} \approx 2.7475$, $R_{30} \approx 2.6139$, and $R_{50} \approx 2.5865$.

(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.



left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$



right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $y = \ln x$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $2.503 > 2.50$ and the right sum with $n = 50$ is about $2.587 < 2.59$, we conclude that $2.50 < L_{50} <$ exact area $< R_{50} < 2.59$, so the exact area is between 2.50 and 2.59.

13. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) = 0.5(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

14. (a) The velocities are given with units mi/h, so we must convert the 10-second intervals to hours:

$$10 \text{ seconds} = \frac{10 \text{ seconds}}{3600 \text{ seconds/h}} = \frac{1}{360} \text{ h}$$

$$\begin{aligned}\text{distance} \approx L_6 &= (182.9 \text{ mi/h})\left(\tfrac{1}{360} \text{ h}\right) + (168.0)\left(\tfrac{1}{360}\right) + (106.6)\left(\tfrac{1}{360}\right) + (99.8)\left(\tfrac{1}{360}\right) \\ &\quad + (124.5)\left(\tfrac{1}{360}\right) + (176.1)\left(\tfrac{1}{360}\right) \\ &= \frac{857.9}{360} \approx 2.383 \text{ miles}\end{aligned}$$

(b) $\text{Distance} \approx R_6 = \left(\frac{1}{360}\right)(168.0 + 106.6 + 99.8 + 124.5 + 176.1 + 175.6) = \dfrac{850.6}{360} \approx 2.363 \text{ miles}$

(c) The velocity is neither increasing nor decreasing on the given interval, so the estimates in parts (a) and (b) are neither upper nor lower estimates.

15. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.
Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

16. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v(t_i)\,\Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$$

17. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

18. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30 - 0}{6} = 5 \text{ s} = \frac{5}{3600} \text{ h} = \frac{1}{720} \text{ h}$.

$$\begin{aligned}M_6 &= \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)] \\ &= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725 \text{ km}\end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

19. $f(t) = -t(t-21)(t+1)$ and $\Delta t = \frac{12-0}{6} = 2$

$$\begin{aligned} M_6 &= 2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5) + 2 \cdot f(7) + 2 \cdot f(9) + 2 \cdot f(11) \\ &= 2 \cdot 40 + 2 \cdot 216 + 2 \cdot 480 + 2 \cdot 784 + 2 \cdot 1080 + 2 \cdot 1320 \\ &= 7840 \text{ (infected cells/mL)} \cdot \text{days} \end{aligned}$$

Thus, the total amount of infection needed to develop symptoms of measles is about 7840 infected cells per mL of blood plasma.

20. (a) Use $\Delta t = 14$ days. The number of people who died of SARS in Singapore between March 1 and May 24, 2003, using left endpoints is

$$L_6 = 14(0.0079 + 0.0638 + 0.1944 + 0.4435 + 0.5620 + 0.4630) = 14(1.7346) = 24.2844 \approx 24 \text{ people}$$

Using right endpoints,

$$R_6 = 14(0.0638 + 0.1944 + 0.4435 + 0.5620 + 0.4630 + 0.2897) = 14(2.0164) = 28.2296 \approx 28 \text{ people}$$

(b) Let t be the number of days since March 1, 2003, $f(t)$ be the number of deaths per day on day t, and the graph of $y = f(t)$ be a reasonable continuous function on the interval $[0, 84]$. Then the number of SARS deaths from $t = a$ to $t = b$ is approximately equal to the area under the curve $y = f(t)$ from $t = a$ to $t = b$.

21. $f(x) = \dfrac{2x}{x^2+1}$, $1 \le x \le 3$. $\Delta x = (3-1)/n = 2/n$ and $x_i = 1 + i\Delta x = 1 + 2i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+2i/n)}{(1+2i/n)^2+1} \cdot \frac{2}{n}.$$

22. $f(x) = x^2 + \sqrt{1+2x}$, $4 \le x \le 7$. $\Delta x = (7-4)/n = 3/n$ and $x_i = 4 + i\,\Delta x = 4 + 3i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[(4+3i/n)^2 + \sqrt{1+2(4+3i/n)}\right] \cdot \frac{3}{n}.$$

23. $f(x) = \sqrt{\sin x}$, $0 \le x \le \pi$. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i = 0 + i\,\Delta x = \pi i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin(\pi i/n)} \cdot \frac{\pi}{n}.$$

24. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y = \sqrt{1+x}$ on the interval $[0, 3]$, since for $y = \sqrt{1+x}$ on $[0, 3]$ with $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{3i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}\,\frac{3}{n}$. Note that this answer is not unique. We could use $y = \sqrt{x}$ on $[1, 4]$ or, in general, $y = \sqrt{x-n}$ on $[n+1, n+4]$, where n is any real number.

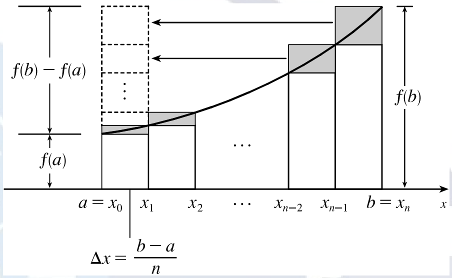
25. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n}\tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \dfrac{\pi/4 - 0}{n} = \dfrac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \dfrac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is $A = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

26. (a) $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$. $\quad A = \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\dfrac{i}{n}\right)^3 \cdot \dfrac{1}{n}$.

(b) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{i^3}{n^3} \cdot \dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n^4} \sum\limits_{i=1}^{n} i^3 = \lim\limits_{n\to\infty} \dfrac{1}{n^4} \left[\dfrac{n(n+1)}{2}\right]^2 = \lim\limits_{n\to\infty} \dfrac{(n+1)^2}{4n^2} = \dfrac{1}{4} \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 = \dfrac{1}{4}$

27. (a) Since f is an increasing function, L_n is an underestimate of A [lower sum] and R_n is an overestimate of A [upper sum]. Thus, A, L_n, and R_n are related by the inequality $L_n < A < R_n$.

(b)
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$
$$\begin{aligned} R_n - L_n &= f(x_n)\Delta x - f(x_0)\Delta x \\ &= \Delta x[f(x_n) - f(x_0)] \\ &= \frac{b-a}{n}[f(b) - f(a)] \end{aligned}$$



In the diagram, $R_n - L_n$ is the sum of the areas of the shaded rectangles. By sliding the shaded rectangles to the left so that they stack on top of the leftmost shaded rectangle, we form a rectangle of height $f(b) - f(a)$ and width $\dfrac{b-a}{n}$.

(c) $A > L_n$, so $R_n - A < R_n - L_n$; that is, $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)]$.

28. $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)] = \dfrac{3-1}{n}[f(3) - f(1)] = \dfrac{2}{n}(e^3 - e)$

Solving $\dfrac{2}{n}(e^3 - e) < 0.0001$ for n gives us $2(e^3 - e) < 0.0001n \;\Rightarrow\; n > \dfrac{2(e^3 - e)}{0.0001} \;\Rightarrow\; n > 347{,}345.1$. Thus, a value of n that assures us that $R_n - A < 0.0001$ is $n = 347{,}346$. [This is not the *least* value of n.]

29. (a) $y = f(x) = x^5$. $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5 \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5.$$

(b) $\sum\limits_{i=1}^{n} i^5 \overset{\text{CAS}}{=} \dfrac{n^2(n+1)^2\left(2n^2 + 2n - 1\right)}{12}$

(c)
$$\begin{aligned} \lim_{n\to\infty} \frac{64}{n^6} \cdot \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12} &= \frac{64}{12} \lim_{n\to\infty} \frac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2 \cdot n^2} \\ &= \frac{16}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(2 + \frac{2}{n} - \frac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3} \end{aligned}$$

30. From Example 3(a), we have $A = \lim\limits_{n\to\infty} \dfrac{2}{n} \sum\limits_{i=1}^{n} e^{-2i/n}$. Using a CAS, $\sum\limits_{i=1}^{n} e^{-2i/n} = \dfrac{e^{-2}\left(e^2 - 1\right)}{e^{2/n} - 1}$ and

$\lim\limits_{n\to\infty} \dfrac{2}{n} \cdot \dfrac{e^{-2}\left(e^2 - 1\right)}{e^{2/n} - 1} = e^{-2}\left(e^2 - 1\right) \approx 0.8647$, whereas the estimate from Example 3(b) using M_{10} was 0.8632.

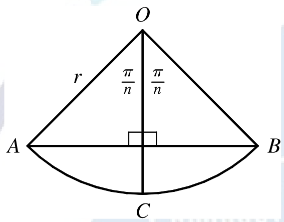
31. $y = f(x) = \cos x$. $\Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n}$$

$$\overset{\text{CAS}}{=} \lim_{n\to\infty} \left[\frac{b \sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n \sin\left(\frac{b}{2n}\right)} - \frac{b}{2n} \right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$.

32. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$. $\triangle AOB$ has area $2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2 \sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

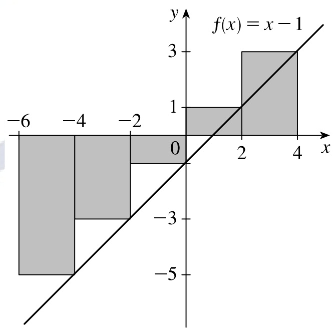
(b) To use Equation 3.3.2, $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \tfrac{1}{2}nr^2\sin(2\pi/n) = \lim_{n\to\infty} \tfrac{1}{2}nr^2 \frac{\sin(2\pi/n)}{2\pi/n} \cdot \frac{2\pi}{n} = \lim_{n\to\infty} \frac{\sin(2\pi/n)}{2\pi/n}\pi r^2.$$ Let $\theta = \dfrac{2\pi}{n}$.

Then as $n\to\infty$, $\theta\to 0$, so $\displaystyle\lim_{n\to\infty}\frac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.

5.2 The Definite Integral

1. $f(x) = x - 1$, $-6 \le x \le 4$. $\Delta x = \dfrac{b-a}{n} = \dfrac{4-(-6)}{5} = 2$.

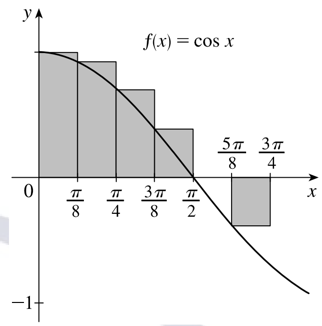


Since we are using right endpoints, $x_i^* = x_i$.

$$\begin{aligned} R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x \\ &= (\Delta x)[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)] \\ &= 2[f(-4) + f(-2) + f(0) + f(2) + f(4)] \\ &= 2[-5 + (-3) + (-1) + 1 + 3] \\ &= 2(-5) = -10 \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

2. $f(x) = \cos x$, $0 \le x \le \frac{3\pi}{4}$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3\pi/4 - 0}{6} = \dfrac{\pi}{8}$.

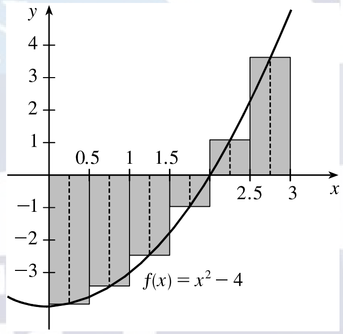


Since we are using left endpoints, $x_i^* = x_{i-1}$.

$$\begin{aligned} L_6 &= \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \\ &= (\Delta x)[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \\ &= \tfrac{\pi}{8}\left[f(0) + f\left(\tfrac{\pi}{8}\right) + f\left(\tfrac{2\pi}{8}\right) + f\left(\tfrac{3\pi}{8}\right) + f\left(\tfrac{4\pi}{8}\right) + f\left(\tfrac{5\pi}{8}\right)\right] \\ &\approx 1.033186 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the area of the rectangle below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis. A sixth rectangle is degenerate, with height 0, and has no area.

3. $f(x) = x^2 - 4$, $0 \le x \le 3$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$.

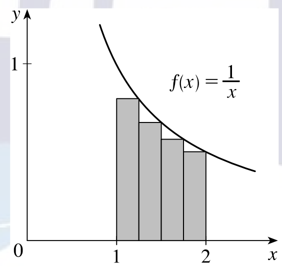


Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

$$\begin{aligned} M_6 &= \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \\ &= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5) + f(\overline{x}_6)] \\ &= \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right)\right] \\ &= \tfrac{1}{2}\left(-\tfrac{63}{16} - \tfrac{55}{16} - \tfrac{39}{16} - \tfrac{15}{16} + \tfrac{17}{16} + \tfrac{57}{16}\right) = \tfrac{1}{2}\left(-\tfrac{98}{16}\right) = -\tfrac{49}{16} \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

4. (a) $f(x) = \dfrac{1}{x}$, $1 \le x \le 2$. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{4} = \dfrac{1}{4}$.

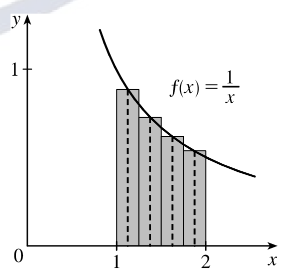


Since we are using right endpoints, $x_i^* = x_i$.

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} f(x_i)\,\Delta x \\ &= (\Delta x)[f(x_1) + f(x_2) + f(x_3) + f(x_4)] \\ &= \tfrac{1}{4}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{6}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{8}{4}\right)\right] \\ &= \tfrac{1}{4}\left[\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7} + \tfrac{1}{2}\right] \\ &\approx 0.634524 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles.

(b) Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.



$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \\ &= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)] \\ &= \tfrac{1}{4}\left[f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + f\left(\tfrac{15}{8}\right)\right] \\ &= \tfrac{1}{4}\left(\tfrac{8}{9} + \tfrac{8}{11} + \tfrac{8}{13} + \tfrac{8}{15}\right) \approx 0.691220 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles.

5. (a) $\int_0^{10} f(x)\,dx \approx R_5 = [f(2) + f(4) + f(6) + f(8) + f(10)]\,\Delta x$

$= [-1 + 0 + (-2) + 2 + 4](2) = 3(2) = 6$

(b) $\int_0^{10} f(x)\,dx \approx L_5 = [f(0) + f(2) + f(4) + f(6) + f(8)]\,\Delta x$

$= [3 + (-1) + 0 + (-2) + 2](2) = 2(2) = 4$

(c) $\int_0^{10} f(x)\,dx \approx M_5 = [f(1) + f(3) + f(5) + f(7) + f(9)]\,\Delta x$

$= [0 + (-1) + (-1) + 0 + 3](2) = 1(2) = 2$

6. (a) $\int_{-2}^{4} g(x)\,dx \approx R_6 = [g(-1) + g(0) + g(1) + g(2) + g(3) + g(4)]\,\Delta x$

$= \left[-\frac{3}{2} + 0 + \frac{3}{2} + \frac{1}{2} + (-1) + \frac{1}{2}\right](1) = 0$

(b) $\int_{-2}^{4} g(x)\,dx \approx L_6 = [g(-2) + g(-1) + g(0) + g(1) + g(2) + g(3)]\,\Delta x$

$= \left[0 + \left(-\frac{3}{2}\right) + 0 + \frac{3}{2} + \frac{1}{2} + (-1)\right](1) = -\frac{1}{2}$

(c) $\int_{-2}^{4} g(x)\,dx \approx M_6 = \left[g\left(-\frac{3}{2}\right) + g\left(-\frac{1}{2}\right) + g\left(\frac{1}{2}\right) + g\left(\frac{3}{2}\right) + g\left(\frac{5}{2}\right) + g\left(\frac{7}{2}\right)\right]\Delta x$

$= \left[-1 + (-1) + 1 + 1 + 0 + \left(-\frac{1}{2}\right)\right](1) = -\frac{1}{2}$

7. Since f is increasing, $L_5 \le \int_{10}^{30} f(x)\,dx \le R_5$.

Lower estimate $= L_5 = \sum_{i=1}^{5} f(x_{i-1})\Delta x = 4[f(10) + f(14) + f(18) + f(22) + f(26)]$

$= 4[-12 + (-6) + (-2) + 1 + 3] = 4(-16) = -64$

Upper estimate $= R_5 = \sum_{i=1}^{5} f(x)\Delta x = 4[f(14) + f(18) + f(22) + f(26) + f(30)]$

$= 4[-6 + (-2) + 1 + 3 + 8] = 4(4) = 16$

8. (a) Using the right endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(5) + f(7) + f(9)] = 2\,(-0.6 + 0.9 + 1.8) = 4.2.$

Since f is *increasing*, using *right* endpoints gives an *overestimate*.

(b) Using the left endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(3) + f(5) + f(7)] = 2\,(-3.4 - 0.6 + 0.9) = -6.2.$

Since f is *increasing*, using *left* endpoints gives an *underestimate*.

(c) Using the midpoint of each interval to approximate $\int_3^9 f(x)\,dx$, we have

$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(4) + f(6) + f(8)] = 2\,(-2.1 + 0.3 + 1.4) = -0.8.$

We cannot say anything about the midpoint estimate compared to the exact value of the integral.

9. $\Delta x = (8-0)/4 = 2$, so the endpoints are 0, 2, 4, 6, and 8, and the midpoints are 1, 3, 5, and 7. The Midpoint Rule gives

$\int_0^8 \sin\sqrt{x}\,dx \approx \sum\limits_{i=1}^{4} f(\bar{x}_i)\,\Delta x = 2\big(\sin\sqrt{1} + \sin\sqrt{3} + \sin\sqrt{5} + \sin\sqrt{7}\,\big) \approx 2(3.0910) = 6.1820.$

10. $\Delta x = (1-0)/5 = \frac{1}{5}$, so the endpoints are 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1, and the midpoints are $\frac{1}{10}$, $\frac{3}{10}$, $\frac{5}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$. The Midpoint Rule gives

$$\int_0^1 \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x = \tfrac{1}{5}\left(\sqrt{\left(\tfrac{1}{10}\right)^3+1} + \sqrt{\left(\tfrac{3}{10}\right)^3+1} + \sqrt{\left(\tfrac{5}{10}\right)^3+1} + \sqrt{\left(\tfrac{7}{10}\right)^3+1} + \sqrt{\left(\tfrac{9}{10}\right)^3+1}\right)$$
$$\approx 1.1097$$

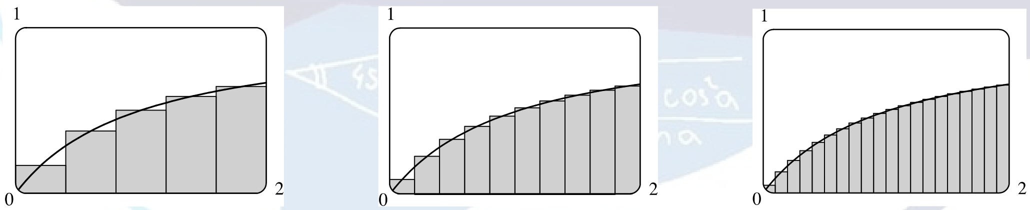
11. $\Delta x = (2-0)/5 = \frac{2}{5}$, so the endpoints are 0, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{5}$, $\frac{8}{5}$, and 2, and the midpoints are $\frac{1}{5}$, $\frac{3}{5}$, $\frac{5}{5}$, $\frac{7}{5}$ and $\frac{9}{5}$. The Midpoint Rule gives

$$\int_0^2 \frac{x}{x+1}\,dx \approx \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x = \frac{2}{5}\left(\frac{\frac{1}{5}}{\frac{1}{5}+1} + \frac{\frac{3}{5}}{\frac{3}{5}+1} + \frac{\frac{5}{5}}{\frac{5}{5}+1} + \frac{\frac{7}{5}}{\frac{7}{5}+1} + \frac{\frac{9}{5}}{\frac{9}{5}+1}\right) = \frac{2}{5}\left(\frac{127}{56}\right) = \frac{127}{140} \approx 0.9071.$$

12. $\Delta x = (\pi - 0)/4 = \frac{\pi}{4}$, so the endpoints are $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, and $\frac{4\pi}{4}$, and the midpoints are $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, and $\frac{7\pi}{8}$. The Midpoint Rule gives

$$\int_0^\pi x\sin^2 x\,dx \approx \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x = \frac{\pi}{4}\left(\frac{\pi}{8}\sin^2\frac{\pi}{8} + \frac{3\pi}{8}\sin^2\frac{3\pi}{8} + \frac{5\pi}{8}\sin^2\frac{5\pi}{8} + \frac{7\pi}{8}\sin^2\frac{7\pi}{8}\right) \approx 2.4674$$

13. In Maple 14, use the commands `with(Student[Calculus1])` and `ReimannSum(x/(x+1),0..2,partition=5,method=midpoint,output=plot)`. In some older versions of Maple, use `with(student)` to load the sum and box commands, then `m:=middlesum(x/(x+1),x=0..2)`, which gives us the sum in summation notation, then `M:=evalf(m)` to get the numerical approximation, and finally `middlebox(x/(x+1),x=0..2)` to generate the graph. The values obtained for $n = 5$, 10, and 20 are 0.9071, 0.9029, and 0.9018, respectively.



14. For $f(x) = x/(x+1)$ on $[0,2]$, we calculate $L_{100} \approx 0.89469$ and $R_{100} \approx 0.90802$. Since f is increasing on $[0,2]$, L_{100} is an underestimate of $\displaystyle\int_0^2 \frac{x}{x+1}\,dx$ and R_{100} is an overestimate. Thus, $0.8946 < \displaystyle\int_0^2 \frac{x}{x+1}\,dx < 0.9081$.

15. We'll create the table of values to approximate $\int_0^\pi \sin x\,dx$ by using the program in the solution to Exercise 5.1.9 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5$, 10, 50, and 100.

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 e^{-x^2}\,dx$ with $n = 5$, 10, 50, and 100.

n	L_n	R_n
5	1.077467	0.684794
10	0.980007	0.783670
50	0.901705	0.862438
100	0.891896	0.872262

The value of the integral lies between 0.872 and 0.892. Note that $f(x) = e^{-x^2}$ is decreasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 e^{-x^2}\,dx$ since f is increasing on $(-1, 0)$.

17. On $[0, 1]$, $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{x_i}}{1+x_i}\,\Delta x = \int_0^1 \frac{e^x}{1+x}\,dx.$

18. On $[2, 5]$, $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} x_i\sqrt{1+x_i^3}\,\Delta x = \int_2^5 x\sqrt{1+x^3}\,dx.$

19. On $[2, 7]$, $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}[5(x_i^*)^3 - 4x_i^*]\,\Delta x = \int_2^7 (5x^3 - 4x)\,dx.$

20. On $[1, 3]$, $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{x_i^*}{(x_i^*)^2+4}\,\Delta x = \int_1^3 \frac{x}{x^2+4}\,dx.$

21. Note that $\Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}$ and $x_i = 2 + i\,\Delta x = 2 + \dfrac{3i}{n}$.

$$\int_2^5 (4-2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[4-2\left(2+\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[-\frac{6i}{n}\right] = \lim_{n\to\infty}\frac{3}{n}\left(-\frac{6}{n}\right)\sum_{i=1}^{n} i = \lim_{n\to\infty}\left(-\frac{18}{n^2}\right)\left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left(-\frac{18}{2}\right)\left(\frac{n+1}{n}\right) = -9\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = -9(1) = -9$$

22. Note that $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$ and $x_i = 1 + i\,\Delta x = 1 + \dfrac{3i}{n}$.

$$\int_1^4 (x^2-4x+2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2 - 4\left(1+\frac{3i}{n}\right)+2\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1+\frac{6i}{n}+\frac{9i^2}{n^2}-4-\frac{12i}{n}+2\right] = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{9i^2}{n^2}-\frac{6i}{n}-1\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2}\frac{n(n+1)}{2} - \frac{3}{n}\cdot n(1)\right]$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\frac{(n+1)(2n+1)}{n^2} - 9\frac{n+1}{n} - 3\right] = \lim_{n\to\infty}\left[\frac{9}{2}\frac{n+1}{n}\frac{2n+1}{n} - 9\left(1+\frac{1}{n}\right) - 3\right]$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 9\left(1+\frac{1}{n}\right) - 3\right] = \frac{9}{2}(1)(2) - 9(1) - 3 = -3$$

23. Note that $\Delta x = \dfrac{0-(-2)}{n} = \dfrac{2}{n}$ and $x_i = -2 + i\,\Delta x = -2 + \dfrac{2i}{n}$.

$$\int_{-2}^{0}(x^2+x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-2+\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[4-\frac{8i}{n}+\frac{4i^2}{n^2}-2+\frac{2i}{n}\right] = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}-\frac{6i}{n}+2\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right] = \lim_{n\to\infty}\left[\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{12}{n^2}\frac{n(n+1)}{2} + \frac{2}{n}\cdot n(2)\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} - 6\,\frac{n+1}{n} + 4\right] = \lim_{n\to\infty}\left[\frac{4}{3}\frac{n+1}{n}\frac{2n+1}{n} - 6\left(1+\frac{1}{n}\right)+4\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 6\left(1+\frac{1}{n}\right)+4\right] = \frac{4}{3}(1)(2) - 6(1) + 4 = \frac{2}{3}$$

24. Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\int_{0}^{2}(2x-x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^3\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i}{n}-\frac{8i^3}{n^3}\right] = \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\left\{\frac{8}{n^2}\,\frac{n(n+1)}{2} - \frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right\} = \lim_{n\to\infty}\left[4\frac{n+1}{n} - 4\,\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right) - 4\,\frac{n+1}{n}\,\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right) - 4\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right]$$

$$= 4(1) - 4(1)(1) = 0$$

25. Note that $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$.

$$\int_{0}^{1}(x^3-3x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - 3\left(\frac{i}{n}\right)^2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{i^3}{n^3}-\frac{3i^2}{n^2}\right] = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^3}\sum_{i=1}^{n} i^3 - \frac{3}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right\} = \lim_{n\to\infty}\left[\frac{1}{4}\,\frac{n+1}{n}\,\frac{n+1}{n} - \frac{1}{2}\,\frac{n+1}{n}\,\frac{2n+1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) - \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{4}(1)(1) - \frac{1}{2}(1)(2) = -\frac{3}{4}$$

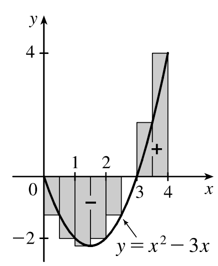
26. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

$$\int_0^4 (x^2-3x)\,dx \approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$

$$= 0.5\{[0.5^2 - 3(0.5)] + [1.0^2 - 3(1.0)] + \cdots + [3.5^2 - 3(3.5)] + [4.0^2 - 3(4.0)]\}$$

$$= \tfrac{1}{2}\left(-\tfrac{5}{4} - 2 - \tfrac{9}{4} - 2 - \tfrac{5}{4} + 0 + \tfrac{7}{4} + 4\right) = -1.5$$

(b)



(c) $\displaystyle\int_0^4 (x^2 - 3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$

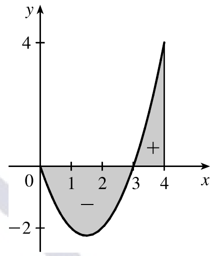
$$= \lim_{n\to\infty} \frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{12}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 24\left(1+\frac{1}{n}\right)\right]$$

$$= \tfrac{32}{3}\cdot 2 - 24 = -\tfrac{8}{3}$$

(d) $\int_0^4 (x^2-3x)\,dx = A_1 - A_2$, where A_1 is the area marked $+$ and A_2 is the area marked $-$.



27. $$\int_a^b x\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n + \frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a\,(b-a) + \lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$= a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)\left(a+\tfrac{1}{2}b-\tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}\left(b^2-a^2\right)$$

28. $$\int_a^b x^2\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}i^2 + \frac{2a\,(b-a)^2}{n^2}\sum_{i=1}^{n}i + \frac{a^2\,(b-a)}{n}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n\,(n+1)\,(2n+1)}{6} + \frac{2a\,(b-a)^2}{n^2}\frac{n\,(n+1)}{2} + \frac{a^2\,(b-a)}{n}n\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + a\,(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right) + a^2\,(b-a)\right]$$

$$= \frac{(b-a)^3}{3} + a\,(b-a)^2 + a^2\,(b-a) = \frac{b^3-3ab^2+3a^2b-a^3}{3} + ab^2 - 2a^2b + a^3 + a^2b - a^3$$

$$= \frac{b^3}{3} - \frac{a^3}{3} - ab^2 + a^2b + ab^2 - a^2b = \frac{b^3-a^3}{3}$$

29. $f(x) = \sqrt{4+x^2}$, $a=1$, $b=3$, and $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$. Using Theorem 4, we get $x_i^* = x_i = 1 + i\,\Delta x = 1 + \dfrac{2i}{n}$, so

$$\int_1^3 \sqrt{4+x^2}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{4+\left(1+\frac{2i}{n}\right)^2}\cdot\frac{2}{n}.$$

30. $f(x) = x^2 + \dfrac{1}{x}$, $a=2$, $b=5$, and $\Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}$. Using Theorem 4, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \dfrac{3i}{n}$, so

$$\int_2^5\left(x^2+\frac{1}{x}\right)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)^2 + \frac{1}{2+\dfrac{3i}{n}}\right]\cdot\frac{3}{n}.$$

31. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

32. $\Delta x = (10-2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\int_2^{10} x^6\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{8i}{n}\right)^6 \left(\frac{8}{n}\right) = 8 \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2+\frac{8i}{n}\right)^6$$

$$\stackrel{\text{CAS}}{=} 8 \lim_{n\to\infty} \frac{1}{n} \cdot \frac{64\left(58{,}593n^6 + 164{,}052n^5 + 131{,}208n^4 - 27{,}776n^2 + 2048\right)}{21n^5}$$

$$\stackrel{\text{CAS}}{=} 8\left(\frac{1{,}249{,}984}{7}\right) = \frac{9{,}999{,}872}{7} \approx 1{,}428{,}553.1$$

33. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)2 = 4$.

(b) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$

trapezoid rectangle triangle

$$= \tfrac{1}{2}(1+3)2 + 3\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 3 = 4+3+3 = 10$$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,

$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2$.

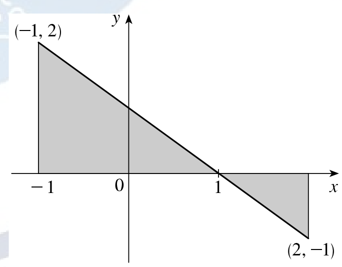
34. (a) $\int_0^2 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 2 = 4$ [area of a triangle]

(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ [negative of the area of a semicircle]

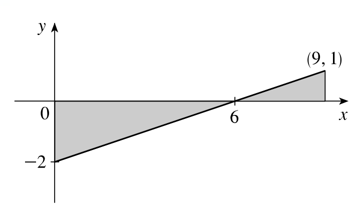
(c) $\int_6^7 g(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ [area of a triangle]

$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

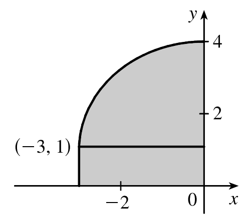
35. $\int_{-1}^2 (1-x)\,dx$ can be interpreted as the difference of the areas of the two shaded triangles; that is, $\frac{1}{2}(2)(2) - \frac{1}{2}(1)(1) = 2 - \frac{1}{2} = \frac{3}{2}$.



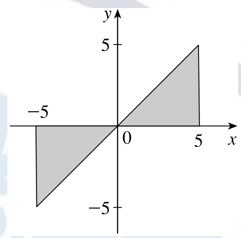
36. $\int_0^9 \left(\frac{1}{3}x - 2\right)dx$ can be interpreted as the difference of the areas of the two shaded triangles; that is, $-\frac{1}{2}(6)(2) + \frac{1}{2}(3)(1) = -6 + \frac{3}{2} = -\frac{9}{2}$.

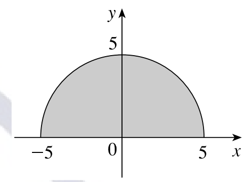


37. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$.

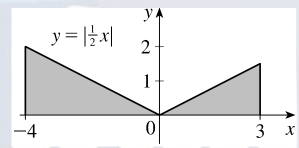


38. $\int_{-5}^{5}\left(x-\sqrt{25-x^2}\right)dx=\int_{-5}^{5}x\,dx-\int_{-5}^{5}\sqrt{25-x^2}\,dx$. By symmetry, the value of the first integral is 0 since the shaded area above the x-axis equals the shaded area below the x-axis. The second integral can be interpreted as one half the area of a circle with radius 5; that is, $\frac{1}{2}\pi(5)^2=\frac{25}{2}\pi$. Thus, the value of the original integral is $0-\frac{25}{2}\pi=-\frac{25}{2}\pi$.

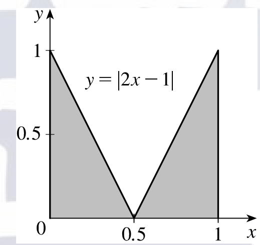




39. $\int_{-4}^{3}\left|\frac{1}{2}x\right|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(4)(2)+\frac{1}{2}(3)\left(\frac{3}{2}\right)=4+\frac{9}{4}=\frac{25}{4}$.



40. $\int_{0}^{1}|2x-1|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1)=\frac{1}{2}$.



41. $\int_{1}^{1}\sqrt{1+x^4}\,dx=0$ since the limits of integration are equal.

42. $\int_{\pi}^{0}\sin^4\theta\,d\theta=-\int_{0}^{\pi}\sin^4\theta\,d\theta$ [because we reversed the limits of integration]

$=-\int_{0}^{\pi}\sin^4 x\,dx$ [we can use any letter without changing the value of the integral]

$=-\frac{3}{8}\pi$ [given value]

43. $\int_{0}^{1}(5-6x^2)\,dx=\int_{0}^{1}5\,dx-6\int_{0}^{1}x^2\,dx=5(1-0)-6\left(\frac{1}{3}\right)=5-2=3$

44. $\int_{1}^{3}(2e^x-1)\,dx=2\int_{1}^{3}e^x\,dx-\int_{1}^{3}1\,dx=2(e^3-e)-1(3-1)=2e^3-2e-2$

45. $\int_{1}^{3}e^{x+2}\,dx=\int_{1}^{3}e^x\cdot e^2\,dx=e^2\int_{1}^{3}e^x\,dx=e^2(e^3-e)=e^5-e^3$

46. $\int_{0}^{\pi/2}(2\cos x-5x)\,dx=\int_{0}^{\pi/2}2\cos x\,dx-\int_{0}^{\pi/2}5x\,dx=2\int_{0}^{\pi/2}\cos x\,dx-5\int_{0}^{\pi/2}x\,dx$

$$=2(1)-5\frac{(\pi/2)^2-0^2}{2}=2-\frac{5\pi^2}{8}$$

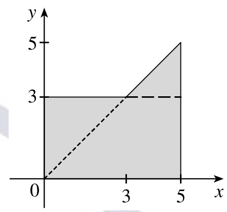
47. $\int_{-2}^{2}f(x)\,dx+\int_{2}^{5}f(x)\,dx-\int_{-2}^{-1}f(x)\,dx=\int_{-2}^{5}f(x)\,dx+\int_{-1}^{-2}f(x)\,dx$ [by Property 5 and reversing limits]

$=\int_{-1}^{5}f(x)\,dx$ [Property 5]

48. $\int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx$, so $\int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx - \int_2^4 f(x)\,dx = 7.3 - 5.9 = 1.4$.

49. $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

50. If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.



51. $\int_0^3 f(x)\,dx$ is clearly less than -1 and has the smallest value. The slope of the tangent line of f at $x = 1$, $f'(1)$, has a value between -1 and 0, so it has the next smallest value. The largest value is $\int_3^8 f(x)\,dx$, followed by $\int_4^8 f(x)\,dx$, which has a value about 1 unit less than $\int_3^8 f(x)\,dx$. Still positive, but with a smaller value than $\int_4^8 f(x)\,dx$, is $\int_0^8 f(x)\,dx$. Ordering these quantities from smallest to largest gives us

$$\int_0^3 f(x)\,dx < f'(1) < \int_0^8 f(x)\,dx < \int_4^8 f(x)\,dx < \int_3^8 f(x)\,dx \text{ or B} < \text{E} < \text{A} < \text{D} < \text{C}$$

52. $F(0) = \int_2^0 f(t)\,dt = -\int_0^2 f(t)\,dt$, so $F(0)$ is negative, and similarly, so is $F(1)$. $F(3)$ and $F(4)$ are negative since they represent negatives of areas below the x-axis. Since $F(2) = \int_2^2 f(t)\,dt = 0$ is the only non-negative value, choice C is the largest.

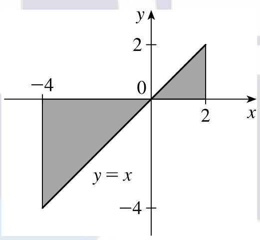
53. $I = \int_{-4}^2 [f(x) + 2x + 5]\,dx = \int_{-4}^2 f(x)\,dx + 2\int_{-4}^2 x\,dx + \int_{-4}^2 5\,dx = I_1 + 2I_2 + I_3$

$I_1 = -3$ [area below x-axis] $+ 3 - 3 = -3$

$I_2 = -\frac{1}{2}(4)(4)$ [area of triangle, see figure] $+ \frac{1}{2}(2)(2)$
$= -8 + 2 = -6$

$I_3 = 5[2 - (-4)] = 5(6) = 30$

Thus, $I = -3 + 2(-6) + 30 = 15$.



54. Using Integral Comparison Property 8, $m \leq f(x) \leq M \;\Rightarrow\; m(2-0) \leq \int_0^2 f(x)\,dx \leq M(2-0) \;\Rightarrow$
$2m \leq \int_0^2 f(x)\,dx \leq 2M$.

55. $x^2 - 4x + 4 = (x-2)^2 \geq 0$ on $[0, 4]$, so $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$ [Property 6].

56. $x^2 \leq x$ on $[0, 1]$, so $\sqrt{1+x^2} \leq \sqrt{1+x}$ on $[0, 1]$. Hence, $\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$ [Property 7].

57. If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1 + x^2 \leq 2$, so $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ and
$1[1-(-1)] \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

58. If $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$, then $\frac{1}{2} \leq \sin x \leq \frac{\sqrt{3}}{2}$ $\left(\sin x \text{ is increasing on } \left[\frac{\pi}{6}, \frac{\pi}{3}\right]\right)$, so

$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$ [Property 8]; that is, $\frac{\pi}{12} \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}\,\pi}{12}$.

59. If $0 \le x \le 1$, then $0 \le x^3 \le 1$, so $0(1-0) \le \int_0^1 x^3\,dx \le 1(1-0)$ [Property 8]; that is, $0 \le \int_0^1 x^3\,dx \le 1$.

60. If $0 \le x \le 3$, then $4 \le x+4 \le 7$ and $\dfrac{1}{7} \le \dfrac{1}{x+4} \le \dfrac{1}{4}$, so $\dfrac{1}{7}(3-0) \le \displaystyle\int_0^3 \frac{1}{x+4}\,dx \le \frac{1}{4}(3-0)$ [Property 8]; that is, $\dfrac{3}{7} \le \displaystyle\int_0^3 \frac{1}{x+4}\,dx \le \frac{3}{4}$.

61. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

62. Let $f(x) = x^3 - 3x + 3$ for $0 \le x \le 2$. Then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, so f is decreasing on $(0,1)$ and increasing on $(1,2)$. f has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum value of f is $f(2) = 5$. Thus, $1 \le x^3 - 3x + 3 \le 5$ for x in $[0,2]$. It follows from Property 8 that $1 \cdot (2-0) \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 5 \cdot (2-0)$; that is, $2 \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 10$.

63. The only critical number of $f(x) = xe^{-x}$ on $[0,2]$ is $x = 1$. Since $f(0) = 0$, $f(1) = e^{-1} \approx 0.368$, and $f(2) = 2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0,2]$ is 0, and the absolute maximum is e^{-1}. By Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2 \;\Rightarrow\; 0(2-0) \le \int_0^2 xe^{-x}\,dx \le e^{-1}(2-0) \;\Rightarrow\; 0 \le \int_0^2 xe^{-x}\,dx \le 2/e$.

64. Let $f(x) = x - 2\sin x$ for $\pi \le x \le 2\pi$. Then $f'(x) = 1 - 2\cos x$ and $f'(x) = 0 \;\Rightarrow\; \cos x = \frac{1}{2} \;\Rightarrow\; x = \frac{5\pi}{3}$. f has the absolute maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ since $f(\pi) = \pi$ and $f(2\pi) = 2\pi$ are both smaller than 6.97. Thus, $\pi \le f(x) \le \frac{5\pi}{3} + \sqrt{3} \;\Rightarrow\; \pi(2\pi - \pi) \le \int_\pi^{2\pi} f(x)\,dx \le \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)$; that is, $\pi^2 \le \int_\pi^{2\pi} (x - 2\sin x)\,dx \le \frac{5}{3}\pi^2 + \sqrt{3}\,\pi$.

65. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

66. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x\sin x \le x \;\Rightarrow\; \int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

67. $\sin x < \sqrt{x} < x$ for $1 \le x \le 2$ and arctan is an increasing function, so $\arctan(\sin x) < \arctan\sqrt{x} < \arctan x$, and hence, $\int_1^2 \arctan(\sin x)\,dx < \int_1^2 \arctan\sqrt{x}\,dx < \int_1^2 \arctan x\,dx$. Thus, $\int_1^2 \arctan x\,dx$ has the largest value.

68. $x^2 < \sqrt{x}$ for $0 < x \le 0.5$ and cosine is a decreasing function on $[0, 0.5]$, so $\cos(x^2) > \cos\sqrt{x}$, and hence, $\int_0^{0.5} \cos(x^2)\,dx > \int_0^{0.5} \cos\sqrt{x}\,dx$. Thus, $\int_0^{0.5} \cos(x^2)\,dx$ is larger.

69. Using right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf(x_i)\,\Delta x = \lim_{n\to\infty} c\sum_{i=1}^n f(x_i)\,\Delta x = c\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = c\int_a^b f(x)\,dx.$$

70. (a) Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \;\Rightarrow\; \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \;\Rightarrow\; |b| \le a$ for all real numbers b and nonnegative numbers a.

(b) $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)\sin 2x|\,dx$ [by part (a)] $= \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \le \int_0^{2\pi} |f(x)|\,dx$ by Property 7,

since $|\sin 2x| \le 1 \quad\Rightarrow\quad |f(x)|\,|\sin 2x| \le |f(x)|$.

71. Suppose that f is integrable on $[0,1]$, that is, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ exists for any choice of x_i^* in $[x_{i-1}, x_i]$. Let *n* denote a positive integer and divide the interval $[0,1]$ into *n* equal subintervals $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$. If we choose x_i^* to be a rational number in the *i*th subinterval, then we obtain the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = 0$, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 0 = 0$. Now suppose we choose x_i^* to be an irrational number. Then we get $\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \sum\limits_{i=1}^{n} 1\cdot\frac{1}{n} = n\cdot\frac{1}{n} = 1$ for each n, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 1 = 1$. Since the value of $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ depends on the choice of the sample points x_i^*, the limit does not exist, and f is not integrable on $[0,1]$.

72. Partition the interval $[0,1]$ into *n* equal subintervals and choose $x_1^* = \frac{1}{n^2}$. Then with $f(x) = \frac{1}{x}$, $\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x \ge f(x_1^*)\Delta x = \frac{1}{1/n^2}\cdot\frac{1}{n} = n$. Thus, $\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ can be made arbitrarily large and hence, f is not integrable on $[0,1]$.

73. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i^4}{n^5} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i^4}{n^4}\cdot\frac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^4\frac{1}{n}$. At this point, we need to recognize the limit as being of the form $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral is $\int_0^1 x^4\,dx$.

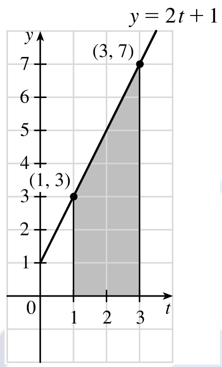
74. $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{1+(i/n)^2} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{1+(i/n)^2}\cdot\frac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = \frac{1}{1+x^2}$. Thus, the definite integral is $\int_0^1 \frac{dx}{1+x^2}$.

75. Choose $x_i = 1 + \frac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \quad [\text{by the hint}] \quad = \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i} - \sum_{i=1}^{n}\frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n\to\infty}\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$

DISCOVERY PROJECT Area Functions

1. (a)

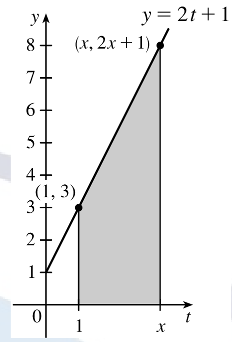


Area of trapezoid $= \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(3 + 7)2$
$= 10$ square units

Or:

Area of rectangle + area of triangle
$= b_r h_r + \frac{1}{2} b_t h_t = (2)(3) + \frac{1}{2}(2)(4) = 10$ square units

(b)

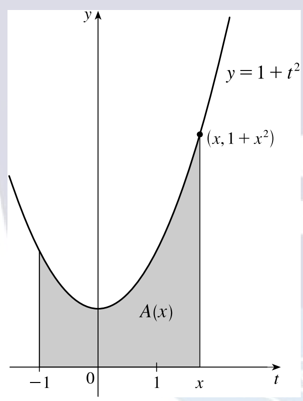


As in part (a),

$A(x) = \frac{1}{2}[3 + (2x + 1)](x - 1) = \frac{1}{2}(2x + 4)(x - 1)$
$= (x + 2)(x - 1) = x^2 + x - 2$ square units

(c) $A'(x) = 2x + 1$. This is the y-coordinate of the point $(x, 2x + 1)$ on the given line.

2. (a)



(b) $A(x) = \int_{-1}^{x} (1 + t^2)\, dt = \int_{-1}^{x} 1\, dt + \int_{-1}^{x} t^2\, dt$ [Property 2]

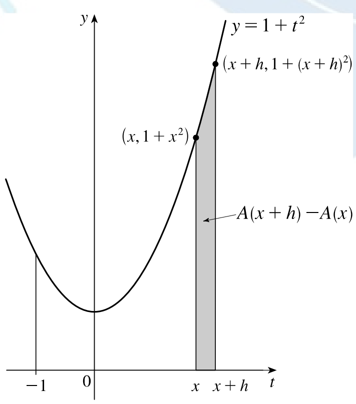
$= 1[x - (-1)] + \dfrac{x^3 - (-1)^3}{3}$ [Property 1 and Exercise 5.2.28]

$= x + 1 + \frac{1}{3}x^3 + \frac{1}{3}$

$= \frac{1}{3}x^3 + x + \frac{4}{3}$

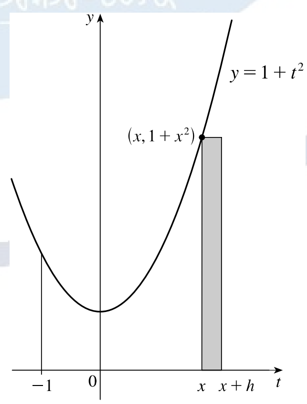
(c) $A'(x) = x^2 + 1$. This is the y-coordinate of the point $(x, 1 + x^2)$ on the given curve.

(d)



$A(x + h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x + h$.

(e)



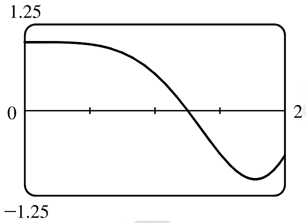
An approximating rectangle is shown in the figure.

It has height $1 + x^2$, width h, and area $h(1 + x^2)$, so

$A(x + h) - A(x) \approx h(1 + x^2) \quad \Rightarrow \quad \dfrac{A(x + h) - A(x)}{h} \approx 1 + x^2.$

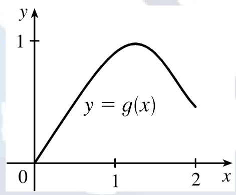
(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the instantaneous rate of change—namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.

3. (a) $f(x) = \cos(x^2)$

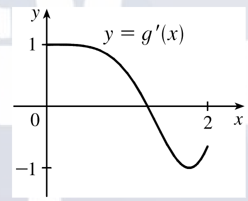


(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from positive to negative; that is, at about $x = 1.25$.

(c) $g(x) = \int_0^x \cos(t^2)\,dt$. Using an integration command, we find that $g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$, $g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$, $g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.



(d) We sketch the graph of g' using the method of Example 1 in Section 2.8. The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.



4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and $f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture that $g'(x) = f(x)$ for any continuous function f. This turns out to be true and is proved in Section 5.3 (the Fundamental Theorem of Calculus).

5.3 The Fundamental Theorem of Calculus

1. One process undoes what the other one does. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it on page 398.

2. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 1 \quad$ [area of triangle] $\quad = \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \quad$ [below the t-axis]

$= \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$.

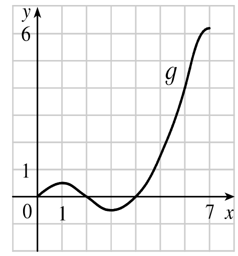
$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5$.

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4$.

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d)



3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$
$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

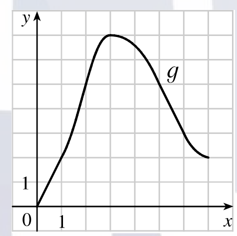
$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the t-axis]
$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

(c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

(d)



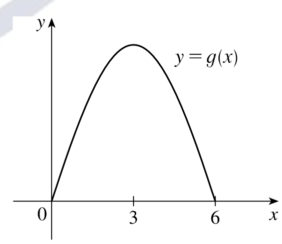
4. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = 0$ since the limits of integration are equal and $g(6) = 0$ since the areas above and below the t-axis are equal.

(b) $g(1)$ is the area under the curve from 0 to 1, which includes two unit squares and about 80% to 90% of a third unit square, so $g(1) \approx 2.8$. Similarly, $g(2) \approx 4.9$ and $g(3) \approx 5.7$. Now $g(3) - g(2) \approx 0.8$, so $g(4) \approx g(3) - 0.8 \approx 4.9$ by the symmetry of f about $x = 3$. Likewise, $g(5) \approx 2.8$.

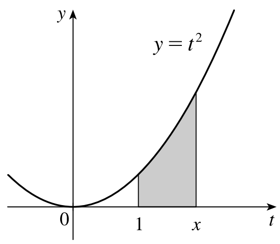
(c) As we go from $x = 0$ to $x = 3$, we are adding area, so g increases on the interval $(0, 3)$.

(d) g increases on $(0, 3)$ and decreases on $(3, 6)$ [where we are subtracting area], so g has a maximum value at $x = 3$.

(e) A graph of g must have a maximum at $x = 3$, be symmetric about $x = 3$, and have zeros at $x = 0$ and $x = 6$.



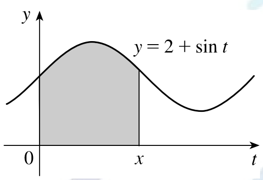
(f) If we sketch the graph of g' by estimating slopes on the graph of g (as in Section 2.8), we get a graph that looks like f (as indicated by FTC1).

5. 

(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \quad\Rightarrow$

$g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \quad\Rightarrow\quad g'(x) = x^2$.

6. 

(a) By FTC1 with $f(t) = 2 + \sin t$ and $a = 0$, $g(x) = \int_0^x (2 + \sin t)\,dt \quad\Rightarrow$

$g'(x) = f(x) = 2 + \sin x$.

(b) Using FTC2,

$$\begin{aligned} g(x) &= \int_0^x (2+\sin t)\,dt = [2t - \cos t]_0^x = (2x - \cos x) - (0 - 1) \\ &= 2x - \cos x + 1 \quad\Rightarrow \end{aligned}$$

$g'(x) = 2 - (-\sin x) + 0 = 2 + \sin x$

7. $f(t) = \sqrt{t + t^3}$ and $g(x) = \int_0^x \sqrt{t+t^3}\,dt$, so by FTC1, $g'(x) = f(x) = \sqrt{x + x^3}$.

8. $f(t) = \ln(1 + t^2)$ and $g(x) = \int_1^x \ln(1+t^2)\,dt$, so by FTC1, $g'(x) = f(x) = \ln(1 + x^2)$.

9. $f(t) = (t - t^2)^8$ and $g(s) = \int_5^s (t - t^2)^8\,dt$, so by FTC1, $g'(s) = f(s) = (s - s^2)^8$.

10. $f(t) = \dfrac{\sqrt{t}}{t+1}$ and $h(u) = \displaystyle\int_0^u \frac{\sqrt{t}}{t+1}\,dt$, so by FTC1, $h'(u) = f(u) = \dfrac{\sqrt{u}}{u+1}$.

11. $F(x) = \displaystyle\int_x^0 \sqrt{1 + \sec t}\,dt = -\int_0^x \sqrt{1+\sec t}\,dt \quad\Rightarrow\quad F'(x) = -\frac{d}{dx}\int_0^x \sqrt{1+\sec t}\,dt = -\sqrt{1 + \sec x}$

12. $R(y) = \displaystyle\int_y^2 t^3 \sin t\,dt = -\int_2^y t^3\sin t\,dt \quad\Rightarrow\quad R'(y) = -\frac{d}{dy}\int_2^y t^3 \sin t\,dt = -y^3 \sin y$

13. Let $u = e^x$. Then $\dfrac{du}{dx} = e^x$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_1^{e^x} \ln t\,dt = \frac{d}{du}\int_1^u \ln t\,dt \cdot \frac{du}{dx} = \ln u \frac{du}{dx} = (\ln e^x)\cdot e^x = xe^x.$$

14. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz = \frac{d}{du}\int_1^u \frac{z^2}{z^4+1}\,dz \cdot \frac{du}{dx} = \frac{u^2}{u^4+1}\,\frac{du}{dx} = \frac{x}{x^2+1}\,\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x^2+1)}.$$

15. Let $u = 3x + 2$. Then $\dfrac{du}{dx} = 3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_1^{3x+2} \frac{t}{1+t^3}\,dt = \frac{d}{du}\int_1^u \frac{t}{1+t^3}\,dt \cdot \frac{du}{dx} = \frac{u}{1+u^3}\,\frac{du}{dx} = \frac{3x+2}{1+(3x+2)^3}\cdot 3 = \frac{3(3x+2)}{1+(3x+2)^3}$$

16. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_0^{x^4} \cos^2\theta\,d\theta = \frac{d}{du}\int_0^u \cos^2\theta\,d\theta \cdot \frac{du}{dx} = \cos^2 u\,\frac{du}{dx} = \cos^2(x^4)\cdot 4x^3.$$

17. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan\theta\, d\theta = -\frac{d}{du}\int_{\pi/4}^{\sqrt{x}} \theta\tan\theta\, d\theta \cdot \frac{du}{dx} = -u\tan u\,\frac{du}{dx} = -\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$$

18. Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{\sin x}^{1} \sqrt{1+t^2}\, dt = \frac{d}{du}\int_{u}^{1} \sqrt{1+t^2}\, dt \cdot \frac{du}{dx} = -\frac{d}{du}\int_{1}^{u} \sqrt{1+t^2}\, dt \cdot \frac{du}{dx}$$
$$= -\sqrt{1+u^2}\,\cos x = -\sqrt{1+\sin^2 x}\,\cos x$$

19. $\int_1^3 (x^2+2x-4)\,dx = \left[\frac{1}{3}x^3 + x^2 - 4x\right]_1^3 = (9+9-12) - \left(\frac{1}{3}+1-4\right) = 6 + \frac{8}{3} = \frac{26}{3}$

20. $\int_{-1}^1 x^{100}\,dx = \left[\frac{1}{101}x^{101}\right]_{-1}^1 = \frac{1}{101} - \left(-\frac{1}{101}\right) = \frac{2}{101}$

21. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right)dt = \left[\frac{1}{5}t^4 - \frac{1}{4}t^3 + \frac{1}{5}t^2\right]_0^2 = \left(\frac{16}{5} - 2 + \frac{4}{5}\right) - 0 = 2$

22. $\int_0^1 (1 - 8v^3 + 16v^7)\,dv = \left[v - 2v^4 + 2v^8\right]_0^1 = (1-2+2) - 0 = 1$

23. $\displaystyle\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^9 = \frac{2}{3}\left[x^{3/2}\right]_1^9 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27-1) = \frac{52}{3}$

24. $\displaystyle\int_1^8 x^{-2/3}\,dx = \left[\frac{x^{1/3}}{1/3}\right]_1^8 = 3\left[x^{1/3}\right]_1^8 = 3(8^{1/3} - 1^{1/3}) = 3(2-1) = 3$

25. $\displaystyle\int_{\pi/6}^{\pi} \sin\theta\,d\theta = \left[-\cos\theta\right]_{\pi/6}^{\pi} = -\cos\pi - \left(-\cos\frac{\pi}{6}\right) = -(-1) - \left(-\sqrt{3}/2\right) = 1 + \sqrt{3}/2$

26. $\int_{-5}^5 e\,dx = \left[ex\right]_{-5}^5 = 5e - (-5e) = 10e$

27. $\displaystyle\int_0^1 (u+2)(u-3)\,du = \int_0^1 (u^2 - u - 6)\,du = \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 - 6u\right]_0^1 = \left(\frac{1}{3} - \frac{1}{2} - 6\right) - 0 = -\frac{37}{6}$

28. $\displaystyle\int_0^4 (4-t)\sqrt{t}\,dt = \int_0^4 (4-t)t^{1/2}\,dt = \int_0^4 (4t^{1/2} - t^{3/2})\,dt = \left[\frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^4 = \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{320-192}{15} = \frac{128}{15}$

29. $\displaystyle\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)dx = \int_1^4 (2x^{-1/2} + x^{3/2})\,dx$

$= \left[4x^{1/2} + \frac{2}{5}x^{5/2}\right]_1^4 = \left[4(2) + \frac{2}{5}(32)\right] - \left(4 + \frac{2}{5}\right) = 8 + \frac{64}{5} - 4 - \frac{2}{5} = \frac{82}{5}$

30. $\int_{-1}^2 (3u-2)(u+1)\,du = \int_{-1}^2 (3u^2 + u - 2)\,du = \left[u^3 + \frac{1}{2}u^2 - 2u\right]_{-1}^2 = (8+2-4) - \left(-1 + \frac{1}{2} + 2\right) = 6 - \frac{3}{2} = \frac{9}{2}$

31. $\displaystyle\int_{\pi/6}^{\pi/2} \csc t\,\cot t\,dt = \left[-\csc t\right]_{\pi/6}^{\pi/2} = \left(-\csc\frac{\pi}{2}\right) - \left(-\csc\frac{\pi}{6}\right) = -1 - (-2) = 1$

32. $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2\theta\,d\theta = \left[-\cot\theta\right]_{\pi/4}^{\pi/3} = \left(-\cot\frac{\pi}{3}\right) - \left(-\cot\frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}} - (-1) = 1 - \frac{1}{\sqrt{3}}$

33. $\int_0^1 (1+r)^3\,dr = \int_0^1 (1+3r+3r^2+r^3)\,dr = \left[r + \frac{3}{2}r^2 + r^3 + \frac{1}{4}r^4\right]_0^1 = \left(1+\frac{3}{2}+1+\frac{1}{4}\right) - 0 = \frac{15}{4}$

34. $\displaystyle\int_0^3 (2\sin x - e^x)\,dx = \Big[-2\cos x - e^x\Big]_0^3 = (-2\cos 3 - e^3) - (-2-1) = 3 - 2\cos 3 - e^3$

35. $\displaystyle\int_1^2 \frac{v^3+3v^6}{v^4} = \int_1^2 \left(\frac{1}{v} + 3v^2\right)\,dv = \big[\ln|v| + v^3\big]_1^2 = (\ln 2 + 8) - (\ln 1 + 1) = \ln 2 + 7$

36. $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \int_1^{18} \sqrt{3}z^{-1/2}\,dz = \sqrt{3}\left[2z^{1/2}\right]_1^{18} = 2\sqrt{3}(18^{1/2} - 1^{1/2}) = 2\sqrt{3}(3\sqrt{2}-1)$

37. $\displaystyle\int_0^1 (x^e + e^x)\,dx = \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1 = \left(\frac{1}{e+1} + e\right) - (0+1) = \frac{1}{e+1} + e - 1$

38. $\int_0^1 \cosh t\,dt = \big[\sinh t\big]_0^1 = \sinh 1 - \sinh 0 = \sinh 1 \quad \left[\text{or } \frac{1}{2}(e - e^{-1})\right]$

39. $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx = \Big[8\arctan x\Big]_{1/\sqrt{3}}^{\sqrt{3}} = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 8\left(\frac{\pi}{6}\right) = \frac{4\pi}{3}$

40. $\displaystyle\int_1^3 \frac{y^3 - 2y^2 - y}{y^2}\,dy = \int_1^3 \left(y - 2 - \frac{1}{y}\right)dy = \left[\frac{1}{2}y^2 - 2y - \ln|y|\right]_1^3 = \left(\frac{9}{2} - 6 - \ln 3\right) - \left(\frac{1}{2} - 2 - 0\right) = -\ln 3$

41. $\displaystyle\int_0^4 2^s\,ds = \left[\frac{1}{\ln 2}2^s\right]_0^4 = \frac{16}{\ln 2} - \frac{1}{\ln 2} = \frac{15}{\ln 2}$

42. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx = \Big[4\arcsin x\Big]_{1/2}^{1/\sqrt{2}} = 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3}$

43. If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$$\int_0^\pi f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^\pi \cos x\,dx = \big[-\cos x\big]_0^{\pi/2} + \big[\sin x\big]_{\pi/2}^\pi = -\cos\frac{\pi}{2} + \cos 0 + \sin\pi - \sin\frac{\pi}{2}$$
$$= -0 + 1 + 0 - 1 = 0$$

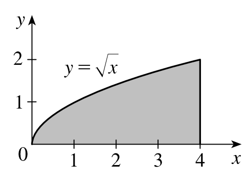
Note that f is integrable by Theorem 3 in Section 5.2.

44. If $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$ then

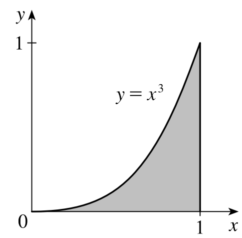
$$\int_{-2}^2 f(x)\,dx = \int_{-2}^0 2\,dx + \int_0^2 (4-x^2)\,dx = \big[2x\big]_{-2}^0 + \left[4x - \frac{1}{3}x^3\right]_0^2 = [0 - (-4)] + \left(\frac{16}{3} - 0\right) = \frac{28}{3}$$

Note that f is integrable by Theorem 3 in Section 5.2.

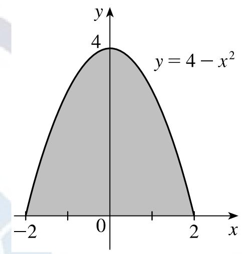
45. Area $\displaystyle = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(8) - 0 = \frac{16}{3}$



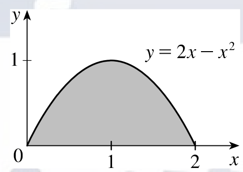
46. Area $= \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$



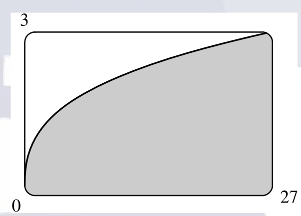
47. Area $= \int_{-2}^2 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$



48. Area $= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

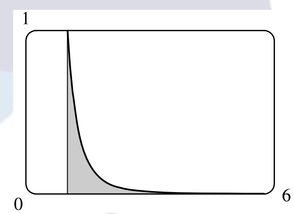


49. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3} dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4} \cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

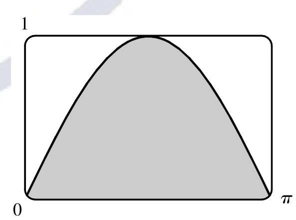


50. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

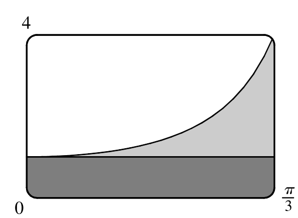
$$\int_1^6 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3 \cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.$$



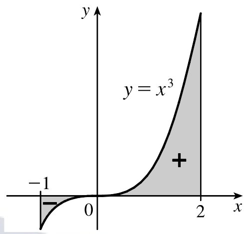
51. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2$.



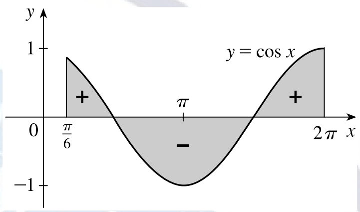
52. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3} + \frac{1}{4}\left(3 \cdot \frac{\pi}{3}\right) \approx 1.8$. The actual area is $\int_0^{\pi/3} \sec^2 x\,dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73$.



53. $\int_{-1}^{2} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



54. $\int_{\pi/6}^{2\pi} \cos x\,dx = \Big[\sin x\Big]_{\pi/6}^{2\pi} = 0 - \frac{1}{2} = -\frac{1}{2}$



55. $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\int_{-2}^{1} x^{-4}\,dx$ does not exist.

56. $f(x) = \dfrac{4}{x^3}$ is not continuous on the interval $[-1, 2]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\,dx$ does not exist.

57. $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^{\pi} \sec\theta\,\tan\theta\,d\theta$ does not exist.

58. $f(x) = \sec^2 x$ is not continuous on the interval $[0, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_0^{\pi} \sec^2 x\,dx$ does not exist.

59. $$g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du \quad\Rightarrow$$

$$g'(x) = -\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x) + \frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x) = -2\cdot\frac{4x^2-1}{4x^2+1} + 3\cdot\frac{9x^2-1}{9x^2+1}$$

60. $$g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt = \int_{1-2x}^{0} t\sin t\,dt + \int_{0}^{1+2x} t\sin t\,dt = -\int_{0}^{1-2x} t\sin t\,dt + \int_{0}^{1+2x} t\sin t\,dt \quad\Rightarrow$$

$$\begin{aligned} g'(x) &= -(1-2x)\sin(1-2x)\cdot\frac{d}{dx}(1-2x) + (1+2x)\sin(1+2x)\cdot\frac{d}{dx}(1+2x) \\ &= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x) \end{aligned}$$

61. $$F(x) = \int_{x}^{x^2} e^{t^2}\,dt = \int_{x}^{0} e^{t^2}\,dt + \int_{0}^{x^2} e^{t^2}\,dt = -\int_{0}^{x} e^{t^2}\,dt + \int_{0}^{x^2} e^{t^2}\,dt \quad\Rightarrow$$

$$F'(x) = -e^{x^2} + e^{(x^2)^2}\cdot\frac{d}{dx}(x^2) = -e^{x^2} + 2xe^{x^4}$$

62. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt = \int_{\sqrt{x}}^{0} \arctan t\ dt + \int_{0}^{2x} \arctan t\ dt = -\int_{0}^{\sqrt{x}} \arctan t\,dt + \int_{0}^{2x} \arctan t\,dt \quad\Rightarrow$

$$F'(x) = -\arctan\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x}) + \arctan 2x\cdot\frac{d}{dx}(2x) = -\frac{1}{2\sqrt{x}}\arctan\sqrt{x} + 2\arctan 2x$$

63. $y = \int_{\cos x}^{\sin x} \ln(1+2v)\,dv = \int_{\cos x}^{0} \ln(1+2v)\,dv + \int_{0}^{\sin x} \ln(1+2v)\,dv$

$$= -\int_{0}^{\cos x} \ln(1+2v)\,dv + \int_{0}^{\sin x} \ln(1+2v)\,dv \quad\Rightarrow$$

$$y' = -\ln(1+2\cos x)\cdot\frac{d}{dx}\cos x + \ln(1+2\sin x)\cdot\frac{d}{dx}\sin x = \sin x\,\ln(1+2\cos x) + \cos x\,\ln(1+2\sin x)$$

64. $f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$ is increasing when $f'(x) = (1-x^2)e^{x^2}$ is positive.

Since $e^{x^2} > 0$, $f'(x) > 0 \quad\Leftrightarrow\quad 1 - x^2 > 0 \quad\Leftrightarrow\quad |x| < 1$, so f is increasing on $(-1, 1)$.

65. $y = \int_0^x \frac{t^2}{t^2+t+2}\,dt \quad\Rightarrow\quad y' = \frac{x^2}{x^2+x+2} \quad\Rightarrow$

$$y'' = \frac{(x^2+x+2)(2x) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{2x^3+2x^2+4x-2x^3-x^2}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2} = \frac{x(x+4)}{(x^2+x+2)^2}.$$

The curve y is concave downward when $y'' < 0$; that is, on the interval $(-4, 0)$.

66. If $F(x) = \int_1^x f(t)\,dt$, then by FTC1, $F'(x) = f(x)$, and also, $F''(x) = f'(x)$. F is concave downward where F'' is negative; that is, where f' is negative. The given graph shows that f is decreasing ($f' < 0$) on the interval $(-1, 1)$.

67. $F(x) = \int_2^x e^{t^2}\,dt \quad\Rightarrow\quad F'(x) = e^{x^2}$, so the slope at $x = 2$ is $e^{2^2} = e^4$. The y-coordinate of the point on F at $x = 2$ is $F(2) = \int_2^2 e^{t^2}\,dt = 0$ since the limits are equal. An equation of the tangent line is $y - 0 = e^4(x-2)$, or $y = e^4x - 2e^4$.

68. $g(y) = \int_3^y f(x)\,dx \quad\Rightarrow\quad g'(y) = f(y)$. Since $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$, $g''(y) = f'(y) = \sqrt{1+\sin^2 y}\cdot\cos y$,

so $g''(\frac{\pi}{6}) = \sqrt{1+\sin^2(\frac{\pi}{6})}\cdot\cos\frac{\pi}{6} = \sqrt{1+(\frac{1}{2})^2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{5}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$.

69. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \quad\Rightarrow\quad f(4) = 17 + 12 = 29$.

70. (a) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad\Rightarrow\quad \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. By Property 5 of definite integrals in Section 5.2,

$\int_0^b e^{-t^2}\,dt = \int_0^a e^{-t^2}\,dt + \int_a^b e^{-t^2}\,dt$, so

$$\int_a^b e^{-t^2}\,dt = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \tfrac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)].$$

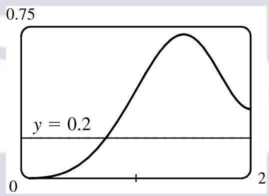
(b) $y = e^{x^2}\operatorname{erf}(x) \quad\Rightarrow\quad y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $= 2xy + \frac{2}{\sqrt{\pi}}$.

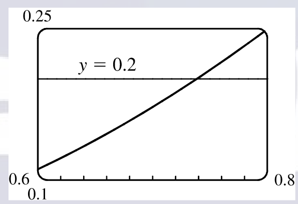
71. (a) The Fresnel function $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0 = S'(x) = \sin\left(\frac{\pi}{2}t^2\right)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$

$x^2 = 2(2n-1) \quad \Leftrightarrow \quad x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow \quad x^2 = 4n \quad \Leftrightarrow \quad x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

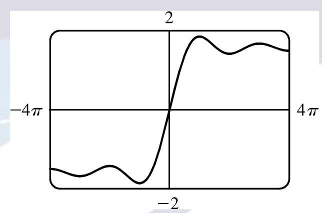
(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right) = \pi x \cos\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \quad \Leftrightarrow \quad 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi$, n any integer $\quad \Leftrightarrow \quad 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0 \quad \Leftrightarrow \quad \left(2n - \frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n - \frac{1}{2}\right)\pi$, n any integer $\quad \Leftrightarrow$ $4n - 3 < x^2 < 4n - 1 \quad \Leftrightarrow \quad \sqrt{4n-3} < |x| < \sqrt{4n-1} \quad \Rightarrow \quad \sqrt{4n-3} < -x < \sqrt{4n-1} \quad \Rightarrow$ $-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $(-\sqrt{3}, -1)$, $(\sqrt{3}, \sqrt{5})$, $(-\sqrt{7}, -\sqrt{5})$, $(\sqrt{7}, 3)$, ...

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt = 0.2$ at $x \approx 0.74$.





72. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



(b) $\mathrm{Si}(x)$ has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $\mathrm{Si}'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve $\text{Si}''(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a rootfinder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate $\text{Si}(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim\limits_{x\to\pm\infty} \text{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim\limits_{x\to\infty} \text{Si}(x) = \frac{\pi}{2}$. Since $\text{Si}(x)$ is an odd function, $\lim\limits_{x\to-\infty} \text{Si}(x) = -\frac{\pi}{2}$. So $\text{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

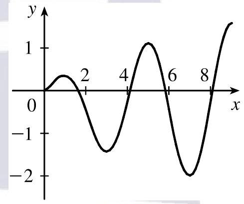
(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 65(c), we graph $y = \text{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

73. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)

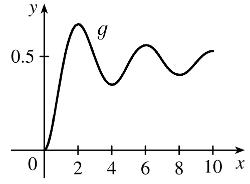


74. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$. So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and $g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



75. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^4}+\frac{i}{n}\right)\frac{1}{n}=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^4+\frac{i}{n}\right]=\int_0^1(x^4+x)\,dx$

$=\left[\frac15x^5+\frac12x^2\right]_0^1=\left(\frac15+\frac12\right)-0=\frac{7}{10}$

76. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac1n}+\sqrt{\frac2n}+\cdots+\sqrt{\frac nn}\right)=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\sqrt{\frac in}=\int_0^1\sqrt{x}\,dx=\left[\frac{2x^{3/2}}{3}\right]_0^1=\frac23-0=\frac23$

77. Suppose $h<0$. Since f is continuous on $[x+h,x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h,x]$ such that $f(u)=m$ and $f(v)=M$, where m and M are the absolute minimum and maximum values of f on $[x+h,x]$. By Property 8 of integrals, $m(-h)\le\int_{x+h}^{x}f(t)\,dt\le M(-h)$; that is, $f(u)(-h)\le-\int_x^{x+h}f(t)\,dt\le f(v)(-h)$.

Since $-h>0$, we can divide this inequality by $-h$: $\displaystyle f(u)\le\frac1h\int_x^{x+h}f(t)\,dt\le f(v)$. By Equation 2,

$\displaystyle\frac{g(x+h)-g(x)}{h}=\frac1h\int_x^{x+h}f(t)\,dt$ for $h\ne0$, and hence $\displaystyle f(u)\le\frac{g(x+h)-g(x)}{h}\le f(v)$, which is Equation 3 in the case where $h<0$.

78. $\displaystyle\frac{d}{dx}\int_{g(x)}^{h(x)}f(t)\,dt=\frac{d}{dx}\left[\int_{g(x)}^{a}f(t)\,dt+\int_a^{h(x)}f(t)\,dt\right]$ [where a is in the domain of f]

$\displaystyle=\frac{d}{dx}\left[-\int_a^{g(x)}f(t)\,dt\right]+\frac{d}{dx}\left[\int_a^{h(x)}f(t)\,dt\right]=-f(g(x))\,g'(x)+f(h(x))\,h'(x)$

$=f(h(x))\,h'(x)-f(g(x))\,g'(x)$

79. (a) Let $f(x)=\sqrt{x}\ \Rightarrow\ f'(x)=1/(2\sqrt{x})>0$ for $x>0\ \Rightarrow\ f$ is increasing on $(0,\infty)$. If $x\ge0$, then $x^3\ge0$, so $1+x^3\ge1$ and since f is increasing, this means that $f(1+x^3)\ge f(1)\ \Rightarrow\ \sqrt{1+x^3}\ge1$ for $x\ge0$. Next let $g(t)=t^2-t\ \Rightarrow\ g'(t)=2t-1\ \Rightarrow\ g'(t)>0$ when $t\ge1$. Thus, g is increasing on $(1,\infty)$. And since $g(1)=0$, $g(t)\ge0$ when $t\ge1$. Now let $t=\sqrt{1+x^3}$, where $x\ge0$. $\sqrt{1+x^3}\ge1$ (from above) $\Rightarrow\ t\ge1\ \Rightarrow\ g(t)\ge0\ \Rightarrow$ $(1+x^3)-\sqrt{1+x^3}\ge0$ for $x\ge0$. Therefore, $1\le\sqrt{1+x^3}\le1+x^3$ for $x\ge0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx\le\int_0^1\sqrt{1+x^3}\,dx\le\int_0^1(1+x^3)\,dx\ \Leftrightarrow$

$[x]_0^1\le\int_0^1\sqrt{1+x^3}\,dx\le\left[x+\frac14x^4\right]_0^1\ \Leftrightarrow\ 1\le\int_0^1\sqrt{1+x^3}\,dx\le1+\frac14=1.25$.

80. (a) For $0\le x\le1$, we have $x^2\le x$. Since $f(x)=\cos x$ is a decreasing function on $[0,1]$, $\cos(x^2)\ge\cos x$.

(b) $\pi/6<1$, so by part (a), $\cos(x^2)\ge\cos x$ on $[0,\pi/6]$. Thus,

$\int_0^{\pi/6}\cos(x^2)\,dx\ge\int_0^{\pi/6}\cos x\,dx=\left[\sin x\right]_0^{\pi/6}=\sin(\pi/6)-\sin0=\frac12-0=\frac12$.

81. $\displaystyle0<\frac{x^2}{x^4+x^2+1}<\frac{x^2}{x^4}=\frac{1}{x^2}$ on $[5,10]$, so

$\displaystyle0\le\int_5^{10}\frac{x^2}{x^4+x^2+1}\,dx<\int_5^{10}\frac{1}{x^2}\,dx=\left[-\frac1x\right]_5^{10}=-\frac{1}{10}-\left(-\frac15\right)=\frac{1}{10}=0.1$.

82. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

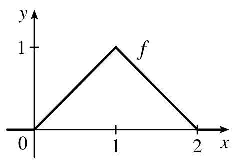
If $1 < x \le 2$, then

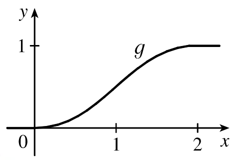
$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2-t)\,dt$$
$$= \tfrac{1}{2}(1)^2 + \left[2t - \tfrac{1}{2}t^2\right]_1^x = \tfrac{1}{2} + \left(2x - \tfrac{1}{2}x^2\right) - \left(2 - \tfrac{1}{2}\right) = 2x - \tfrac{1}{2}x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)





(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

83. Using FTC1, we differentiate both sides of $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\dfrac{f(x)}{x^2} = 2\dfrac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \displaystyle\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow 6 + 0 = 2\sqrt{a} \Rightarrow$ $3 = \sqrt{a} \Rightarrow a = 9$.

84. $B = 3A \Rightarrow \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \Rightarrow [e^x]_0^b = 3\,[e^x]_0^a \Rightarrow e^b - 1 = 3(e^a - 1) \Rightarrow e^b = 3e^a - 2 \Rightarrow$ $b = \ln(3e^a - 2)$

85. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right] = \dfrac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\dfrac{1}{t^2}\left[A + \displaystyle\int_0^t f(s)\,ds\right] + \dfrac{1}{t}f(t)$.

$$C'(t) = 0 \Rightarrow t\,f(t) = A + \int_0^t f(s)\,ds \Rightarrow f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t).$$

86. (a) $C(t) = \dfrac{1}{t}\displaystyle\int_0^t [f(s) + g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t) = \dfrac{1}{t}[f(t) + g(t)] - \dfrac{1}{t^2}\displaystyle\int_0^t [f(s) + g(s)]\,ds$. Set $C'(t) = 0$: $\dfrac{1}{t}[f(t) + g(t)] - \dfrac{1}{t^2}\displaystyle\int_0^t [f(s) + g(s)]\,ds = 0 \Rightarrow$

$$[f(t) + g(t)] - \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds = 0 \Rightarrow [f(t) + g(t)] - C(t) = 0 \Rightarrow C(t) = f(t) + g(t).$$

(b) For $0 \le t \le 30$, we have $D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s\right) ds = \left[\frac{V}{15}s - \frac{V}{900}s^2\right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$.

So $D(t) = V \quad \Rightarrow \quad \frac{V}{15}t - \frac{V}{900}t^2 = V \quad \Rightarrow \quad 60t - t^2 = 900 \quad \Rightarrow \quad t^2 - 60t + 900 = 0 \quad \Rightarrow$

$(t-30)^2 = 0 \quad \Rightarrow \quad t = 30$. So the length of time T is 30 months.

(c) $C(t) = \frac{1}{t}\int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12{,}900}s^2\right) ds = \frac{1}{t}\left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38{,}700}s^3\right]_0^t$

$$= \frac{1}{t}\left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38{,}700}t^3\right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 \quad \Rightarrow$$

$C'(t) = -\frac{V}{900} + \frac{V}{19{,}350}t = 0$ when $\frac{1}{19{,}350}t = \frac{1}{900} \quad \Rightarrow \quad t = 21.5$.

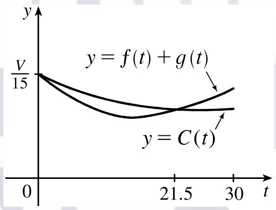
$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \frac{V}{15} \approx 0.06667V$, and

$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2$, so $C(t) = f(t) + g(t) \quad \Leftrightarrow$

$$\frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12{,}900}t^2 \quad \Leftrightarrow$$

$$t^2\left(\frac{1}{12{,}900} - \frac{1}{38{,}700}\right) = t\left(\frac{1}{450} - \frac{1}{900}\right) \quad \Leftrightarrow \quad t = \frac{1/900}{2/38{,}700} = \frac{43}{2} = 21.5.$$



This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.

5.4 Indefinite Integrals and the Net Change Theorem

1. $$\frac{d}{dx}\left[-\frac{\sqrt{1+x^2}}{x} + C\right] = \frac{d}{dx}\left[-\frac{(1+x^2)^{1/2}}{x} + C\right] = -\frac{x \cdot \frac{1}{2}(1+x^2)^{-1/2}(2x) - (1+x^2)^{1/2} \cdot 1}{(x)^2} + 0$$
$$= -\frac{(1+x^2)^{-1/2}\left[x^2 - (1+x^2)\right]}{x^2} = -\frac{-1}{(1+x^2)^{1/2}x^2} = \frac{1}{x^2\sqrt{1+x^2}}$$

2. $$\frac{d}{dx}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right) = \frac{1}{2} + \frac{1}{4}\cos 2x \cdot 2 + 0 = \frac{1}{2} + \frac{1}{2}\cos 2x$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\left(2\cos^2 x - 1\right) = \tfrac{1}{2} + \cos^2 x - \tfrac{1}{2} = \cos^2 x$$

3. $$\frac{d}{dx}(\tan x - x + C) = \sec^2 x - 1 + 0 = \tan^2 x$$

4. $$\frac{d}{dx}\left[\frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C\right] = \frac{2}{15b^2}\left[(3bx-2a)\tfrac{3}{2}(a+bx)^{1/2}(b) + (a+bx)^{3/2}(3b) + 0\right]$$
$$= \frac{2}{15b^2}(3b)(a+bx)^{1/2}\left[(3bx-2a)\tfrac{1}{2} + (a+bx)\right]$$
$$= \frac{2}{5b}(a+bx)^{1/2}\left(\frac{5}{2}bx\right) = x\sqrt{a+bx}$$

5. $\displaystyle\int (x^{1.3} + 7x^{2.5})\,dx = \frac{1}{2.3}x^{2.3} + \frac{7}{3.5}x^{3.5} + C = \frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$

6. $\int \sqrt[4]{x^5}\,dx = \int x^{5/4}\,dx = \frac{4}{9}x^{9/4} + C$

7. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right)\,dx = 5x + \frac{2}{3}\cdot\frac{1}{3}x^3 + \frac{3}{4}\cdot\frac{1}{4}x^4 + C = 5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$

8. $\int (u^6 - 2u^5 - u^3 + \frac{2}{7})\,du = \frac{1}{7}u^7 - 2\cdot\frac{1}{6}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C = \frac{1}{7}u^7 - \frac{1}{3}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C$

9. $\displaystyle\int (u+4)(2u+1)\,du = \int (2u^2 + 9u + 4)\,du = 2\,\frac{u^3}{3} + 9\,\frac{u^2}{2} + 4u + C = \frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$

10. $\int \sqrt{t}(t^2 + 3t + 2)\,dt = \int t^{1/2}(t^2 + 3t + 2)\,dt = \int (t^{5/2} + 3t^{3/2} + 2t^{1/2})\,dt$

$= \frac{2}{7}t^{7/2} + 3\cdot\frac{2}{5}t^{5/2} + 2\cdot\frac{2}{3}t^{3/2} + C = \frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C$

11. $\displaystyle\int \frac{1 + \sqrt{x} + x}{x}\,dx = \int \left(\frac{1}{x} + \frac{\sqrt{x}}{x} + \frac{x}{x}\right)dx = \int \left(\frac{1}{x} + x^{-1/2} + 1\right)dx$

$= \ln|x| + 2x^{1/2} + x + C = \ln|x| + 2\sqrt{x} + x + C$

12. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2+1}\right)dx = \frac{x^3}{3} + x + \tan^{-1}x + C$

13. $\int (\sin x + \sinh x)\,dx = -\cos x + \cosh x + C$

14. $\displaystyle\int \left(\frac{1+r}{r}\right)^2 dr = \int \frac{1 + 2r + r^2}{r^2}\,dr = \int (r^{-2} + 2r^{-1} + 1)\,dr = -r^{-1} + 2\ln|r| + r + C = -\frac{1}{r} + 2\ln|r| + r + C$

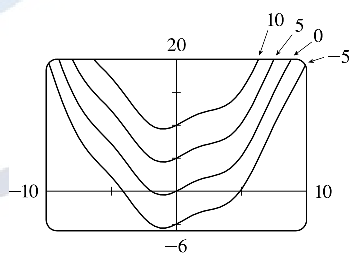
15. $\int (2 + \tan^2\theta)\,d\theta = \int [2 + (\sec^2\theta - 1)]\,d\theta = \int (1 + \sec^2\theta)\,d\theta = \theta + \tan\theta + C$

16. $\int \sec t\,(\sec t + \tan t)\,dt = \int (\sec^2 t + \sec t\,\tan t)\,dt = \tan t + \sec t + C$

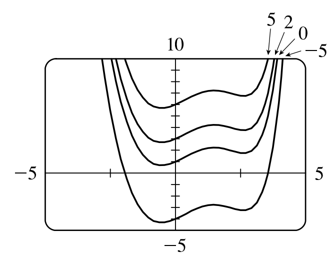
17. $\displaystyle\int 2^t(1 + 5^t)\,dt = \int (2^t + 2^t\cdot 5^t)\,dt = \int (2^t + 10^t)\,dt = \frac{2^t}{\ln 2} + \frac{10^t}{\ln 10} + C$

18. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\,\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x + C$

19. $\int \left(\cos x + \frac{1}{2}x\right)dx = \sin x + \frac{1}{4}x^2 + C$. The members of the family in the figure correspond to $C = -5$, 0, 5, and 10.



20. $\int (e^x - 2x^2)\,dx = e^x - \frac{2}{3}x^3 + C$. The members of the family in the figure correspond to $C = -5$, 0, 2, and 5.



21. $\int_{-2}^{3}(x^2-3)\,dx = \left[\frac{1}{3}x^3 - 3x\right]_{-2}^{3} = (9-9) - \left(-\frac{8}{3}+6\right) = \frac{8}{3} - \frac{18}{3} = -\frac{10}{3}$

22. $\int_{1}^{2}(4x^3 - 3x^2 + 2x)\,dx = \left[x^4 - x^3 + x^2\right]_{1}^{2} = (16-8+4) - (1-1+1) = 12 - 1 = 11$

23. $\int_{-2}^{0}\left(\frac{1}{2}t^4 + \frac{1}{4}t^3 - t\right)dt = \left[\frac{1}{10}t^5 + \frac{1}{16}t^4 - \frac{1}{2}t^2\right]_{-2}^{0} = 0 - \left[\frac{1}{10}(-32) + \frac{1}{16}(16) - \frac{1}{2}(4)\right] = -\left(-\frac{16}{5} + 1 - 2\right) = \frac{21}{5}$

24. $\int_{0}^{3}(1 + 6w^2 - 10w^4)\,dw = \left[w + 2w^3 - 2w^5\right]_{0}^{3} = (3 + 54 - 486) - 0 = -429$

25. $\int_{0}^{2}(2x-3)(4x^2+1)\,dx = \int_{0}^{2}(8x^3 - 12x^2 + 2x - 3)\,dx = \left[2x^4 - 4x^3 + x^2 - 3x\right]_{0}^{2} = (32 - 32 + 4 - 6) - 0 = -2$

26. $\int_{-1}^{1} t(1-t)^2\,dt = \int_{-1}^{1} t(1 - 2t + t^2)\,dt = \int_{-1}^{1}(t - 2t^2 + t^3)\,dt = \left[\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right]_{-1}^{1}$

$= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) = -\frac{4}{3}$

27. $\int_{0}^{\pi}(5e^x + 3\sin x)\,dx = \left[5e^x - 3\cos x\right]_{0}^{\pi} = \left[5e^{\pi} - 3(-1)\right] - \left[5(1) - 3(1)\right] = 5e^{\pi} + 1$

28. $\displaystyle\int_{1}^{2}\left(\frac{1}{x^2} - \frac{4}{x^3}\right)dx = \int_{1}^{2}(x^{-2} - 4x^{-3})\,dx = \left[\frac{x^{-1}}{-1} - \frac{4x^{-2}}{-2}\right]_{1}^{2} = \left[-\frac{1}{x} + \frac{2}{x^2}\right]_{1}^{2} = \left(-\frac{1}{2} + \frac{1}{2}\right) - (-1 + 2) = -1$

29. $\displaystyle\int_{1}^{4}\left(\frac{4+6u}{\sqrt{u}}\right)du = \int_{1}^{4}\left(\frac{4}{\sqrt{u}} + \frac{6u}{\sqrt{u}}\right)du = \int_{1}^{4}(4u^{-1/2} + 6u^{1/2})\,du = \left[8u^{1/2} + 4u^{3/2}\right]_{1}^{4} = (16+32) - (8+4) = 36$

30. $\displaystyle\int_{0}^{1}\frac{4}{1+p^2}\,dp = \left[4\arctan p\right]_{0}^{1} = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

31. $\int_{0}^{1} x\left(\sqrt[3]{x} + \sqrt[4]{x}\right)dx = \int_{0}^{1}(x^{4/3} + x^{5/4})\,dx = \left[\frac{3}{7}x^{7/3} + \frac{4}{9}x^{9/4}\right]_{0}^{1} = \left(\frac{3}{7} + \frac{4}{9}\right) - 0 = \frac{55}{63}$

32. $\displaystyle\int_{1}^{4}\frac{\sqrt{y} - y}{y^2}\,dy = \int_{1}^{4}\left(\frac{\sqrt{y}}{y^2} - \frac{y}{y^2}\right)dy = \int_{1}^{4}(y^{-3/2} - y^{-1})\,dy = \left[-2y^{-1/2} - \ln|y|\right]_{1}^{4} = \left[-\frac{2}{\sqrt{y}} - \ln|y|\right]_{1}^{4}$

$= (-1 - \ln 4) - (-2 - \ln 1) = 1 - \ln 4$

33. $\displaystyle\int_{1}^{2}\left(\frac{x}{2} - \frac{2}{x}\right)dx = \left[\frac{1}{4}x^2 - 2\ln|x|\right]_{1}^{2} = (1 - 2\ln 2) - \left(\frac{1}{4} - 2\ln 1\right) = \frac{3}{4} - 2\ln 2$

34. $\displaystyle\int_{0}^{1}(5x - 5^x)\,dx = \left[\frac{5}{2}x^2 - \frac{5^x}{\ln 5}\right]_{0}^{1} = \left(\frac{5}{2} - \frac{5}{\ln 5}\right) - \left(0 - \frac{1}{\ln 5}\right) = \frac{5}{2} - \frac{4}{\ln 5}$

35. $\displaystyle\int_{0}^{1}(x^{10} + 10^x)\,dx = \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_{0}^{1} = \left(\frac{1}{11} + \frac{10}{\ln 10}\right) - \left(0 + \frac{1}{\ln 10}\right) = \frac{1}{11} + \frac{9}{\ln 10}$

36. $\displaystyle\int_{0}^{\pi/4}\sec\theta\,\tan\theta\,d\theta = \left[\sec\theta\right]_{0}^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$

37. $\displaystyle\int_{0}^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_{0}^{\pi/4}\left(\frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_{0}^{\pi/4}(\sec^2\theta + 1)\,d\theta$

$= \left[\tan\theta + \theta\right]_{0}^{\pi/4} = \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (0 + 0) = 1 + \frac{\pi}{4}$

38. $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\sin\theta\,\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \sin\theta\,d\theta$

$= \big[-\cos\theta\big]_0^{\pi/3} = -\frac{1}{2} - (-1) = \frac{1}{2}$

39. $\displaystyle\int_1^8 \frac{2+t}{\sqrt[3]{t^2}}\,dt = \int_1^8 \left(\frac{2}{t^{2/3}} + \frac{t}{t^{2/3}}\right) dt = \int_1^8 (2t^{-2/3} + t^{1/3})\,dt = \left[2\cdot 3t^{1/3} + \tfrac{3}{4}t^{4/3}\right]_1^8 = (12+12) - \left(6+\tfrac{3}{4}\right) = \tfrac{69}{4}$

40. $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx = \int_{-10}^{10} \frac{2e^x}{\dfrac{e^x - e^{-x}}{2} + \dfrac{e^x + e^{-x}}{2}}\,dx = \int_{-10}^{10} \frac{2e^x}{e^x}\,dx = \int_{-10}^{10} 2\,dx = \big[2x\big]_{-10}^{10} = 20 - (-20) = 40$

41. $\displaystyle\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}} = \Big[\arcsin r\Big]_0^{\sqrt{3}/2} = \arcsin\!\left(\sqrt{3}/2\right) - \arcsin 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

42. $\displaystyle\int_1^2 \frac{(x-1)^3}{x^2}\,dx = \int_1^2 \frac{x^3 - 3x^2 + 3x - 1}{x^2}\,dx = \int_1^2 \left(x - 3 + \frac{3}{x} - \frac{1}{x^2}\right) dx = \left[\tfrac{1}{2}x^2 - 3x + 3\ln|x| + \frac{1}{x}\right]_1^2$

$= \left(2 - 6 + 3\ln 2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{2} - 3 + 0 + 1\right) = 3\ln 2 - 2$

43. $\displaystyle\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt = \int_0^{1/\sqrt{3}} \frac{t^2-1}{(t^2+1)(t^2-1)}\,dt = \int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\,dt = \big[\arctan t\big]_0^{1/\sqrt{3}} = \arctan\!\left(1/\sqrt{3}\right) - \arctan 0$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

44. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$

Thus, $\int_0^2 |2x-1|\,dx = \int_0^{1/2}(1-2x)\,dx + \int_{1/2}^2 (2x-1)\,dx = \big[x - x^2\big]_0^{1/2} + \big[x^2 - x\big]_{1/2}^2$

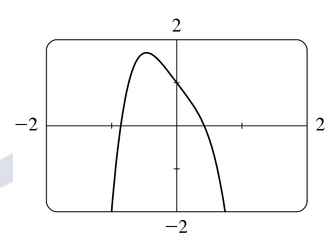
$= \left(\frac{1}{2} - \frac{1}{4}\right) - 0 + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4} + 2 - \left(-\frac{1}{4}\right) = \frac{5}{2}$

45. $\int_{-1}^2 (x - 2|x|)\,dx = \int_{-1}^0 [x - 2(-x)]\,dx + \int_0^2 [x - 2(x)]\,dx = \int_{-1}^0 3x\,dx + \int_0^2 (-x)\,dx = 3\left[\frac{1}{2}x^2\right]_{-1}^0 - \left[\frac{1}{2}x^2\right]_0^2$

$= 3\left(0 - \frac{1}{2}\right) - (2 - 0) = -\frac{7}{2} = -3.5$

46. $\int_0^{3\pi/2} |\sin x|\;dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2}(-\sin x)\,dx = \big[-\cos x\big]_0^{\pi} + \big[\cos x\big]_{\pi}^{3\pi/2} = [1 - (-1)] + [0 - (-1)] = 2 + 1 = 3$

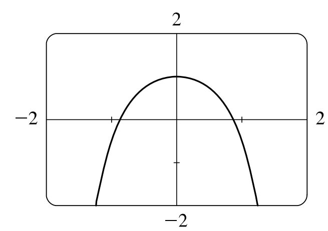
47. The graph shows that $y = 1 - 2x - 5x^4$ has x-intercepts at $x = a \approx -0.86$ and at $x = b \approx 0.42$. So the area of the region that lies under the curve and above the x-axis is



$\int_a^b (1 - 2x - 5x^4)\,dx = \big[x - x^2 - x^5\big]_a^b$

$= (b - b^2 - b^5) - (a - a^2 - a^5) \approx 1.36$

48. The graph shows that $y = (x^2+1)^{-1} - x^4$ has x-intercepts at $x = a \approx -0.87$ and at $x = b \approx 0.87$. So the area of the region that lies under the curve and above the x-axis is



$\int_a^b \left[(x^2+1)^{-1} - x^4\right]\,dx = \left[\tan^{-1} x - \frac{1}{5}x^5\right]_a^b$

$= \left(\tan^{-1} b - \frac{1}{5}b^5\right) - \left(\tan^{-1} a - \frac{1}{5}a^5\right)$

≈ 1.23

49. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

50. $y = \sqrt[4]{x} \quad \Rightarrow \quad x = y^4$, so $A = \int_0^1 y^4\,dy = \left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{5}$.

51. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

52. $\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

53. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

54. By the Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

55. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

56. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

57. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters (or joules). (A newton-meter is abbreviated N·m.)

58. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted $(\text{lb/ft})/\text{ft}$. The unit of measurement for $\int_2^8 a(x)\,dx$ is the product of pounds per foot and feet; that is, pounds.

59. (a) Displacement $= \int_0^3 (3t - 5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) Distance traveled $= \int_0^3 |3t - 5|\,dt = \int_0^{5/3} (5 - 3t)\,dt + \int_{5/3}^3 (3t - 5)\,dt$

$$= \left[5t - \tfrac{3}{2}t^2\right]_0^{5/3} + \left[\tfrac{3}{2}t^2 - 5t\right]_{5/3}^3 = \tfrac{25}{3} - \tfrac{3}{2}\cdot\tfrac{25}{9} + \tfrac{27}{2} - 15 - \left(\tfrac{3}{2}\cdot\tfrac{25}{9} - \tfrac{25}{3}\right) = \tfrac{41}{6}\text{ m}$$

60. (a) Displacement $= \int_2^4 (t^2 - 2t - 3)\,dt = \left[\frac{1}{3}t^3 - t^2 - 3t\right]_2^4 = \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{8}{3} - 4 - 6\right) = \frac{2}{3}$ m

(b) $v(t) = t^2 - 2t - 3 = (t + 1)(t - 3)$, so $v(t) < 0$ for $-1 < t < 3$, but on the interval $[2, 4]$, $v(t) < 0$ for $2 \le t < 3$.

Distance traveled $= \int_2^4 \left|t^2 - 2t - 3\right|\,dt = \int_2^3 -(t^2 - 2t - 3)\,dt + \int_3^4 (t^2 - 2t - 3)\,dt$

$$= \left[-\tfrac{1}{3}t^3 + t^2 + 3t\right]_2^3 + \left[\tfrac{1}{3}t^3 - t^2 - 3t\right]_3^4$$

$$= (-9 + 9 + 9) - \left(-\tfrac{8}{3} + 4 + 6\right) + \left(\tfrac{64}{3} - 16 - 12\right) - (9 - 9 - 9) = 4\text{ m}$$

61. (a) $v'(t) = a(t) = t + 4 \quad\Rightarrow\quad v(t) = \frac{1}{2}t^2 + 4t + C \quad\Rightarrow\quad v(0) = C = 5 \quad\Rightarrow\quad v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) Distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right| dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right) dt = \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10}$

$= \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

62. (a) $v'(t) = a(t) = 2t + 3 \quad\Rightarrow\quad v(t) = t^2 + 3t + C \quad\Rightarrow\quad v(0) = C = -4 \quad\Rightarrow\quad v(t) = t^2 + 3t - 4$

(b) Distance traveled $= \int_0^3 \left|t^2 + 3t - 4\right| dt = \int_0^3 |(t+4)(t-1)|\,dt = \int_0^1 \left(-t^2 - 3t + 4\right) dt + \int_1^3 \left(t^2 + 3t - 4\right) dt$

$= \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t\right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t\right]_1^3$

$= \left(-\frac{1}{3} - \frac{3}{2} + 4\right) + \left(9 + \frac{27}{2} - 12\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{89}{6}$ m

63. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 \left(9 + 2\sqrt{x}\right) dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

64. By the Net Change Theorem, the amount of water that flows from the tank during the first 10 minutes is

$\int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = \left[200t - 2t^2\right]_0^{10} = (2000 - 200) - 0 = 1800$ liters.

65. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour. So the distance traveled is

$\int_0^{100} v(t)\,dt \approx \frac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \frac{1}{180}(38 + 58 + 51 + 53 + 47) = \frac{247}{180} \approx 1.4$ miles.

66. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\,dt = Q(6)$ since $Q(0) = 0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6. $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{6} = 1$.

$R_6 = \sum_{i=1}^{6} r(t_i)\,\Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230$ tonnes.

$L_6 = \sum_{i=1}^{6} r(t_{i-1})\,\Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172$ tonnes.

(b) $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{3} = 2$. $Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$ tonnes.

67. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} (3 - 0.01x + 0.000006x^2)\,dx = \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000}$$

$$= 60{,}000 - 2{,}000 = \$58{,}000$$

68. By the Net Change Theorem, the amount of water after four days is

$$25{,}000 + \int_0^4 r(t)\,dt \approx 25{,}000 + M_4 = 25{,}000 + \frac{4-0}{4}\left[r(0.5) + r(1.5) + r(2.5) + r(3.5)\right]$$

$$\approx 25{,}000 + [1500 + 1770 + 740 + (-690)] = 28{,}320 \text{ liters}$$

69. To use the Midpoint Rule, we'll use the midpoint of each of three 2-second intervals.

$v(6) - v(0) = \int_0^6 a(t)\,dt \approx [a(1) + a(3) + a(5)]\,\dfrac{6-0}{3} \approx (0.6 + 10 + 9.3)(2) = 39.8 \text{ ft/s}$

70. Use the midpoint of each of four 2-day intervals. Let $t = 0$ correspond to July 18 and note that the inflow rate, $r(t)$, is in ft^3/s.

Amount of water $= \int_0^8 r(t)\,dt \approx [r(1) + r(3) + r(5) + r(7)]\,\dfrac{8-0}{4} \approx [6401 + 4249 + 3821 + 2628](2) = 34{,}198.$

Now multiply by the number of seconds in a day, $24 \cdot 60^2$, to get 2,954,707,200 ft^3.

71. Let $P(t)$ denote the bacteria population at time t (in hours). By the Net Change Theorem,

$P(1) - P(0) = \displaystyle\int_0^1 P'(t)\,dt = \int_0^1 (1000 \cdot 2^t)\,dt = \left[1000\,\frac{2^t}{\ln 2}\right]_0^1 = \frac{1000}{\ln 2}\,(2^1 - 2^0) = \frac{1000}{\ln 2} \approx 1443.$

Thus, the population after one hour is $4000 + 1443 = 5443$.

72. Let $M(t)$ denote the number of megabits transmitted at time t (in hours) [note that $D(t)$ is measured in megabits/second]. By the Net Change Theorem and the Midpoint Rule,

$$\begin{aligned} M(8) - M(0) &= \textstyle\int_0^8 3600D(t)\,dt \approx 3600 \cdot \frac{8-0}{4}[D(1) + D(3) + D(5) + D(7)] \\ &\approx 7200(0.32 + 0.50 + 0.56 + 0.83) = 7200(2.21) = 15{,}912 \text{ megabits} \end{aligned}$$

73. Power is the rate of change of energy with respect to time; that is, $P(t) = E'(t)$. By the Net Change Theorem and the Midpoint Rule,

$$\begin{aligned} E(24) - E(0) &= \int_0^{24} P(t)\,dt \approx \frac{24-0}{12}[P(1) + P(3) + P(5) + \cdots + P(21) + P(23)] \\ &\approx 2(16{,}900 + 16{,}400 + 17{,}000 + 19{,}800 + 20{,}700 + 21{,}200 \\ &\qquad + 20{,}500 + 20{,}500 + 21{,}700 + 22{,}300 + 21{,}700 + 18{,}900) \\ &= 2(237{,}600) = 475{,}200 \end{aligned}$$

Thus, the energy used on that day was approximately 4.75×10^5 megawatt-hours.

74. (a) From Exercise 4.1.74(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $h(125) - h(0) = \int_0^{125} v(t)\,dt = \left[0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t\right]_0^{125} \approx 206{,}407 \text{ ft}$

5.5 The Substitution Rule

1. Let $u = 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so $\int \cos 2x\,dx = \int \cos u\,\left(\frac{1}{2}\,du\right) = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$.

2. Let $u = -x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so $\int xe^{-x^2}\,dx = \int e^u\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$\displaystyle\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\,\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C.$

4. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \sin^2\theta\,\cos\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C$.

5. Let $u = x^4 - 5$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,du$, so

$$\int \frac{x^3}{x^4-5}\,dx = \int \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|x^4-5\right| + C.$$

6. Let $u = 2t+1$. Then $du = 2\,dt$ and $dt = \frac{1}{2}\,du$, so $\int \sqrt{2t+1}\,dt = \int \sqrt{u}\,\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(2t+1)^{3/2} + C$.

7. Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so

$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\,\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$.

8. Let $u = x^3$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so $\int x^2 e^{x^3}\,dx = \int e^u\,\left(\frac{1}{3}\,du\right) = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$.

9. Let $u = 1 - 2x$. Then $du = -2\,dx$ and $dx = -\frac{1}{2}\,du$, so

$\int (1-2x)^9\,dx = \int u^9\,\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\cdot\frac{1}{10}u^{10} + C = -\frac{1}{20}(1-2x)^{10} + C$.

10. Let $u = 1 + \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$\int \sin t\,\sqrt{1+\cos t}\,dt = \int \sqrt{u}\,(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1+\cos t)^{3/2} + C$.

11. Let $u = \frac{\pi}{2}t$. Then $du = \frac{\pi}{2}\,dt$ and $dt = \frac{2}{\pi}\,du$, so $\int \cos\!\left(\frac{\pi}{2}t\right)\,dt = \int \cos u\,\left(\frac{2}{\pi}\,du\right) = \frac{2}{\pi}\sin u + C = \frac{2}{\pi}\sin\!\left(\frac{\pi}{2}t\right) + C$.

12. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so $\int \sec^2\,2\theta\,d\theta = \int \sec^2 u\,\left(\frac{1}{2}\,du\right) = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2\theta + C$.

13. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5-3x} = \int \frac{1}{u}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5-3x| + C.$$

14. Let $u = 4 - y^3$. Then $du = -3y^2\,dy$ and $y^2\,dy = -\frac{1}{3}\,du$, so

$\int y^2(4-y^3)^{2/3}\,dy = \int u^{2/3}\,\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\cdot\frac{3}{5}u^{5/3} + C = -\frac{1}{5}(4-y^3)^{5/3} + C$.

15. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$\int \cos^3\theta\,\sin\theta\,d\theta = \int u^3(-du) = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4\theta + C$.

16. Let $u = -5r$. Then $du = -5\,dr$ and $dr = -\frac{1}{5}\,du$, so $\int e^{-5r}\,dr = \int e^u\left(-\frac{1}{5}\,du\right) = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5r} + C$.

17. Let $x = 1 - e^u$. Then $dx = -e^u\,du$ and $e^u\,du = -dx$, so

$$\int \frac{e^u}{(1-e^u)^2}\,du = \int \frac{1}{x^2}(-dx) = -\int x^{-2}\,dx = -(-x^{-1}) + C = \frac{1}{x} + C = \frac{1}{1-e^u} + C.$$

18. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $2\,du = \dfrac{1}{\sqrt{x}}\,dx$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = -2\cos u + C = -2\cos\sqrt{x} + C.$$

19. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$, so

$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx = \int \frac{\frac{1}{3}\,du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\cdot 2u^{1/2} + C = \frac{2}{3}\sqrt{3ax+bx^3} + C.$$

20. Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and $\frac{1}{3}\,du = z^2\,dz + C$, so

$$\int \frac{z^2}{z^3+1}\,dz = \int \frac{1}{u}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln|z^3+1| + C.$$

21. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C.$

22. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $-du = \sin x\,dx$, so

$\int \sin x\,\sin(\cos x)\,dx = \int \sin u\,(-du) = (-\cos u)(-1) + C = \cos(\cos x) + C.$

23. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so $\int \sec^2\theta\,\tan^3\theta\,d\theta = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4\theta + C.$

24. Let $u = x + 2$. Then $du = dx$ and $x = u - 2$, so

$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\,du = \int (u^{3/2} - 2u^{1/2})\,du = \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$

25. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$

Or: Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C.$

26. Let $u = ax + b$. Then $du = a\,dx$ and $dx = (1/a)\,du$, so

$$\int \frac{dx}{ax+b} = \int \frac{(1/a)\,du}{u} = \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C.$$

27. Let $u = x^3 + 3x$. Then $du = (3x^2+3)\,dx$ and $\frac{1}{3}\,du = (x^2+1)\,dx$, so

$\int (x^2+1)(x^3+3x)^4\,dx = \int u^4\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{15}(x^3+3x)^5 + C.$

28. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so $\int e^{\cos t}\sin t\,dt = \int e^u\,(-du) = -e^u + C = -e^{\cos t} + C.$

29. Let $u = 5^t$. Then $du = 5^t\ln 5\,dt$ and $5^t\,dt = \dfrac{1}{\ln 5}\,du$, so

$$\int 5^t\sin(5^t)\,dt = \int \sin u\left(\frac{1}{\ln 5}\,du\right) = -\frac{1}{\ln 5}\cos u + C = -\frac{1}{\ln 5}\cos(5^t) + C.$$

30. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -1u^{-1} + C = -\frac{1}{\tan x} + C = -\cot x + C.$$

Or: $\displaystyle\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \left(\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}\right)dx = \int \csc^2 x\,dx = -\cot x + C$

31. Let $u = \arctan x$. Then $du = \dfrac{1}{x^2+1}\,dx$, so $\displaystyle\int \frac{(\arctan x)^2}{x^2+1}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\arctan x)^3 + C.$

32. Let $u = x^2 + 4$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4\right| + C = \frac{1}{2}\ln(x^2+4) + C \quad [\text{since } x^2+4>0].$$

33. Let $u = 1 + 5t$. Then $du = 5\,dt$ and $dt = \frac{1}{5}\,du$, so

$\int \cos(1+5t)\,dt = \int \cos u\left(\frac{1}{5}\,du\right) = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(1+5t) + C.$

34. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\,du\right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C$$

35. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\,\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

36. Let $u = 2^t + 3$. Then $du = 2^t \ln 2\,dt$ and $2^t\,dt = \dfrac{1}{\ln 2}\,du$, so

$$\int \frac{2^t}{2^t+3}\,dt = \int \frac{1}{u}\left(\frac{1}{\ln 2}\,du\right) = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln(2^t+3) + C.$$

37. Let $u = \sinh x$. Then $du = \cosh x\,dx$, so $\int \sinh^2 x\,\cosh x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sinh^3 x + C$.

38. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

39. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

40. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

41. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

42. Let $u = \ln t$. Then $du = \dfrac{1}{t}\,dt$, so $\displaystyle\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du = \sin u + C = \sin(\ln t) + C$.

43. Let $u = \sin^{-1} x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1} x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|\sin^{-1} x\right| + C$.

44. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(x^2) + C$.

45. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int \frac{1+x}{1+x^2}\,dx &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C \\ &= \tan^{-1} x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].\end{aligned}$$

46. Let $u = 2 + x$. Then $du = dx$, $x = u - 2$, and $x^2 = (u-2)^2$, so

$$\int x^2\sqrt{2+x}\,dx = \int (u-2)^2\sqrt{u}\,du = \int (u^2 - 4u + 4)u^{1/2}\,du = \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du$$
$$= \tfrac{2}{7}u^{7/2} - \tfrac{8}{5}u^{5/2} + \tfrac{8}{3}u^{3/2} + C = \tfrac{2}{7}(2+x)^{7/2} - \tfrac{8}{5}(2+x)^{5/2} + \tfrac{8}{3}(2+x)^{3/2} + C$$

47. Let $u = 2x + 5$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u-5)$, so

$$\int x(2x+5)^8\,dx = \int \tfrac{1}{2}(u-5)u^8\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{4}\int (u^9 - 5u^8)\,du$$
$$= \tfrac{1}{4}\left(\tfrac{1}{10}u^{10} - \tfrac{5}{9}u^9\right) + C = \tfrac{1}{40}(2x+5)^{10} - \tfrac{5}{36}(2x+5)^9 + C$$

48. Let $u = x^2 + 1$ [so $x^2 = u - 1$]. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u-1)\sqrt{u}\,\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int (u^{3/2} - u^{1/2})\,du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

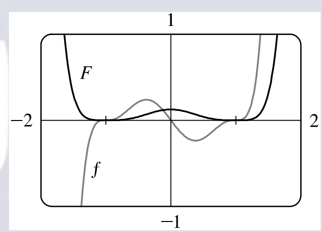
Or: Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1 \;\Rightarrow\; 2u\,du = 2x\,dx \;\Rightarrow\; u\,du = x\,dx$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u^2-1)\,u\cdot u\,du = \int (u^4 - u^2)\,du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

Note: This answer can be written as $\frac{1}{15}\sqrt{x^2+1}(3x^4 + x^2 - 2) + C$.

49. $f(x) = x(x^2-1)^3$. $\quad u = x^2 - 1 \;\Rightarrow\; du = 2x\,dx$, so

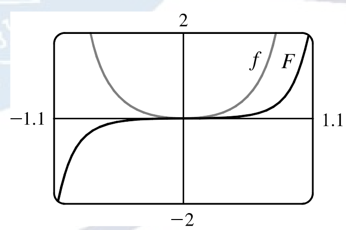
$$\int x(x^2-1)^3\,dx = \int u^3\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{8}u^4 + C = \tfrac{1}{8}(x^2-1)^4 + C$$



Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).

50. $f(\theta) = \tan^2\theta\,\sec^2\theta$. $u = \tan\theta \;\Rightarrow\; du = \sec^2\theta\,d\theta$, so

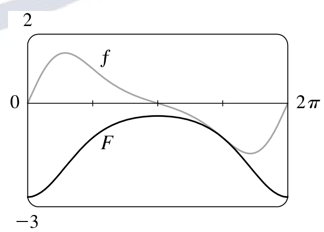
$$\int \tan^2\theta\,\sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$



Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.

51. $f(x) = e^{\cos x}\sin x$. $u = \cos x \;\Rightarrow\; du = -\sin x\,dx$, so

$$\int e^{\cos x}\sin x\,dx = \int e^u\,(-du) = -e^u + C = -e^{\cos x} + C$$

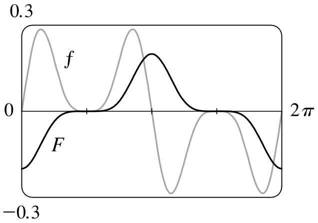


Note that at $x = \pi$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and at $x = 2\pi$, f changes from negative to positive and F has a local minimum.

52. $f(x) = \sin x \cos^4 x$. $u = \cos x \quad \Rightarrow \quad du = -\sin x\, dx$, so

$$\int \sin x \cos^4 x\, dx = \int u^4\,(-du) = -\tfrac{1}{5}u^5 + C = -\tfrac{1}{5}\cos^5 x + C$$

Note that at $x = \pi$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and at $x = 2\pi$, f changes from negative to positive and F has a local minimum.



53. Let $u = \frac{\pi}{2}t$, so $du = \frac{\pi}{2}\,dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \frac{\pi}{2}$. Thus,

$$\int_0^1 \cos(\pi t/2)\,dt = \int_0^{\pi/2} \cos u\,\left(\tfrac{2}{\pi}\,du\right) = \tfrac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \tfrac{2}{\pi}\left(\sin\tfrac{\pi}{2} - \sin 0\right) = \tfrac{2}{\pi}(1-0) = \tfrac{2}{\pi}$$

54. Let $u = 3t - 1$, so $du = 3\,dt$. When $t = 0$, $u = -1$; when $t = 1$, $u = 2$. Thus,

$$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^2 u^{50}\,\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\left[\tfrac{1}{51}u^{51}\right]_{-1}^2 = \tfrac{1}{153}\left[2^{51} - (-1)^{51}\right] = \tfrac{1}{153}(2^{51}+1)$$

55. Let $u = 1 + 7x$, so $du = 7\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 u^{1/3}\left(\tfrac{1}{7}\,du\right) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}(8^{4/3} - 1^{4/3}) = \tfrac{3}{28}(16-1) = \tfrac{45}{28}$$

56. Let $u = 5x + 1$, so $du = 5\,dx$. When $x = 0$, $u = 1$; when $x = 3$, $u = 16$. Thus,

$$\int_0^3 \frac{dx}{5x+1} = \int_1^{16} \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\Big[\ln|u|\Big]_1^{16} = \frac{1}{5}(\ln 16 - \ln 1) = \frac{1}{5}\ln 16.$$

57. Let $u = \cos t$, so $du = -\sin t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{6}$, $u = \sqrt{3}/2$. Thus,

$$\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt = \int_1^{\sqrt{3}/2} \frac{1}{u^2}\,(-du) = \left[\frac{1}{u}\right]_1^{\sqrt{3}/2} = \frac{2}{\sqrt{3}} - 1.$$

58. Let $u = \frac{1}{2}t$, so $du = \frac{1}{2}\,dt$. When $t = \frac{\pi}{3}$, $u = \frac{\pi}{6}$; when $t = \frac{2\pi}{3}$, $u = \frac{\pi}{3}$. Thus,

$$\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right)dt = \int_{\pi/6}^{\pi/3} \csc^2 u\,(2\,du) = 2\Big[-\cot u\Big]_{\pi/6}^{\pi/3} = -2\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{6}\right)$$

$$= -2\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -2\left(\tfrac{1}{3}\sqrt{3} - \sqrt{3}\right) = \tfrac{4}{3}\sqrt{3}$$

59. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

60. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

61. $\int_{-\pi/4}^{\pi/4}(x^3 + x^4\tan x)\,dx = 0$ by Theorem 7(b), since $f(x) = x^3 + x^4\tan x$ is an odd function.

62. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x\,\sin(\sin x)\,dx = \int_0^1 \sin u\,du = \Big[-\cos u\Big]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

63. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3-1) = 3.$$

64. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

65. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x\sqrt{x^2 + a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2} = \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

66. $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx = 0$ by Theorem 7(b), since $f(x) = x^4 \sin x$ is an odd function.

67. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

68. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{1+2x}} = \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9$$
$$= \tfrac{1}{6}[(27-9) - (1-3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

69. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

70. Let $u = (x-1)^2$, so $du = 2(x-1)\,dx$. When $x = 0$, $u = 1$; when $x = 2$, $u = 1$. Thus,

$$\int_0^2 (x-1)e^{(x-1)^2}\,dx = \int_1^1 e^u\left(\tfrac{1}{2}\,du\right) = 0 \text{ since the limits are equal.}$$

71. Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z + 1}{e^z + z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \left[\ln|u|\right]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

72. Let $u = \dfrac{2\pi t}{T} - \alpha$, so $du = \dfrac{2\pi}{T}\,dt$. When $t = 0$, $u = -\alpha$; when $t = \dfrac{T}{2}$, $u = \pi - \alpha$. Thus,

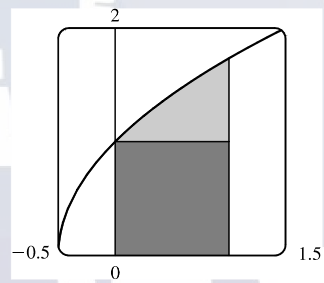
$$\int_0^{T/2} \sin\left(\frac{2\pi t}{T} - \alpha\right)dt = \int_{-\alpha}^{\pi-\alpha} \sin u\left(\frac{T}{2\pi}\,du\right) = \frac{T}{2\pi}\left[-\cos u\right]_{-\alpha}^{\pi-\alpha} = -\frac{T}{2\pi}[\cos(\pi-\alpha) - \cos(-\alpha)]$$
$$= -\frac{T}{2\pi}(-\cos\alpha - \cos\alpha) = -\frac{T}{2\pi}(-2\cos\alpha) = \frac{T}{\pi}\cos\alpha$$

73. Let $u = 1 + \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow 2\sqrt{x}\,du = dx \Rightarrow 2(u-1)\,du = dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$. Thus,

$$\int_0^1 \frac{dx}{(1+\sqrt{x}\,)^4} = \int_1^2 \frac{1}{u^4}\cdot[2(u-1)\,du] = 2\int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du = 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$
$$= 2\left[\left(-\tfrac{1}{8} + \tfrac{1}{24}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] = 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}$$

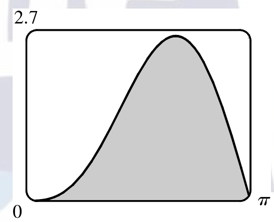
74. If $f(x) = \sin \sqrt[3]{x}$, then $f(-x) = \sin \sqrt[3]{-x} = \sin(-\sqrt[3]{x}\,) = -\sin \sqrt[3]{x} = -f(x)$, so f is an odd function. Now $I = \int_{-2}^{3} \sin \sqrt[3]{x}\,dx = \int_{-2}^{2} \sin \sqrt[3]{x}\,dx + \int_{2}^{3} \sin \sqrt[3]{x}\,dx = I_1 + I_2$. $I_1 = 0$ by Theorem 7(b). To estimate I_2, note that $2 \le x \le 3 \Rightarrow \sqrt[3]{2} \le \sqrt[3]{x} \le \sqrt[3]{3}$ $[\approx 1.44]$ $\Rightarrow 0 \le \sqrt[3]{x} \le \frac{\pi}{2}$ $[\approx 1.57]$ $\Rightarrow \sin 0 \le \sin \sqrt[3]{x} \le \sin \frac{\pi}{2}$ [since sine is increasing on this interval] $\Rightarrow 0 \le \sin \sqrt[3]{x} \le 1$. By comparison property 8, $0(3-2) \le I_2 \le 1(3-2) \Rightarrow 0 \le I_2 \le 1 \Rightarrow 0 \le I \le 1$.

75. From the graph, it appears that the area under the curve is about $1 + \big(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\big)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x+1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and $A = \int_1^3 \sqrt{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^3 = \frac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \frac{1}{3} \approx 1.399$.



76. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4. The exact area is given by

$$A = \int_0^\pi (2\sin x - \sin 2x)\,dx = -2\left[\cos x\right]_0^\pi - \int_0^\pi \sin 2x\,dx$$
$$= -2(-1-1) - 0 = 4$$



Note: $\int_0^\pi \sin 2x\,dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^{\pi} \sin 2x\,dx = -\int_0^{\pi/2} \sin 2x\,dx$.

77. First write the integral as a sum of two integrals:
$I = \int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 7(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

78. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so
$I = \int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1. So $I = \frac{1}{2}\cdot\frac{1}{4}\left(\pi\cdot 1^2\right) = \frac{1}{8}\pi$.

79. **First Figure** Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$$

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 ue^u\,du.$

Third Figure Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$A_3 = \int_0^{\pi/2} e^{\sin x}\sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$$

Since $A_1 = A_2 = A_3$, all three areas are equal.

80. Let $u = \dfrac{\pi t}{12}$. Then $du = \dfrac{\pi}{12}\,dt$ and

$$\int_0^{24} R(t)\,dt = \int_0^{24}\left[85 - 0.18\cos\left(\frac{\pi t}{12}\right)\right]dt = \int_0^{2\pi}(85 - 0.18\cos u)\left(\frac{12}{\pi}\,du\right) = \frac{12}{\pi}\left[85u - 0.18\sin u\right]_0^{2\pi}$$

$$= \frac{12}{\pi}\left[(85\cdot 2\pi - 0) - (0 - 0)\right] = 2040 \text{ kcal}$$

81. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$$\int_0^{60} r(t)\,dt = \int_0^{60} 100e^{-0.01t}\,dt \qquad [u = -0.01t, du = -0.01dt]$$

$$= 100\int_0^{-0.6} e^u(-100\,du) = -10{,}000\left[e^u\right]_0^{-0.6} = -10{,}000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ liters}$$

82. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$, $\int_0^3 r(t)\,dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$n(3) = 400 + \int_0^3 r(t)\,dt = 400 + \int_0^3 ae^{bt}\,dt = 400 + \frac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \frac{a}{b}\left(e^{3b} - 1\right)$$

$$\approx 400 + 11{,}313 = 11{,}713 \text{ bacteria}$$

83. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}\,u\right)du = \int_0^{2\pi t/5}\frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) \qquad \left[\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du\right]$$

$$= \frac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right) + 1\right] = \frac{5}{4\pi}\left[1 - \cos\left(\frac{2\pi}{5}t\right)\right] \text{ liters}$$

84. The rate G is measured in kilograms per year. Integrating from $t = 0$ years (2000) to $t = 20$ years (2020) will give us the net change in biomass from 2000 to 2020.

$$\int_0^{20}\frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}\,dt = \int_6^{1+5e^{-12}}\frac{60{,}000}{u^2}\left(-\tfrac{1}{3}\,du\right) \qquad \left[\begin{array}{l} u = 1 + 5e^{-0.6t}, \\ du = -3e^{-0.6t}dt \end{array}\right]$$

$$= \left[\frac{20{,}000}{u}\right]_6^{1+5e^{-12}} = \frac{20{,}000}{1 + 5e^{-12}} - \frac{20{,}000}{6} \approx 16{,}666$$

Thus, the predicted biomass for the year 2020 is approximately $25{,}000 + 16{,}666 = 41{,}666$ kg.

85. $$\int_0^{30} u(t)\,dt = \int_0^{30} \frac{r}{V}\,C_0 e^{-rt/V}\,dt = C_0 \int_1^{e^{-30r/V}} (-dx) \qquad \left[\begin{matrix} x = e^{-rt/V}, \\ dx = -\frac{r}{V} e^{-rt/V}\,dt \end{matrix}\right]$$

$$= C_0\Big[-x\Big]_1^{e^{-30r/V}} = C_0(-e^{-30r/V} + 1)$$

The integral $\int_0^{30} u(t)\,dt$ represents the total amount of urea removed from the blood in the first 30 minutes of dialysis.

86. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right] dt$

$$= 5000\left[t + 100(t+10)^{-1}\right]_2^4 = 5000\left[\left(4 + \tfrac{100}{14}\right) - \left(2 + \tfrac{100}{12}\right)\right] \approx 4048$$

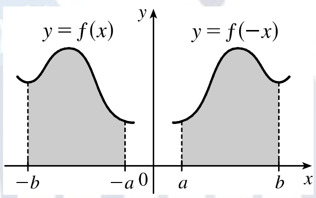
87. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

88. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 xf(x^2)\,dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

89. Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$

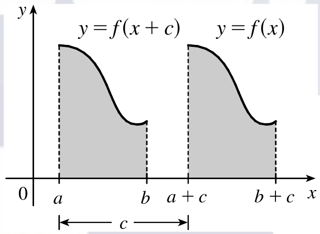
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



90. Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.



91. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a\,u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx$.

92. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\int_0^\pi xf(\sin x)\,dx = -\int_\pi^0 (\pi - u)\,f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u)\,f(\sin u)\,du$$

$$= \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u\,f(\sin u)\,du = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x\,f(\sin x)\,dx \quad \Rightarrow$$

$2\int_0^\pi x\,f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

93. $\dfrac{x\sin x}{1+\cos^2 x} = x \cdot \dfrac{\sin x}{2 - \sin^2 x} = x\,f(\sin x)$, where $f(t) = \dfrac{t}{2 - t^2}$. By Exercise 92,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^1$$

$$= \frac{\pi}{2}[\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

94. (a) $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[u = \frac{\pi}{2} - x,\ du = -dx\right]$

$$= \int_{\pi/2}^{0} f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u)\,du = \int_0^{\pi/2} f(\sin x)\,dx$$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

(b) In part (a), take $f(x) = x^2$, so $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \sin^2 x\,dx$. Now

$\int_0^{\pi/2} \cos^2 x\,dx + \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,dx = \int_0^{\pi/2} 1\,dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$,

so $2\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2} \quad\Rightarrow\quad \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4} \quad \left[= \int_0^{\pi/2} \sin^2 x\,dx\right]$.

5 Review

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.
2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.
3. True by Property 3 of the Integral in Section 5.2.
4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$, $\int_0^1 x\,f(x)\,dx = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$ (a constant) while $x\int_0^1 1\,dx = x\,[x]_0^1 = x \cdot 1 = x$ (a variable).
5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.
6. True by the Net Change Theorem.
7. True by Comparison Property 7 of the Integral in Section 5.2.
8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.
9. True. The integrand is an odd function that is continuous on $[-1, 1]$.
10. True. $\int_{-5}^{5} \left(ax^2 + bx + c\right) dx = \int_{-5}^{5} \left(ax^2 + c\right) dx + \int_{-5}^{5} bx\,dx$

    $= 2\int_0^5 \left(ax^2 + c\right) dx + 0 \quad$ [because $ax^2 + c$ is even and bx is odd]
11. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.
12. True by FTC1.
13. True by Property 5 of Integrals.
14. False. For example, $\int_0^1 \left(x - \frac{1}{2}\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_0^1 = \left(\frac{1}{2} - \frac{1}{2}\right) - (0 - 0) = 0$, but $f(x) = x - \frac{1}{2} \neq 0$.

15. False. $\int_a^b f(x)\,dx$ is a constant, so $\dfrac{d}{dx}\left(\int_a^b f(x)\,dx\right) = 0$, not $f(x)$ [unless $f(x) = 0$]. Compare the given statement carefully with FTC1, in which the upper limit in the integral is x.

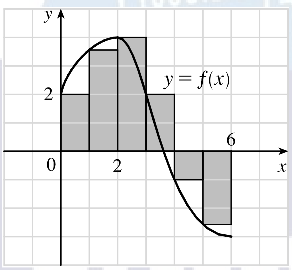
16. False. See the paragraph before Note 4 and Figure 4 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \le 2$.

17. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

18. False. For example, if $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } -1 \le x < 0 \end{cases}$ then f has a jump discontinuity at 0, but $\int_{-1}^1 f(x)\,dx$ exists and is equal to 1.

EXERCISES

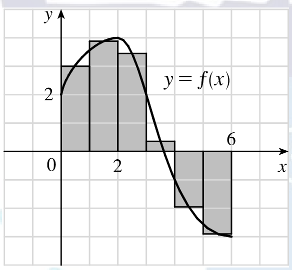
1. (a)



$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$
$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

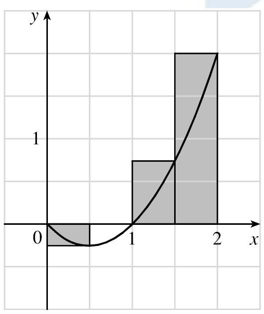
(b)



$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(\overline{x}_1)\cdot 1 + f(\overline{x}_2)\cdot 1 + f(\overline{x}_3)\cdot 1 + f(\overline{x}_4)\cdot 1 + f(\overline{x}_5)\cdot 1 + f(\overline{x}_6)\cdot 1$$
$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$
$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \quad \Rightarrow$

$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$
$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the *x*-axis minus the area of the rectangle below the *x*-axis. (The second rectangle vanishes.)

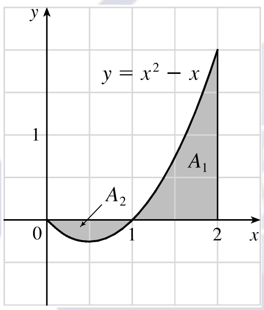
(b) $\int_0^2 (x^2 - x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \qquad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[\frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} - 2 \cdot \frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 2\left(1 + \frac{1}{n}\right)\right] = \tfrac{4}{3} \cdot 1 \cdot 2 - 2 \cdot 1 = \tfrac{2}{3}$$

(c) $\int_0^2 (x^2 - x)\,dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^2 = \left(\tfrac{8}{3} - 2\right) = \tfrac{2}{3}$

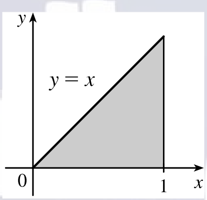
(d) 

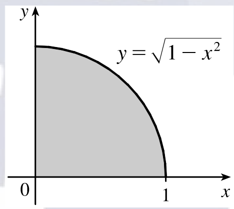
$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1 - x^2}\,dx = I_1 + I_2$.

I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle.

Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.





4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\,\Delta x = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -(-1) - (-1) = 2$.

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad\Rightarrow\quad 10 = 7 + \int_4^6 f(x)\,dx \quad\Rightarrow\quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

6. (a) $\int_1^5 (x + 2x^5)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n},\ x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n} \overset{\text{CAS}}{=} \lim_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n}$$

$$= 5220$$

(b) $\int_1^5 (x + 2x^5)\,dx = \left[\tfrac{1}{2}x^2 + \tfrac{2}{6}x^6\right]_1^5 = \left(\tfrac{25}{2} + \tfrac{15{,}625}{3}\right) - \left(\tfrac{1}{2} + \tfrac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By the Net Change Theorem (FTC2), $\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx = \left[e^{\arctan x}\right]_0^1 = e^{\pi/4} - 1$

(b) $\frac{d}{dx}\int_0^1 e^{\arctan x}\, dx = 0$ since this is the derivative of a constant.

(c) By FTC1, $\frac{d}{dx}\int_0^x e^{\arctan t}\, dt = e^{\arctan x}$.

9. $g(4) = \int_0^4 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$

$= -\frac{1}{2}\cdot 1\cdot 2 \quad \left[\begin{array}{l}\text{area of triangle,}\\ \textit{below } t\text{-axis}\end{array}\right] \quad + \frac{1}{2}\cdot 1\cdot 2 + 1\cdot 2 + \frac{1}{2}\cdot 1\cdot 2 = 3$

By FTC1, $g'(x) = f(x)$, so $g'(4) = f(4) = 0$.

10. $g(x) = \int_0^x f(t)\,dt \quad\Rightarrow\quad g'(x) = f(x) \quad$ [by FTC1] $\quad\Rightarrow\quad g''(x) = f'(x)$, so $g''(4) = f'(4) = -2$, which is the slope of the line segment at $x = 4$.

11. $\int_1^2 \left(8x^3 + 3x^2\right) dx = \left[8\cdot\frac{1}{4}x^4 + 3\cdot\frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = \left(2\cdot 2^4 + 2^3\right) - (2+1) = 40 - 3 = 37$

12. $\int_0^T \left(x^4 - 8x + 7\right) dx = \left[\frac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\frac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \frac{1}{5}T^5 - 4T^2 + 7T$

13. $\int_0^1 \left(1 - x^9\right) dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = \left(1 - \frac{1}{10}\right) - 0 = \frac{9}{10}$

14. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,

$\int_0^1 (1-x)^9\,dx = \int_1^0 u^9(-du) = \int_0^1 u^9\,du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}(1-0) = \frac{1}{10}$.

15. $\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du = \int_1^9 \left(u^{-1/2} - 2u\right) du = \left[2u^{1/2} - u^2\right]_1^9 = (6 - 81) - (2 - 1) = -76$

16. $\int_0^1 \left(\sqrt[4]{u} + 1\right)^2\,du = \int_0^1 \left(u^{1/2} + 2u^{1/4} + 1\right) du = \left[\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right]_0^1 = \left(\frac{2}{3} + \frac{8}{5} + 1\right) - 0 = \frac{49}{15}$

17. Let $u = y^2 + 1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,

$\int_0^1 y(y^2+1)^5\,dy = \int_1^2 u^5\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64 - 1) = \frac{63}{12} = \frac{21}{4}$.

18. Let $u = 1 + y^3$, so $du = 3y^2\,dy$ and $y^2\,dy = \frac{1}{3}\,du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,

$\int_0^2 y^2\sqrt{1+y^3}\,dy = \int_1^9 u^{1/2}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$.

19. $\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$; that is, f is discontinuous on the interval $[1, 5]$.

20. Let $u = 3\pi t$, so $du = 3\pi\,dt$. When $t = 0$, $u = 1$; when $t = 1$, $u = 3\pi$. Thus,

$\int_0^1 \sin(3\pi t)\,dt = \int_0^{3\pi} \sin u\left(\frac{1}{3\pi}\,du\right) = \frac{1}{3\pi}\left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1-1) = \frac{2}{3\pi}$.

21. Let $u = v^3$, so $du = 3v^2\,dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,

$\int_0^1 v^2 \cos(v^3)\,dv = \int_0^1 \cos u\,\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1.$

22. $\displaystyle\int_{-1}^{1} \frac{\sin x}{1+x^2}\,dx = 0$ by Theorem 5.5.7(b), since $f(x) = \dfrac{\sin x}{1+x^2}$ is an odd function.

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2+\cos t}\,dt = 0$ by Theorem 5.5.7(b), since $f(t) = \dfrac{t^4\tan t}{2+\cos t}$ is an odd function.

24. Let $u = e^x$, so $du = e^x\,dx$. When $x = 0, u = 1$; when $x = 1, u = e$. Thus,

$\displaystyle\int_0^1 \frac{e^x}{1+e^{2x}}\,dx = \int_1^e \frac{1}{1+u^2}\,du = \left[\arctan u\right]_1^e = \arctan e - \arctan 1 = \arctan e - \frac{\pi}{4}.$

25. $\displaystyle\int \left(\frac{1-x}{x}\right)^2 dx = \int\left(\frac{1}{x} - 1\right)^2 dx = \int\left(\frac{1}{x^2} - \frac{2}{x} + 1\right)dx = -\frac{1}{x} - 2\ln|x| + x + C$

26. $\displaystyle\int_1^{10} \frac{x}{x^2-4}\,dx$ does not exist because the function $f(x) = \dfrac{x}{x^2-4}$ has an infinite discontinuity at $x = 2$; that is, f is discontinuous on the interval $[1, 10]$.

27. Let $u = x^2 + 4x$. Then $du = (2x+4)\,dx = 2(x+2)\,dx$, so

$\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2+4x} + C.$

28. Let $u = 1 + \cot x$. Then $du = -\csc^2 x\,dx$, so $\displaystyle\int \frac{\csc^2 x}{1+\cot x}\,dx = \int \frac{1}{u}\,(-du) = -\ln|u| + C = -\ln|1+\cot x| + C.$

29. Let $u = \sin \pi t$. Then $du = \pi\,\cos \pi t\,dt$, so $\int \sin \pi t\,\cos \pi t\,dt = \int u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}\cdot\frac{1}{2}u^2 + C = \frac{1}{2\pi}(\sin \pi t)^2 + C.$

30. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\int \sin x\,\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C.$

31. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

32. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C.$

33. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx$, so

$\int \tan x\,\ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$

34. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\sin^{-1}\left(x^2\right) + C.$

35. Let $u = 1 + x^4$. Then $du = 4x^3\,dx$, so $\displaystyle\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \tfrac{1}{4}\ln\left(1+x^4\right) + C.$

36. Let $u = 1 + 4x$. Then $du = 4\,dx$, so $\int \sinh(1+4x)\,dx = \frac{1}{4}\int \sinh u\,du = \frac{1}{4}\cosh u + C = \frac{1}{4}\cosh(1+4x) + C.$

37. Let $u = 1 + \sec\theta$. Then $du = \sec\theta\,\tan\theta\,d\theta$, so

$$\int \frac{\sec\theta\,\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{1+\sec\theta}\,(\sec\theta\,\tan\theta\,d\theta) = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$$

38. Let $u = 1 + \tan t$, so $du = \sec^2 t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt = \int_1^2 u^3\,du = \left[\tfrac{1}{4}u^4\right]_1^2 = \tfrac{1}{4}\left(2^4 - 1^4\right) = \tfrac{1}{4}(16-1) = \tfrac{15}{4}.$$

39. Since $x^2 - 4 < 0$ for $0 \le x < 2$ and $x^2 - 4 > 0$ for $2 < x \le 3$, we have $\left|x^2 - 4\right| = -(x^2-4) = 4 - x^2$ for $0 \le x < 2$ and $\left|x^2 - 4\right| = x^2 - 4$ for $2 < x \le 3$. Thus,

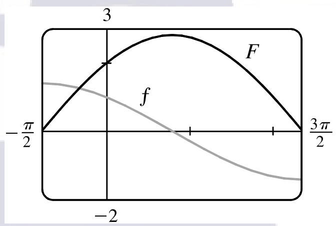
$$\begin{aligned}\int_0^3 \left|x^2-4\right| dx &= \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3 \\ &= \left(8 - \tfrac{8}{3}\right) - 0 + (9-12) - \left(\tfrac{8}{3} - 8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}\end{aligned}$$

40. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $\left|\sqrt{x} - 1\right| = -\left(\sqrt{x} - 1\right) = 1 - \sqrt{x}$ for $0 \le x < 1$ and $\left|\sqrt{x} - 1\right| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

$$\begin{aligned}\int_0^4 \left|\sqrt{x} - 1\right| dx &= \int_0^1 \left(1 - \sqrt{x}\right) dx + \int_1^4 \left(\sqrt{x} - 1\right) dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4 \\ &= \left(1 - \tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2\end{aligned}$$

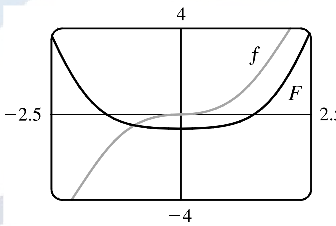
41. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$



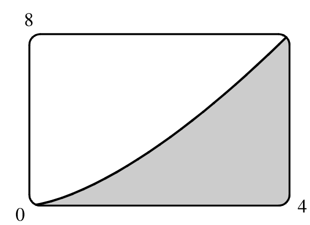
42. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C \\ &= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2-2) + C\end{aligned}$$

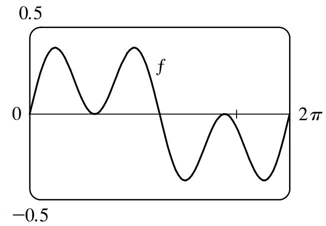


43. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \tfrac{2}{5}(4)^{5/2} = \tfrac{64}{5} = 12.8.$$



44. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \, \sin x \, dx$ is equal to 0. To evaluate the integral, let $u = \cos x \quad \Rightarrow \quad du = -\sin x \, dx$. Thus, $I = \int_1^1 u^2 \, (-du) = 0$.



45. $F(x) = \displaystyle\int_0^x \frac{t^2}{1+t^3}\, dt \quad \Rightarrow \quad F'(x) = \frac{d}{dx}\int_0^x \frac{t^2}{1+t^3}\, dt = \frac{x^2}{1+x^3}$

46. $F(x) = \displaystyle\int_x^1 \sqrt{t+\sin t}\, dt = -\int_1^x \sqrt{t+\sin t}\, dt \quad \Rightarrow \quad F'(x) = -\frac{d}{dx}\int_1^x \sqrt{t+\sin t}\, dt = -\sqrt{x+\sin x}$

47. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_0^{x^4} \cos(t^2)\, dt = \frac{d}{du}\int_0^u \cos(t^2)\, dt \cdot \frac{du}{dx} = \cos(u^2)\,\frac{du}{dx} = 4x^3 \cos(x^8).$$

48. Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_1^{\sin x} \frac{1-t^2}{1+t^4}\, dt = \frac{d}{du}\int_1^u \frac{1-t^2}{1+t^4}\, dt \cdot \frac{du}{dx} = \frac{1-u^2}{1+u^4}\cdot\frac{du}{dx} = \frac{1-\sin^2 x}{1+\sin^4 x}\cdot \cos x = \frac{\cos^3 x}{1+\sin^4 x}$$

49. $y = \displaystyle\int_{\sqrt{x}}^x \frac{e^t}{t}\, dt = \int_{\sqrt{x}}^1 \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt = -\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{d}{dx}\left(\displaystyle\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt\right) + \dfrac{d}{dx}\left(\displaystyle\int_1^x \frac{e^t}{t}\, dt\right)$. Let $u = \sqrt{x}$. Then

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt = \frac{d}{dx}\int_1^u \frac{e^t}{t}\, dt = \frac{d}{du}\left(\int_1^u \frac{e^t}{t}\, dt\right)\frac{du}{dx} = \frac{e^u}{u}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x},$$

so $\dfrac{dy}{dx} = -\dfrac{e^{\sqrt{x}}}{2x} + \dfrac{e^x}{x} = \dfrac{2e^x - e^{\sqrt{x}}}{2x}$.

50. $y = \int_{2x}^{3x+1} \sin(t^4)dt = \int_{2x}^0 \sin(t^4)\, dt + \int_0^{3x+1} \sin(t^4)\, dt = \int_0^{3x+1} \sin(t^4)\, dt - \int_0^{2x} \sin(t^4)\, dt \quad \Rightarrow$

$y' = \sin\left[(3x+1)^4\right]\cdot \dfrac{d}{dx}(3x+1) - \sin\left[(2x)^4\right]\cdot\dfrac{d}{dx}(2x) = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$

51. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \quad \Rightarrow \quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so $2(3-1) \le \int_1^3 \sqrt{x^2+3}\, dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\, dx \le 4\sqrt{3}$.

52. If $3 \le x \le 5$, then $4 \le x+1 \le 6$ and $\dfrac{1}{6} \le \dfrac{1}{x+1} \le \dfrac{1}{4}$, so $\dfrac{1}{6}(5-3) \le \displaystyle\int_3^5 \frac{1}{x+1}\, dx \le \frac{1}{4}(5-3)$;

that is, $\dfrac{1}{3} \le \displaystyle\int_3^5 \frac{1}{x+1}\, dx \le \frac{1}{2}$.

53. $0 \le x \le 1 \quad \Rightarrow \quad 0 \le \cos x \le 1 \quad \Rightarrow \quad x^2 \cos x \le x^2 \quad \Rightarrow \quad \int_0^1 x^2 \cos x\, dx \le \int_0^1 x^2\, dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

54. On the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\dfrac{\sin x}{x}$ is decreasing. Therefore, the largest value of $\dfrac{\sin x}{x}$ on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ is $\dfrac{\sin(\pi/4)}{\pi/4} = \dfrac{\sqrt{2}/2}{\pi/4} = \dfrac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \dfrac{2\sqrt{2}}{\pi}$ we get $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

55. $\cos x \le 1 \quad\Rightarrow\quad e^x \cos x \le e^x \quad\Rightarrow\quad \int_0^1 e^x \cos x\,dx \le \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$

56. For $0 \le x \le 1$, $0 \le \sin^{-1} x \le \frac{\pi}{2}$, so $\int_0^1 x \sin^{-1} x\,dx \le \int_0^1 x\left(\frac{\pi}{2}\right) dx = \left[\frac{\pi}{4}x^2\right]_0^1 = \frac{\pi}{4}$.

57. $\Delta x = (3 - 0)/6 = \frac{1}{2}$, so the endpoints are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, and 3, and the midpoints are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}$, and $\frac{11}{4}$.

The Midpoint Rule gives

$$\int_0^3 \sin(x^3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{1}{2}\left[\sin\left(\tfrac{1}{4}\right)^3 + \sin\left(\tfrac{3}{4}\right)^3 + \sin\left(\tfrac{5}{4}\right)^3 + \sin\left(\tfrac{7}{4}\right)^3 + \sin\left(\tfrac{9}{4}\right)^3 + \sin\left(\tfrac{11}{4}\right)^3\right] \approx 0.280981.$$

58. (a) Displacement $= \int_0^5 (t^2 - t)\,dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$ meters

(b) Distance traveled $= \int_0^5 |t^2 - t|\,dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t - t^2)\,dt + \int_1^5 (t^2 - t)\,dt$

$$= \left[\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3\right]_0^1 + \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2\right]_1^5 = \tfrac{1}{2} - \tfrac{1}{3} - 0 + \left(\tfrac{125}{3} - \tfrac{25}{2}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right) = \tfrac{177}{6} = 29.5 \text{ meters}$$

59. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem, $\int_0^8 r(t)\,dt = b(8) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2008.

60. Distance covered $= \int_0^{5.0} v(t)\,dt \approx M_5 = \frac{5.0 - 0}{5}[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5)]$

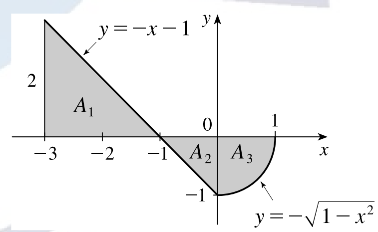
$$= 1(4.67 + 8.86 + 10.22 + 10.67 + 10.81) = 45.23 \text{ m}$$

61. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24 - 0}{6} = 4$. The increase in the bee population was

$$\int_0^{24} r(t)\,dt \approx M_6 = 4[r(2) + r(6) + r(10) + r(14) + r(18) + r(22)]$$
$$\approx 4[50 + 1000 + 7000 + 8550 + 1350 + 150] = 4(18{,}100) = 72{,}400$$

62. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1 - x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So

$$\int_{-3}^{1} f(x)\,dx = A_1 - A_2 - A_3 = 2 - \tfrac{1}{2} - \tfrac{\pi}{4} = \tfrac{1}{4}(6 - \pi)$$



63. Let $u = 2\sin\theta$. Then $du = 2\cos\theta\,d\theta$ and when $\theta = 0$, $u = 0$; when $\theta = \frac{\pi}{2}$, $u = 2$. Thus,

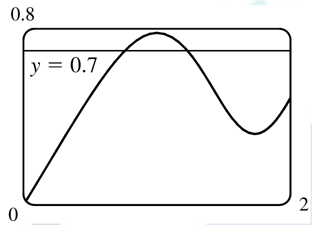
$$\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta = \int_0^2 f(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^2 f(u)\,du = \tfrac{1}{2}\int_0^2 f(x)\,dx = \tfrac{1}{2}(6) = 3.$$

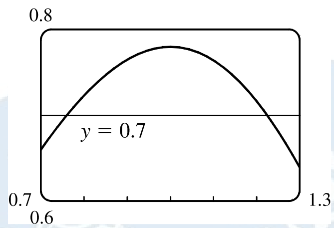
64. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus,

$C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi \quad\Leftrightarrow\quad 0 \le |x| < 1$ or $\sqrt{4n - 1} < |x| < \sqrt{4n + 1}$, n any positive integer. So C is increasing on the intervals $(-1, 1)$, $\left(\sqrt{3}, \sqrt{5}\right)$, $\left(-\sqrt{5}, -\sqrt{3}\right)$, $\left(\sqrt{7}, 3\right)$, $\left(-3, -\sqrt{7}\right)$, ...

(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right) \Rightarrow C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2} \cdot 2x\right) = -\pi x \sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where $(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\Leftrightarrow \sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer. Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $(-\sqrt{2}, 0)$, $(\sqrt{2}, 2)$, $(-\sqrt{6}, -2)$, $(\sqrt{6}, 2\sqrt{2})$, $\ldots$.

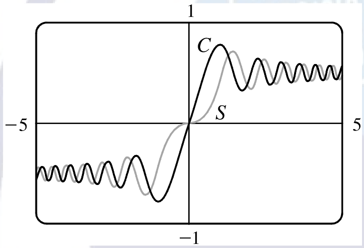
(c)





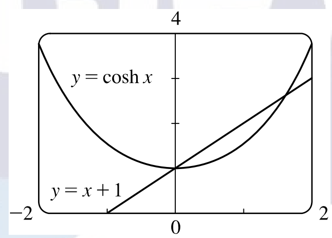
From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.3.3 and Exercise 5.3.65 for a discussion of $S(x)$.

65. Area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1 $\Rightarrow$ $\int_0^1 \sinh cx\, dx = 1 \Rightarrow \frac{1}{c}\left[\cosh cx\right]_0^1 = 1 \Rightarrow \frac{1}{c}(\cosh c - 1) = 1 \Rightarrow$ $\cosh c - 1 = c \Rightarrow \cosh c = c + 1$. From the graph, we get $c = 0$ and $c \approx 1.6161$, but $c = 0$ isn't a solution for this problem since the curve $y = \sinh cx$ becomes $y = 0$ and the area under it is 0. Thus, $c \approx 1.6161$.



66. Both numerator and denominator approach 0 as $a \to 0$, so we use l'Hospital's Rule. (Note that we are differentiating *with respect to* a, since that is the quantity which is changing.) We also use FTC1:

$$\lim_{a\to 0} T(x,t) = \lim_{a\to 0} \frac{C\int_0^a e^{-(x-u)^2/(4kt)}du}{a\sqrt{4\pi kt}} \overset{\text{H}}{=} \lim_{a\to 0} \frac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

67. Using FTC1, we differentiate both sides of the given equation, $\int_1^x f(t)\,dt = (x-1)e^{2x} + \int_1^x e^{-t} f(t)\,dt$, and get

$$f(x) = e^{2x} + 2(x-1)e^{2x} + e^{-x}f(x) \Rightarrow f(x)(1 - e^{-x}) = e^{2x} + 2(x-1)e^{2x} \Rightarrow f(x) = \frac{e^{2x}(2x-1)}{1-e^{-x}}.$$

68. The second derivative is the derivative of the first derivative, so we'll apply the Net Change Theorem with $F = h'$. $\int_1^2 h''(u)\,du = \int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

69. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

70. Let $F(x) = \displaystyle\int_2^x \sqrt{1+t^3}\,dt$. Then $F'(2) = \displaystyle\lim_{h\to 0}\frac{F(2+h)-F(2)}{h} = \lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$, and $F'(x) = \sqrt{1+x^3}$, so

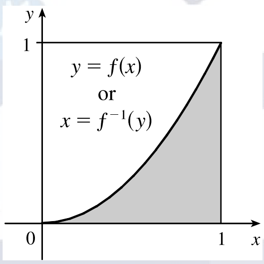
$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3.$$

71. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

72. $$\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

73. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



PROBLEMS PLUS

1. Differentiating both sides of the equation $x\sin\pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin\pi x + \pi x\cos\pi x = 2xf(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi\cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi\cdot 1) = \frac{\pi}{2}$.

2. The area A under the curve $y = x + 1/x$ from $x = a$ to $x = a + 1.5$ is given by $A(a) = \displaystyle\int_a^{a+1.5}\left(x + \frac{1}{x}\right)dx$.

   To find the minimum value of A, we'll differentiate A using FTC1 and set the derivative equal to 0.

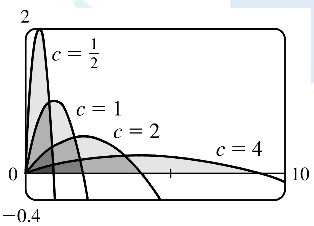
$$\begin{aligned} A'(a) &= \frac{d}{da}\int_a^{a+1.5}\left(x + \frac{1}{x}\right)dx \\ &= \frac{d}{da}\int_a^{1}\left(x + \frac{1}{x}\right)dx + \frac{d}{da}\int_1^{a+1.5}\left(x + \frac{1}{x}\right)dx \\ &= -\frac{d}{da}\int_1^{a}\left(x + \frac{1}{x}\right)dx + \frac{d}{da}\int_1^{a+1.5}\left(x + \frac{1}{x}\right)dx \\ &= -\left(a + \frac{1}{a}\right) + \left(a + 1.5 + \frac{1}{a+1.5}\right) = 1.5 + \frac{1}{a+1.5} - \frac{1}{a} \end{aligned}$$

   $A'(a) = 0 \;\Leftrightarrow\; 1.5 + \dfrac{1}{a+1.5} - \dfrac{1}{a} = 0 \;\Leftrightarrow\; 1.5a(a+1.5) + a - (a+1.5) = 0 \;\Leftrightarrow$ $1.5a^2 + 2.25a - 1.5 = 0$ $\left[\text{multiply by } \frac{4}{3}\right]$ $\Leftrightarrow\; 2a^2 + 3a - 2 = 0 \;\Leftrightarrow\; (2a-1)(a+2) = 0 \;\Leftrightarrow\; a = \frac{1}{2}$ or $a = -2$. Since $a > 0, a = \frac{1}{2}$. $A''(a) = -\dfrac{1}{(a+1.5)^2} + \dfrac{1}{a^2} > 0$, so

   $A\left(\frac{1}{2}\right) = \displaystyle\int_{1/2}^{2}\left(x + \frac{1}{x}\right)dx = \left[\tfrac{1}{2}x^2 + \ln|x|\right]_{1/2}^{2} = (2 + \ln 2) - \left(\tfrac{1}{8} - \ln 2\right) = \tfrac{15}{8} + 2\ln 2$ is the minimum value of A.

3. For $I = \int_0^4 xe^{(x-2)^4}\,dx$, let $u = x - 2$ so that $x = u + 2$ and $dx = du$. Then $I = \int_{-2}^{2}(u+2)e^{u^4}\,du = \int_{-2}^{2} ue^{u^4}\,du + \int_{-2}^{2} 2e^{u^4}\,du = 0$ [by 5.5.7(b)] $+2\int_0^4 e^{(x-2)^4}\,dx = 2k$.

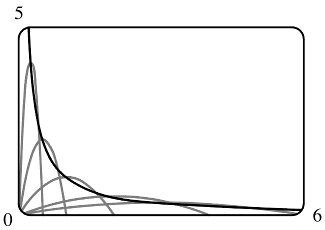
4. (a)



   From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c.

   (b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \;\Leftrightarrow\; 2cx - x^2 = 0 \;\Leftrightarrow\; x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

   $A = \displaystyle\int_0^{2c}\frac{2cx - x^2}{c^3}\,dx = \frac{1}{c^3}\left[cx^2 - \tfrac{1}{3}x^3\right]_0^{2c} = \frac{1}{c^3}\left[c(2c)^2 - \tfrac{1}{3}(2c)^3\right] = \frac{1}{c^3}\left[4c^3 - \tfrac{8}{3}c^3\right] = \tfrac{4}{3}$, a constant.

(c) 

The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

5. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \displaystyle\int_0^{\cos x} [1 + \sin(t^2)]\,dt$. Using FTC1 and the Chain Rule (twice) we have

$f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,[1+\sin(\cos^2 x)](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \displaystyle\int_0^0 [1+\sin(t^2)]\,dt = 0$, so

$f'\left(\frac{\pi}{2}\right) = \dfrac{1}{\sqrt{1+0}}\,(1+\sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt = x^2 \int_0^x \sin(t^2)\,dt$, then $f'(x) = x^2 \sin(x^2) + 2x \int_0^x \sin(t^2)\,dt$, by the Product Rule and FTC1.

7. By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y) = e^y$,

$$\lim_{x\to 0} \frac{\int_0^x (1-\tan 2t)^{1/t}\,dt}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{(1-\tan 2x)^{1/x}}{1} = \exp\left(\lim_{x\to 0} \frac{\ln(1-\tan 2x)}{x}\right)$$

$$\overset{\text{H}}{=} \exp\left(\lim_{x\to 0} \frac{-2\sec^2 2x}{1-\tan 2x}\right) = \exp\left(\frac{-2\cdot 1^2}{1-0}\right) = e^{-2}$$

8. The area $A(t) = \int_0^t \sin(x^2)\,dx$, and the area $B(t) = \frac{1}{2} t \sin(t^2)$. Since $\lim\limits_{t\to 0^+} A(t) = 0 = \lim\limits_{t\to 0^+} B(t)$, we can use l'Hospital's Rule:

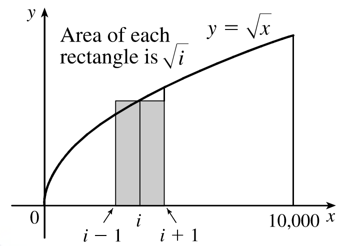
$$\lim_{t\to 0^+} \frac{A(t)}{B(t)} \overset{\text{H}}{=} \lim_{t\to 0^+} \frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + \frac{1}{2}t[2t\cos(t^2)]} \qquad \text{[by FTC1 and the Product Rule]}$$

$$\overset{\text{H}}{=} \lim_{t\to 0^+} \frac{2t\cos(t^2)}{t\cos(t^2) - 2t^3\sin(t^2) + 2t\cos(t^2)} = \lim_{t\to 0^+} \frac{2\cos(t^2)}{3\cos(t^2) - 2t^2\sin(t^2)} = \frac{2}{3-0} = \frac{2}{3}$$

9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \iff x = 2$ or $x = -1$. $f(x) \ge 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\,dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \ge 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n\to\infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^{10{,}000} = \tfrac{2}{3}(1{,}000{,}000) \approx 666{,}667.$$



Alternate method: We can use graphical methods as follows:

From the figure we see that $\int_{i-1}^{i} \sqrt{x}\, dx < \sqrt{i} < \int_{i}^{i+1} \sqrt{x}\, dx$, so

$\int_0^{10,000} \sqrt{x}\, dx < \sum_{i=1}^{10,000} \sqrt{i} < \int_1^{10,001} \sqrt{x}\, dx$. Since

$\int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C$, we get $\int_0^{10,000} \sqrt{x}\, dx = 666{,}666.\overline{6}$ and

$\int_1^{10,001} \sqrt{x}\, dx = \frac{2}{3}[(10{,}001)^{3/2} - 1] \approx 666{,}766$.

Hence, $666{,}666.\overline{6} < \sum_{i=1}^{10,000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$\sum_{i=1}^{10,000} \approx \dfrac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716$. The actual value is about 666,716.46.

11. (a) We can split the integral $\int_0^n [\![x]\!]\, dx$ into the sum $\sum_{i=1}^{n} \left[\int_{i-1}^{i} [\![x]\!]\, dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i - (i-1)] = (i-1)$. So the original integral is equal to $\sum_{i=1}^{n} (i-1) = \sum_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\, dx = \int_0^b [\![x]\!]\, dx - \int_0^a [\![x]\!]\, dx$.

Now $\int_0^b [\![x]\!]\, dx = \int_0^{[\![b]\!]} [\![x]\!]\, dx + \int_{[\![b]\!]}^b [\![x]\!]\, dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!] - 1)\,[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_0^b [\![x]\!]\, dx = \frac{1}{2}([\![b]\!] - 1)\,[\![b]\!] + [\![b]\!]\,(b - [\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly $\int_0^a [\![x]\!]\, dx = \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

Therefore, $\int_a^b [\![x]\!]\, dx = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

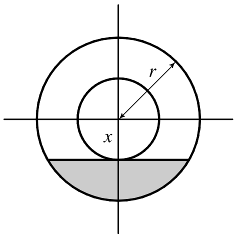
12. By FTC1, $\dfrac{d}{dx}\displaystyle\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\, du\right) dt = \int_1^{\sin x} \sqrt{1+u^4}\, du$. Again using FTC1,

$$\frac{d^2}{dx^2}\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\, du\right) dt = \frac{d}{dx}\int_1^{\sin x} \sqrt{1+u^4}\, du = \sqrt{1+\sin^4 x}\cos x.$$

13. Let $Q(x) = \displaystyle\int_0^x P(t)\, dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3 + \frac{d}{4}t^4\right]_0^x = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4$. Then $Q(0) = 0$, and $Q(1) = 0$ by the given condition, $a + \dfrac{b}{2} + \dfrac{c}{3} + \dfrac{d}{4} = 0$. Also, $Q'(x) = P(x) = a + bx + cx^2 + dx^3$ by FTC1. By Rolle's Theorem, applied to Q on $[0,1]$, there is a number r in $(0,1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$. Thus, the equation $P(x) = 0$ has a root between 0 and 1.

More generally, if $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and if $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$, then the equation $P(x) = 0$ has a root between 0 and 1. The proof is the same as before:

Let $Q(x) = \displaystyle\int_0^x P(t)\, dt = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^n$. Then $Q(0) = Q(1) = 0$ and $Q'(x) = P(x)$. By Rolle's Theorem applied to Q on $[0,1]$, there is a number r in $(0,1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$.

14. 

Let x be the distance between the center of the disk and the surface of the liquid. The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2\int_x^r \sqrt{r^2 - t^2}\,dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2\int_x^r \sqrt{r^2 - t^2}\,dt$, $0 \le x \le r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$.

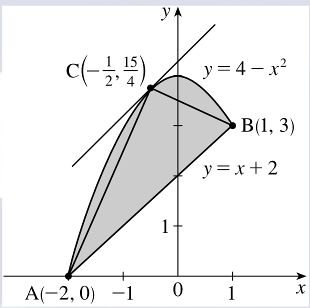
Now $A'(x) > 0 \Rightarrow -2\pi x + 2\sqrt{r^2 - x^2} > 0 \Rightarrow \sqrt{r^2 - x^2} > \pi x \Rightarrow r^2 - x^2 > \pi^2 x^2 \Rightarrow$ $r^2 > \pi^2 x^2 + x^2 \Rightarrow r^2 > x^2(\pi^2 + 1) \Rightarrow x^2 < \dfrac{r^2}{\pi^2 + 1} \Rightarrow x < \dfrac{r}{\sqrt{\pi^2 + 1}}$, and we'll call this value x^*. Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

15. Note that $\dfrac{d}{dx}\left(\displaystyle\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \displaystyle\int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$
$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

16. The parabola $y = 4 - x^2$ and the line $y = x + 2$ intersect when $4 - x^2 = x + 2 \Leftrightarrow x^2 + x - 2 = 0 \Leftrightarrow (x+2)(x-1) = 0 \Leftrightarrow$ $x = -2$ or 1. So the point A is $(-2, 0)$ and B is $(1, 3)$. The slope of the line $y = x + 2$ is 1 and the slope of the parabola $y = 4 - x^2$ at x-coordinate x is $-2x$. These slopes are equal when $x = -\frac{1}{2}$, so the point C is $\left(-\frac{1}{2}, \frac{15}{4}\right)$.



The area A_1 of the parabolic segment is the area under the parabola from $x = -2$ to $x = 1$, minus the area under the line $y = x + 2$ from -2 to 1. Thus,

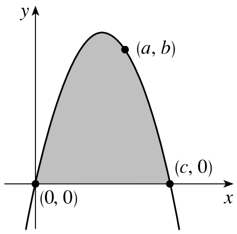
$$A_1 = \int_{-2}^{1}(4 - x^2)\,dx - \int_{-2}^{1}(x+2)\,dx = \left[4x - \tfrac{1}{3}x^3\right]_{-2}^{1} - \left[\tfrac{1}{2}x^2 + 2x\right]_{-2}^{1}$$
$$= \left[\left(4 - \tfrac{1}{3}\right) - \left(-8 + \tfrac{8}{3}\right)\right] - \left[\left(\tfrac{1}{2} + 2\right) - (2 - 4)\right] = 9 - \tfrac{9}{2} = \tfrac{9}{2}.$$

The area A_2 of the inscribed triangle is the area under the line segment AC plus the area under the line segment CB minus the area under the line segment AB. The line through A and C has slope $\dfrac{15/4 - 0}{-1/2 + 2} = \dfrac{5}{2}$ and equation $y - 0 = \frac{5}{2}(x + 2)$, or $y = \frac{5}{2}x + 5$. The line through C and B has slope $\dfrac{3 - 15/4}{1 + 1/2} = -\dfrac{1}{2}$ and equation $y - 3 = -\frac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{7}{2}$.

Thus,

$$A_2 = \int_{-2}^{-1/2}\left(\tfrac{5}{2}x + 5\right)dx + \int_{-1/2}^{1}\left(-\tfrac{1}{2}x + \tfrac{7}{2}\right)dx - \int_{-2}^{1}(x+2)\,dx = \left[\tfrac{5}{4}x^2 + 5x\right]_{-2}^{-1/2} + \left[-\tfrac{1}{4}x^2 + \tfrac{7}{2}x\right]_{-1/2}^{1} - \tfrac{9}{2}$$
$$= \left[\left(\tfrac{5}{16} - \tfrac{5}{2}\right) - (5 - 10)\right] + \left[\left(-\tfrac{1}{4} + \tfrac{7}{2}\right) - \left(-\tfrac{1}{16} - \tfrac{7}{4}\right)\right] - \tfrac{9}{2} = \tfrac{45}{16} + \tfrac{81}{16} - \tfrac{72}{16} = \tfrac{54}{16} = \tfrac{27}{8}$$

Archimedes' result states that $A_1 = \frac{4}{3}A_2$, which is verified in this case since $\frac{4}{3} \cdot \frac{27}{8} = \frac{9}{2}$.

17. 

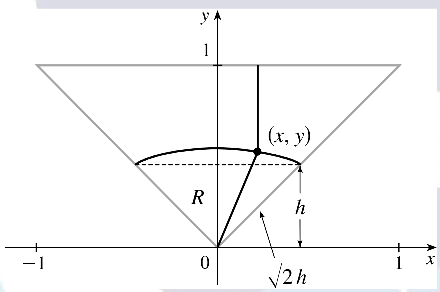
Let c be the nonzero x-intercept so that the parabola has equation $f(x) = kx(x - c)$, or $y = kx^2 - ckx$, where $k < 0$. The area A under the parabola is

$$A = \int_0^c kx(x-c)\,dx = k\int_0^c (x^2 - cx)\,dx = k\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}cx^2\right]_0^c$$
$$= k\left(\tfrac{1}{3}c^3 - \tfrac{1}{2}c^3\right) = -\tfrac{1}{6}kc^3$$

The point (a, b) is on the parabola, so $f(a) = b \Rightarrow b = ka(a - c) \Rightarrow$ $k = \dfrac{b}{a(a-c)}$. Substituting for k in A gives $A(c) = -\dfrac{b}{6a} \cdot \dfrac{c^3}{a-c} \Rightarrow$

$$A'(c) = -\frac{b}{6a} \cdot \frac{(a-c)(3c^2) - c^3(-1)}{(a-c)^2} = -\frac{b}{6a} \cdot \frac{c^2[3(a-c)+c]}{(a-c)^2} = -\frac{bc^2(3a-2c)}{6a(a-c)^2}$$

Now $A' = 0 \Rightarrow c = \frac{3}{2}a$. Since $A'(c) < 0$ for $a < c < \frac{3}{2}a$ and $A'(c) > 0$ for $c > \frac{3}{2}a$, so A has an absolute minimum when $c = \frac{3}{2}a$. Substituting for c in k gives us $k = \dfrac{b}{a\left(a - \frac{3}{2}a\right)} = -\dfrac{2b}{a^2}$, so $f(x) = -\dfrac{2b}{a^2}\,x\left(x - \frac{3}{2}a\right)$, or $f(x) = -\dfrac{2b}{a^2}\,x^2 + \dfrac{3b}{a}\,x$. Note that the vertex of the parabola is $\left(\frac{3}{4}a, \frac{9}{8}b\right)$ and the minimal area under the parabola is $A\left(\frac{3}{2}a\right) = \frac{9}{8}ab$.

18. 

We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$. So the distances are equal if $\sqrt{x^2+y^2} = 1 - y \Leftrightarrow x^2 + y^2 = 1 - 2y + y^2 \Leftrightarrow y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we have to find the y-coordinate h of the horizontal line separating them. From the diagram, $1 - h = \sqrt{2}\,h \Leftrightarrow h = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h) = h^2$, and the area of the crescent-shaped section is

$$\int_{-h}^{h}\left[\tfrac{1}{2}(1-x^2) - h\right]dx = 2\int_0^h\left(\tfrac{1}{2} - h - \tfrac{1}{2}x^2\right)dx = 2\left[\left(\tfrac{1}{2} - h\right)x - \tfrac{1}{6}x^3\right]_0^h = h - 2h^2 - \tfrac{1}{3}h^3.$$

So the area of the whole region is

$$4\left[\left(h - 2h^2 - \tfrac{1}{3}h^3\right) + h^2\right] = 4h\left(1 - h - \tfrac{1}{3}h^2\right) = 4\left(\sqrt{2}-1\right)\left[1 - \left(\sqrt{2}-1\right) - \tfrac{1}{3}\left(\sqrt{2}-1\right)^2\right]$$
$$= 4\left(\sqrt{2}-1\right)\left(1 - \tfrac{1}{3}\sqrt{2}\right) = \tfrac{4}{3}\left(4\sqrt{2} - 5\right)$$

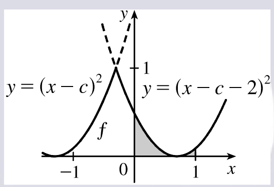
19. $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}}+\frac{1}{\sqrt{n}\,\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\,\sqrt{n+n}}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\qquad\left[\text{where } f(x)=\frac{1}{\sqrt{1+x}}\right]$$
$$=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=\left[2\sqrt{1+x}\right]_0^1=2\left(\sqrt{2}-1\right)$$

20. Note that the graphs of $(x-c)^2$ and $[(x-c)-2]^2$ intersect when $|x-c|=|x-c-2|$ $\Leftrightarrow$ $c-x=x-c-2$ $\Leftrightarrow$ $x=c+1$. The integration will proceed differently depending on the value of c.

Case 1: $-2\le c<-1$

In this case, $f_c(x)=(x-c-2)^2$ for $x\in[0,1]$, so

$$g(c)=\int_0^1(x-c-2)^2\,dx=\tfrac{1}{3}\left[(x-c-2)^3\right]_0^1=\tfrac{1}{3}\left[(-c-1)^3-(-c-2)^3\right]$$
$$=\tfrac{1}{3}(3c^2+9c+7)=c^2+3c+\tfrac{7}{3}=\left(c+\tfrac{3}{2}\right)^2+\tfrac{1}{12}$$



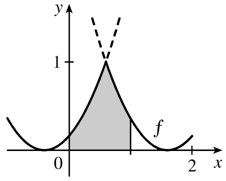
This is a parabola; its maximum value for $-2\le c<-1$ is $g(-2)=\frac{1}{3}$, and its minimum value is $g\left(-\frac{3}{2}\right)=\frac{1}{12}$.

Case 2: $-1\le c<0$

In this case, $f_c(x)=\begin{cases}(x-c)^2 & \text{if } 0\le x\le c+1\\ (x-c-2)^2 & \text{if } c+1<x\le 1\end{cases}$

Therefore,

$$g(c)=\int_0^1 f_c(x)\,dx=\int_0^{c+1}(x-c)^2\,dx+\int_{c+1}^1(x-c-2)^2\,dx$$
$$=\tfrac{1}{3}\left[(x-c)^3\right]_0^{c+1}+\tfrac{1}{3}\left[(x-c-2)^3\right]_{c+1}^1=\tfrac{1}{3}\left[1+c^3+(-c-1)^3-(-1)\right]$$
$$=-c^2-c+\tfrac{1}{3}=-\left(c+\tfrac{1}{2}\right)^2+\tfrac{7}{12}$$

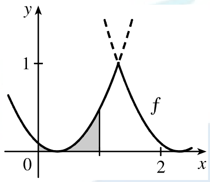


Again, this is a parabola, whose maximum value for $-1\le c<0$ is $g\left(-\frac{1}{2}\right)=\frac{7}{12}$, and whose minimum value on this c-interval is $g(-1)=\frac{1}{3}$.

Case 3: $0 \le c \le 2$

In this case, $f_c(x) = (x-c)^2$ for $x \in [0,1]$, so

$$g(c) = \int_0^1 (x-c)^2\,dx = \tfrac{1}{3}\left[(x-c)^3\right]_0^1 = \tfrac{1}{3}\left[(1-c)^3 - (-c)^3\right]$$
$$= c^2 - c + \tfrac{1}{3} = \left(c - \tfrac{1}{2}\right)^2 + \tfrac{1}{12}$$



This parabola has a maximum value of $g(2) = \frac{7}{3}$ and a minimum value of $g\left(\frac{1}{2}\right) = \frac{1}{12}$.

We conclude that $g(c)$ has an absolute maximum value of $g(2) = \frac{7}{3}$, and absolute minimum values of $g\left(-\frac{3}{2}\right) = g\left(\frac{1}{2}\right) = \frac{1}{12}$.