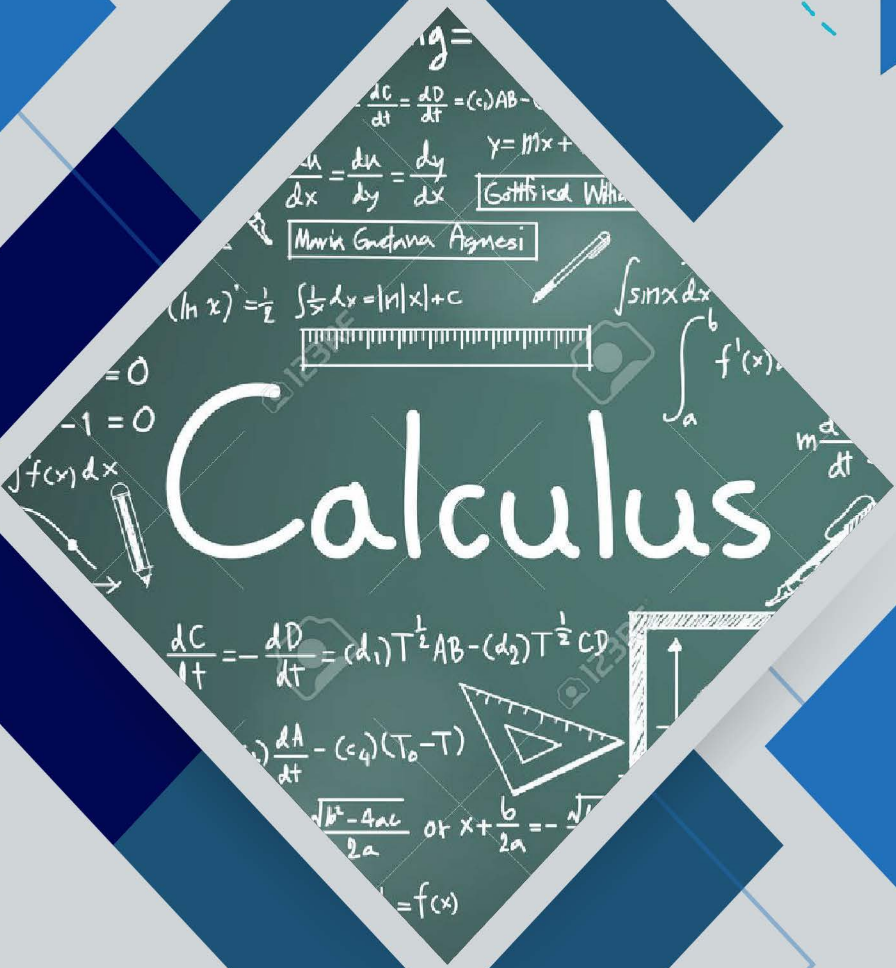


اسألني عن الكالكولاس

اسألني عن الهندسة

Introduction

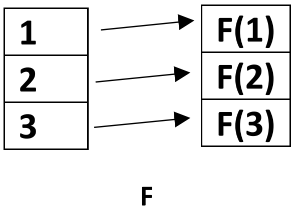
*يمكن تمثيل الاقترانات ب 4 طرق, هي:-

1- عن طريق جدول من القيم (numerically)

2- كلاميا عن طريق وصف الاقترانات (verbally)

3- جبريا عن طريق صيغة رياضية (algebraically)

4- بصريا عن طريق الرسم بيانيا (visually)



*الية عمل الاقترانات:

1

مدخلات ← اقتران ← مخرجات

س ← ق ← ق(س)

x → F → F(x)

للاقتران $f(x) = x^2$ يمكن تمثيله بعدة طرق :-

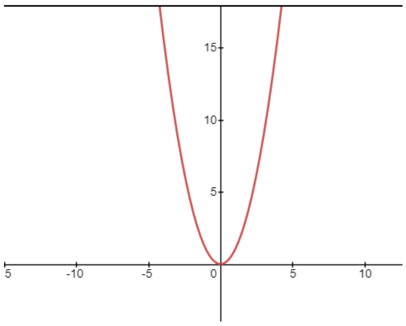
1

x	0	1	2	-3	-5
F(x)	0	1	4	9	25

2

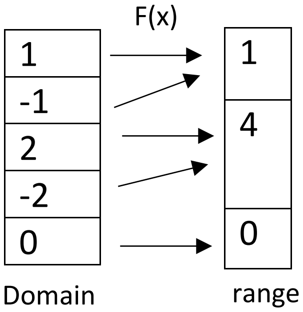
$$f(x) = x^2$$

3



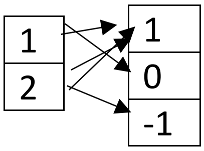
4 X square

بالعودة الى المثال السابق $F(x)=x^2$ واخذ قيم عشوائية للمتغير (إكس) كمدخلات نتجت قيم ل F(x) وهي المخرجات, ففي هذه الحالة يطلق على قيم (إكس) اسم المجال (Domain) وقيم F(x) اسم المدى (range)



نلاحظ أن كل عنصر في المجال ارتبط فقط بصورة واحدة في المدى وهذا هو الشرط الذي يميز الإقترانات عن العلاقات الرياضية أي العكس غير صحيح

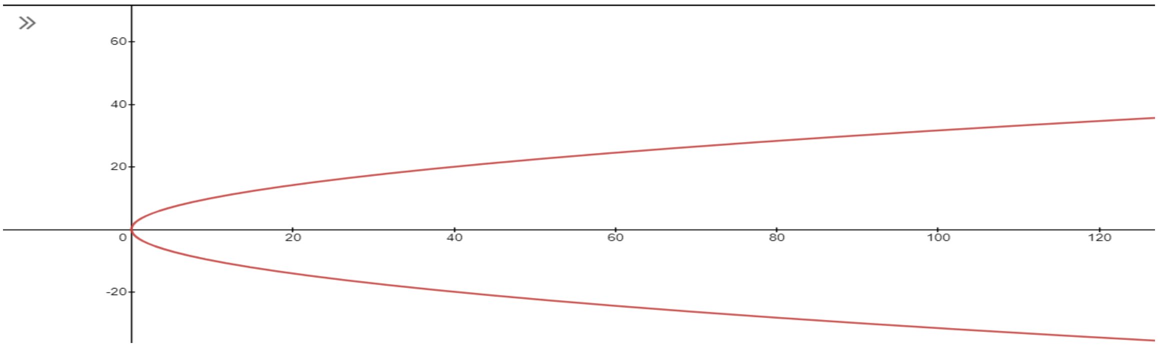
- الإقترانات ← كل عنصر في المجال يرتبط بعنصر واحد فقط في المدى
- العلاقات الرياضية ← قد يرتبط كل عنصر في المجال بعنصرين أو اكثر في المدى



لاحظ ان لقيم المجال اكثر من صورة في المدى

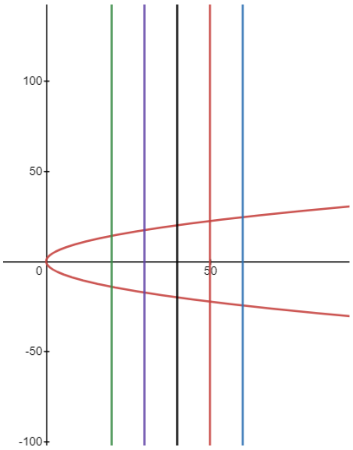
اذا لتحقيق شرط الاقتران فانه يجب ان يكون لكل عنصر في المجال صورة واحدة فقط في المدى ولضمان ذلك فإننا نستخدم اختبار الخط العمودي (Vertical line test)

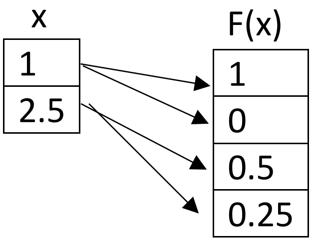
فمثلا



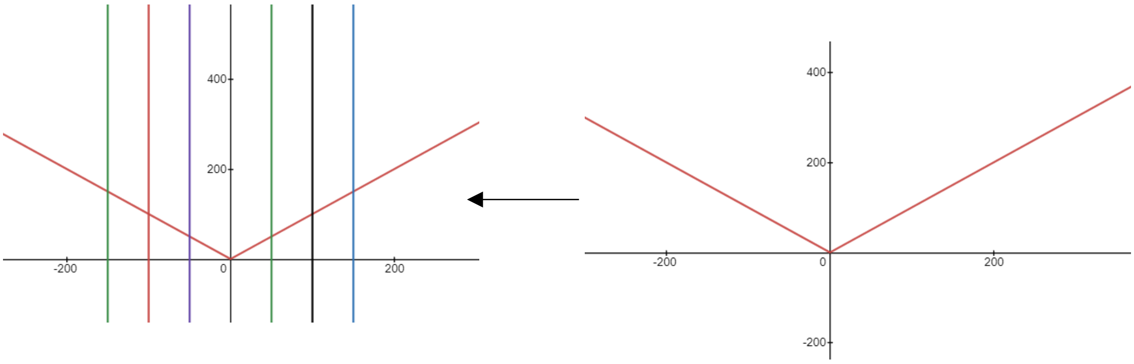
نلاحظ في الشكل التالي ان الخط العمودي يقطع F(x) في اكثر من

نقطة عندما x>0 مما يعني وجود اكثر من صورة لكل عنصر في المجال في تلك الفترة, الامر الذي يتعارض مع تعريف الاقتران



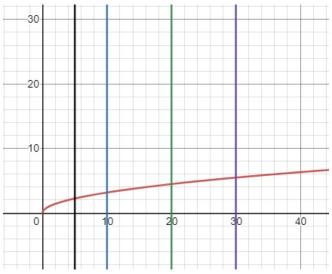


اي يجب ان يقطع الخط العمودي الاقتران في نقطة واحدة فقط لكل عنصر في المجال



Using vertical line test to select the graphs that are functions:

1



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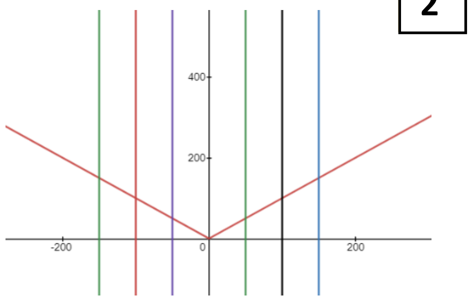

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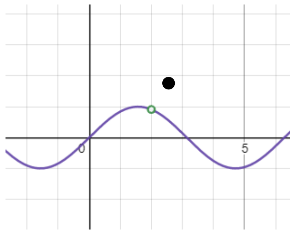

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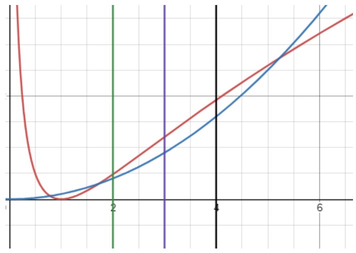

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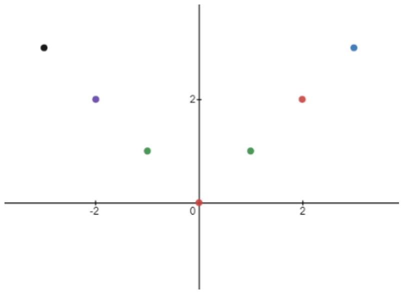

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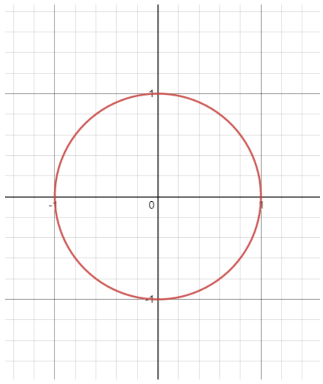

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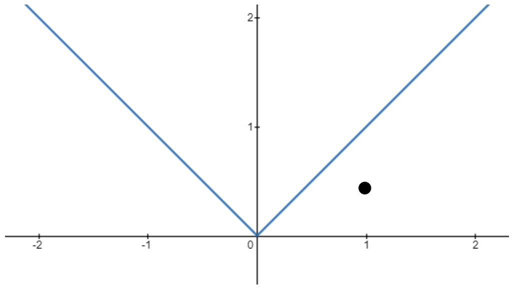


1,2,3 اقترانات لان الخط العمودي يقطع المنحنى في
نقطة واحدة لكل قيم إكس في المنحنى

4,6,7 ليست اقترانات لان الخط العمودي يقطع المنحنى
في اكثر من نقطة

ليس اقتران على الرغم من ان الخط العمودي يقطع
5 المنحنى في نقطة واحدة فقط وسبب ذلك ان الشكل
يمثل تمثيلا لعدة ازواج من النقاط X,f(x) وليس اقترانا

المجال والمدى

(domain & range)

المجال **(domain)**: قيم x المسموح تعويضها داخل الاقتران F(x)

المدى **(range)** : نتيجة تعويض قيم المجال داخل الاقتران F(x)

-how to find domain and range?

- → **Skew (الرسم)**
- → **functions (اقترانات)**

ملاحظة

دائما اذا اعطاك رسمة جاهزة وطلب منك المجال والمدى,

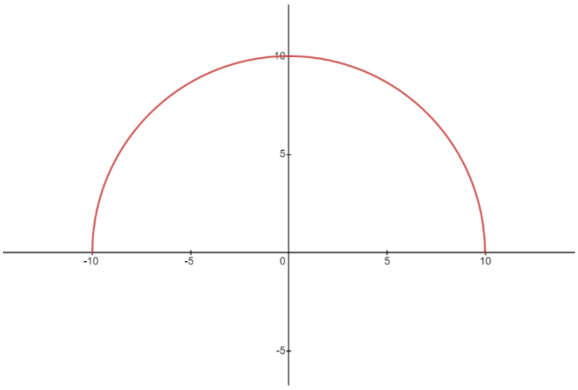
فيجب عليك الاعتماد على الاتي:

المجال **(domain)** : امتداد ال x-axis يعني الرسمة من وين لوين على محور السينات

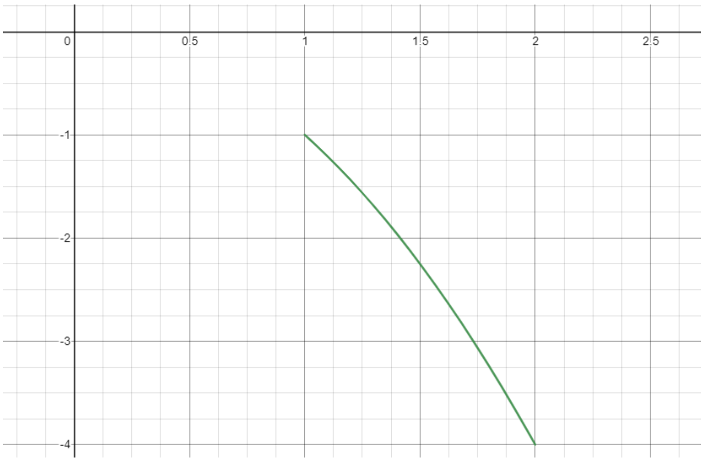
Domain ∈ [-10,10]

المدى **(range)** : امتداد ال y-axis يعني الرسمة من وين لوين على محور الصادات **range ∈ [0,10]**

1 Skew

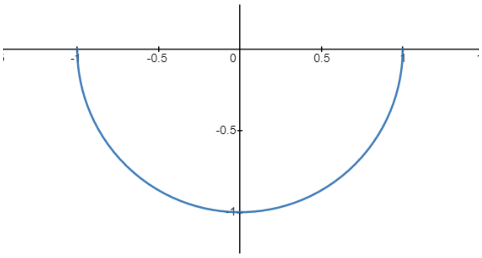


Ex: find the domain and range of the following:



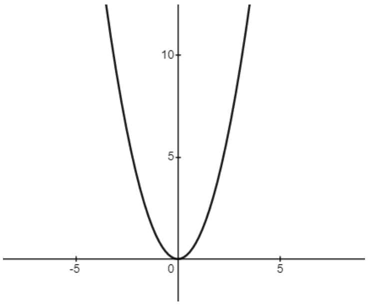
Domain ∈ [1,2]

range ∈ [-4,-1]



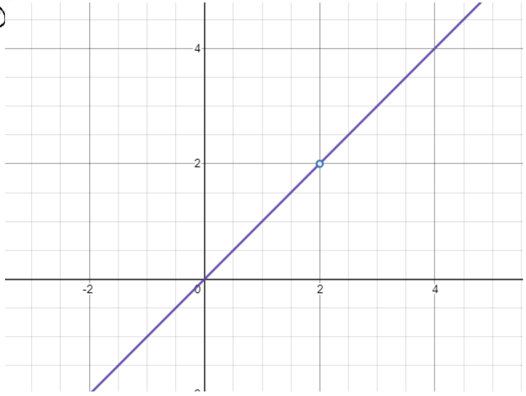
Domain ∈ [-1,1]

range ∈ [-1,0]



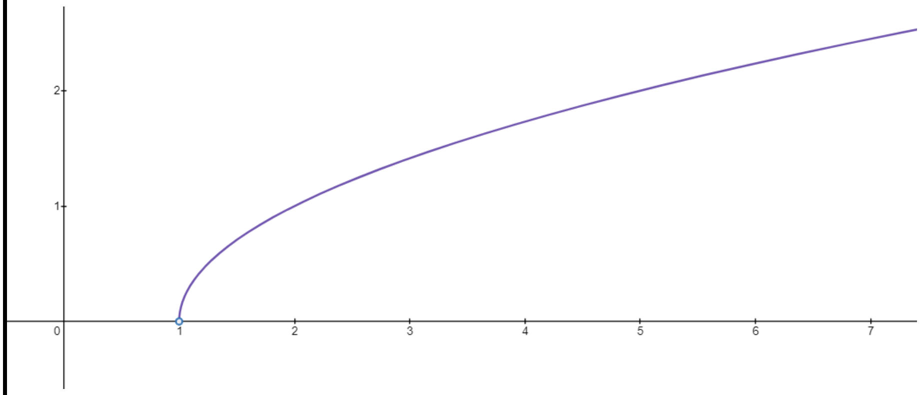
Domain ∈ $(-\infty, \infty)$ لانه مكمل وعم بزيد

range ∈ $[\mathbf{0}, \infty)$ لانه طالع للمالانهاية



Domain $R-\{2\}$

range $\in(-\infty,\infty)$



domain $\in(1,\infty)$

range $\in(0,\infty)$

Circle (الدائرة)

$$(x-a)^2+(y-b)^2=r^2$$

-center (المركز): (a,b)

-radius (نصف القطر): r

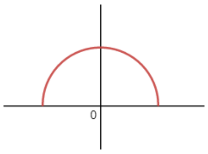
-special case if the center is (0,0)

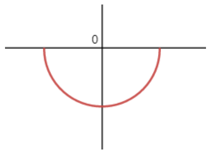
$$x^2+y^2=r^2$$

$$y^2=r^2-x^2$$

$$y=\pm\sqrt{r^2-x^2}$$

$$y=\sqrt{r^2-x^2} \qquad y=-\sqrt{r^2-x^2}$$





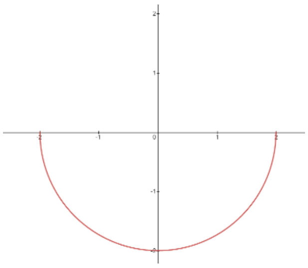
نصف دائرة علوي بسبب الموجب

نصف دائرة سفلي بسبب السالب

Ex: plot the following functions and find the domain and range:

1) $y = -\sqrt{4 - x^2}$

في سالب يعني نصف دائرة سفلي ونصف قطرها (2) ولا تنسى انه (2 وليس 4) والمركز (0,0)

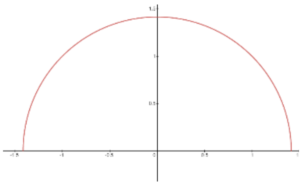


Domain ∈ [-2,2]

Range ∈ [-2,0]

2) $\sqrt{2 - x^2}$

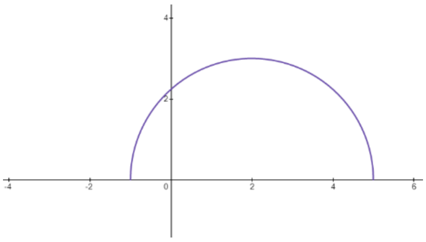
في موجب يعني نصف دائرة علوي ونصف قطرها هنا ليس 2 بل جذر 2 والمركز (0,0)



Domain ∈ $\left[-\sqrt{2},\sqrt{2}\right]$

Range ∈ $\left[0,\sqrt{2}\right]$

3) $\sqrt{9 - (x - 2)^2}$



Domain ∈ [-1,5]

Range ∈ $\left[0,\sqrt{3}\right]$

في موجب يعني نصف دائرة علوي ونصف قطرها 3 ومركزها (2,0)

طبعا هذا كل شيء يلزمك اذا جاءك رسمة وطلب منك Domain & range

وطبعا هنالك طرق اخرى ستأخذها لاحقا

Domain of functions

Function diagram

x	→	F(x)
a	→	F(a)
b	→	F(b)
c	→	F(c)

Input → domain

Output → range

Domain: قيم المدخلات التي يعطي تعويضها اعداد حقيقية

يستثنى من الاعداد الحقيقية (ما داخل الجذر الزوجي سالب, صفار المقام, موجب وسالب المالانهاية)

1-polynomial functions (كثيرات الحدود)

domain → R (real numbers)

2-roots → odd roots → R (وايضا في حال ما في مقام او قوة سالبة)

→ Even roots → when the value inside the root is bigger than or equal to zero

3-rational functions → $\frac{f(x)}{g(x)}$ **→ the domain of f(x) ∩ the domain of g(x) where g(x)≠0**

4- logarithmic functions $\log_a \overset{\text{Positive number}}{f(x)}$

Domain is when f(x) is bigger than zero and f(x) is a real number

Note: - when a=e the form becomes ln(f(x))

5- exponential function $a^{f(x)} \longrightarrow$ a>0

If a =0, f(x) must be bigger than zero

The domain is the domain of f(x)

6-f(x) $\overset{\text{قسمة}}{\underset{\text{x}}{\pm}}$ g(x)

The domain is the domain of f(x) ∩ the domain of g(x)

في حال وجود جمع او طرح او ضرب اقترانات او وجود اقتران داخل اقتران يكون المجال الكلي هو فترة تقاطع مجالات الإقترانات

Ex: - find the domain of each function

1- $f(x) = x^2 + x + 1$ the domain is R

2- $f(x) = \sqrt[3]{x^2 + 1}$ the domain is R

3- $\sqrt{x^2 - 1}$ the domain is $x^2 \geq 1 \longrightarrow \quad x \geq 1, x \leq -1$

4- $\frac{1}{\sqrt[3]{(x+1)^2}}$ the domain is R - {-1}

5- $\sqrt{2}$ the domain is R

6- $\sqrt{x^2 + 6x + 8}$ the domain is $x \geq -2, x \leq -4$

7- $\left(\sqrt[4]{x}\right)^4$ the domain is $x \geq 0$

8- $\sqrt[4]{x^4}$ the domain is R

9- $\sqrt{\left(\frac{x+9}{x-7}\right)}$, $\frac{x+9}{x-7} \geq 0$ The domain is $x \geq -9, x < 7$

$10 -$ $\frac{\sqrt{(x+9)}}{\sqrt{x-7}}$ the domain is $x > 7$

$11 - \sqrt{\sqrt[4]{x^2 - 1} - \sqrt{15}}$ The domain is $x \geq 4 \cup x \leq -4$

$12 - \sqrt[6]{\sqrt[4]{x^2 - 2x + 1} - 1}$ The domain is $[2, \infty) \cup (-\infty, -2]$

13- $\frac{x-2}{1-\frac{2}{x}}$ the domain is R-{0,2}

14- $\frac{\sqrt{x-2}}{x-4}$ the domain is $[2,\infty)-\{4\}$

15- $\frac{x^2-4x+4}{x+5+\frac{6}{x}}$ the domain is R-{0,-2,-3}

16- $\log\sqrt{x^2-1}$ the domain is $(-\infty,-1)\cup(1,\infty)$

17- $\log_3\sqrt{\frac{x}{x^3-1}}$ The domain is $(-\infty,0),(1,\infty)$

18- $4\log_5\sqrt[4]{x}$ the domain is $(0,\infty)$

19- $\log_{22}\sqrt{x}+\sqrt{1-x}$ The domain is [0,1] - {$\frac{1}{2}$}

20- $\log_9 x+2-\log_9 x-2$ The domain is $(2,\infty)$

21- $\log_2\frac{x+2}{x-2}$ Domain is $(-\infty,-2)\cup(2,\infty)$

***ملاحظة في المجال لا تستخدم خصائص الاقتران او لا تبسطه**

22- $\log\frac{x+2}{x-2}\neq\log x+2-\log x-2$

23- $\frac{\sqrt{x}}{x-3}$ the domain is $[0,\infty)-\{3\}$

23- $e^{\sqrt{x}}$ the domain is $[0,\infty)$

24- $\log x-1$ the domain is $(1,\infty)$

25- $ln\ x$ The domain is $x\geq 0,-x\leq 0$

26- $\ln(\sqrt{x^2}-1)$ The domain is R-[-1,1]

ملاحظة لإكمال المربع

$$x^2+6x$$

$$(ax+b)^2$$

$$a^2x^2+2abx+b^2$$

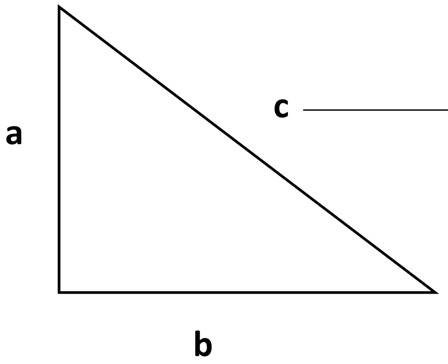
نقوم بقسمة معامل اكس على 2 ثم نربع هذا الرقم ونزيده ونطرحه من المعادلة

$$(x^2+6x+9)-9$$

فتصبح

$$(x+3)^2-9$$

Trigonometric Functions



$\boldsymbol{c^2 = a^2 + b^2}$

$\boldsymbol{Sinx=\frac{a}{c}}$

$\boldsymbol{Cosx=\frac{b}{c}}$

$\boldsymbol{Tanx=\frac{a}{b}}$

$\boldsymbol{Secx=\frac{c}{b}}$

$\boldsymbol{Cscx=\frac{c}{a}}$

$\boldsymbol{Cotx=\frac{b}{a}}$

Some important identities:

1-$sin^2 ax + cos^2 ax = 1$

2- $1 + cot^2(ax) = \csc^2 ax$

3- $1 + \tan^2 ax = \sec^2 ax$

4-sin2x=2sinxcosx

5- cos2x=$\cos^2 x - \sin^2 x$

= $2\cos^2 x - 1$

= $1 - 2\sin^2 x$

6-$\sin x \pm y = sinxcosy \pm cosxsiny$

7-$cosx \pm y = cosxcosy \mp sinxsiny$

- csc x = $\frac{1}{\sin x}$

D= R - {أصفار المقام}

D= R - {0, π , 2π , 3π }

D ∈ - {nπ }

R (غير مطالب فيها)

- sec x = $\frac{1}{\cos x}$

D= R - {أصفار المقام}

D= R - { $\frac{\pi}{2}$, $\frac{3\pi}{2}$, …}

D ∈ R - { (2n+1) $\frac{\pi}{2}$ }

R (غير مطالب فيها)

- cot = $\frac{\cos x}{\sin x}$

D ∈ R - { nπ }

R (غير مطالب فيها)

- sin x, cos x $\longrightarrow$ period = 2π

  sin(x) = sin(x+2π) = sin(x+4π) …

  cos(x) = cos(x+2π) = cos(x+4π) …

- tan x $\longrightarrow$ period = π

  tan(x) = tan(x - π)= tan(x+π) = tan(x+2π) = tan(x+3π)

Ex : If sin θ = $\frac{1}{3}$, $\frac{\pi}{2} \le \theta \le \pi$, find sin2θ

Sol:

$\sin 2\theta = \underset{\checkmark\checkmark}{\underline{2\sin\theta}}\ \underset{??}{\underline{\cos\theta}}$

$$sin^2\theta + cos^2\theta = 1$$

$$\frac{1}{9} + cos^2\theta = 1$$

$$cos^2\theta = 1 - \frac{1}{9}$$

$$cos^2\theta = \frac{9}{9} - \frac{1}{9}$$

$$cos^2\theta = \frac{8}{9}$$

$$cos\ \theta = \frac{+\sqrt{8}}{3}\ ,\ \frac{-\sqrt{8}}{3}$$

$\longrightarrow$ Who is the correct ??

$\theta \in [\frac{\pi}{2}, \pi]$ $\xrightarrow{\text{ربع ثاني}}$ $\xrightarrow{\quad}$ $cos\ \theta \le 0\ (-)$

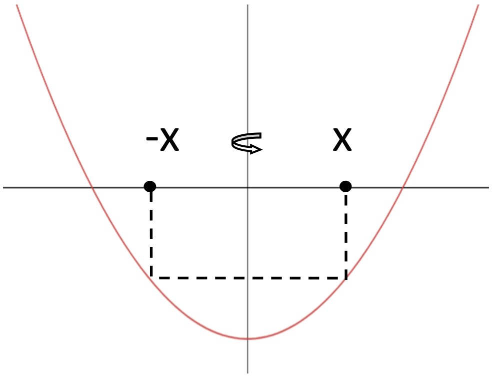
$$cos\ \theta = \frac{-\sqrt{8}}{3}$$

$$sin\ 2\theta = 2sin\theta\ cos\theta$$

$$= 2\ .\ \frac{1}{3}\ .\ \frac{-\sqrt{8}}{3} = \frac{-2\sqrt{8}}{9}\ or\ \frac{-2\sqrt{2*4}}{9} = \frac{-4\sqrt{2}}{9}$$

Even and odd function

*)Even function ⟶ graph is symmetric about y-axis



- **If we reflect the graph about y-axis, we get the same graph**

- شروطه:

1)متماثل حول y-axis

2) F(x)=F(-x)

3) معرف على الفترة

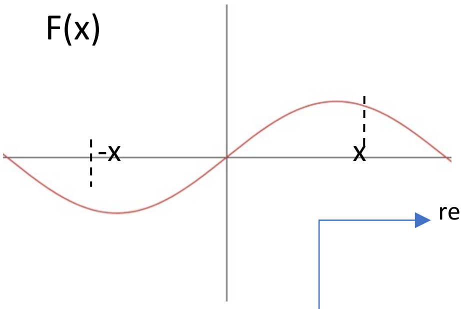
[-a, a] or (-a, a) [A

[-b,-a] ∪ [a , b] [B or

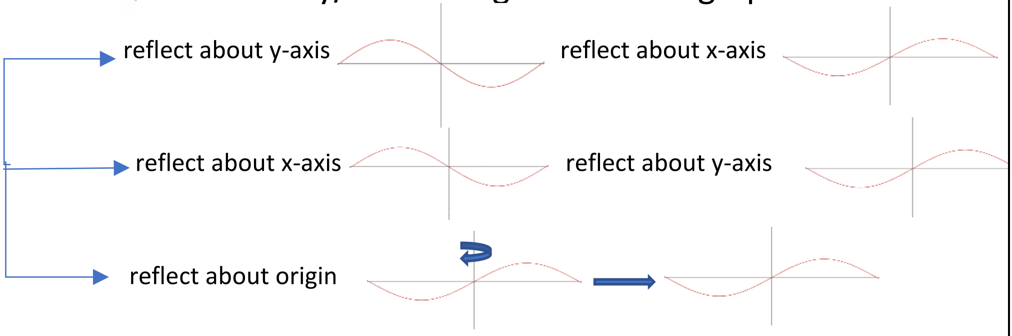
(-b,-a) ∪ (a , b) or

وعلى نفس النمط

*)odd function ⟶ the graph is symmetric about the origin



- If we reflect the graph about x/y axis, then reflect about y/x axis we get the same graph



- شروطه:

1) متماثل حول نقطة الأصل.

2) F(x) = - F(-X)

3) معرف على الفترة:

[-a ,a]or (-a, a) [A

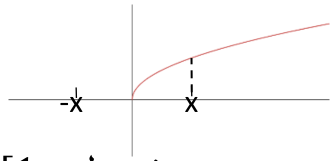
[-b,-a] ∪ [a , b] [B or

(-b,-a) ∪ (a , b) or

و على نفس النمط

- F(x) = F(-X)

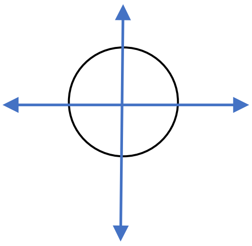
*) neither odd nor even function



شروطه: [1 $f(x) \neq f(-x)$ [2 $f(x) \neq -f(-x)$

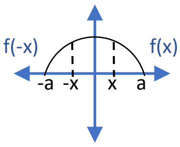
*) we can know the odd, even, neither function from the graph :

1) f(x) graph ⟶



⟶ f(x) is not a function

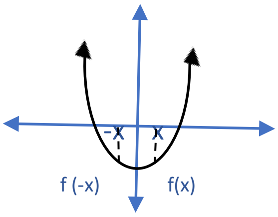
2) f(x) graph ⟶



⟶ f(x) is even function

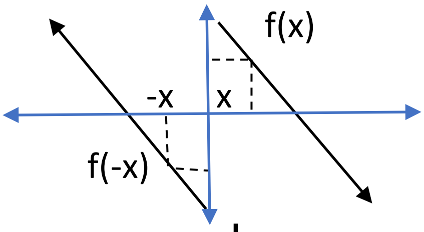
حقق كل الشروط

3) f(x) graph ⟶



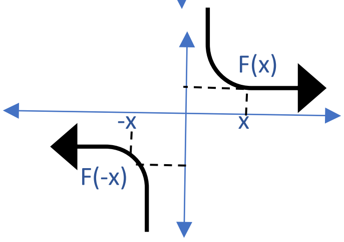
⟶ f(x) is even function

4) f(x) graph ⟶



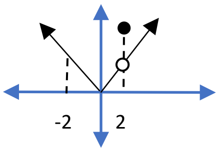
⟶ f(x) is odd function

5) f(x) graph ⟶



⟶ f(x) is odd function

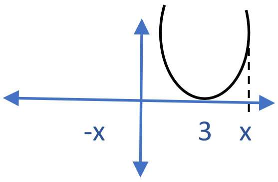
6) f(x) graph



⟶ f(x) is neither because

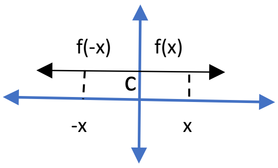
$f(2) \neq f(-2) and\ f(2) \neq -f(-2)$

7) f(x) graph



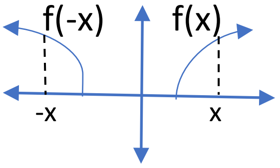
⟶ f(x) is neither

8) f(x) graph



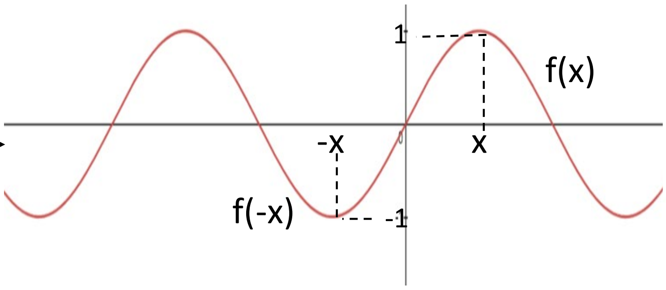
→ f(x) is even function

9) f(x) graph



→ f(x) is even function

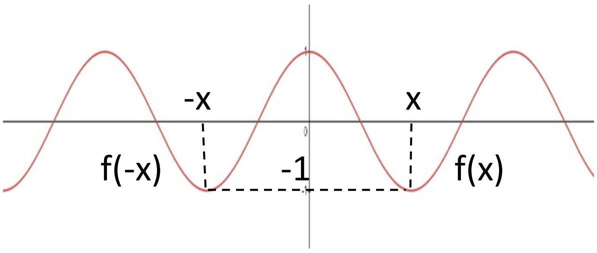
10) f(x) = sin x →



f(x) is odd function

f(x) = - f(-x)

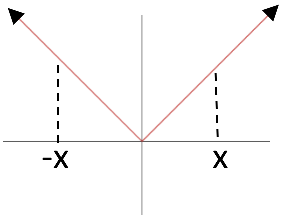
11) f(x) = cos x →



f(x) is even function

f(x) = f(-x)

12) f(x) = |x| →



f(x) is even function

Find if the functions are odd, even function ?

*) f(x) = x^2 → f(−x) = $(-x)^2 = x^2$ → f(x) = f(-x) → **even**

*) f(x) = 1 → f(-x) = 1 → f(x) = f(-x) → **even**

x^0

*) f(x) = $x^2 \pm 1$ → f(-x) = $(-x)^2 \pm 1 = x^2 \pm 1$ → f(x)=f(-x) → **even**

even ± even = even

*) f(x) = $\frac{x^2}{x^4}$, x≠ 0 → f(-x) = $\frac{(-x)^2}{(-x)^4} = \frac{x^2}{x^4}$ → f(x) = f(-x) → **even**, $\frac{x^2}{x^4} = \frac{even}{even} = even$

*) f(x)=$x^2(x^4+1)$ → f(-x) = $(-x)^2((-x)^4+1) = x^2(x^4+1)$ → f(x)=f(-x) → **even**

even * even = even

*) f(x) = x → f(-x) = -x → f(x) = -f(-x) → **odd**

*) f(x) = x^3 → f(-x) = $(-x)^3 = -x^3$ → f(x) = -f(-x) → **odd**

*) f(x)= $x^3 \pm x$ → f(-x) = $(-x)^3 \pm (-x) = -x^3 \pm -x = -(x^3 \pm x)$ → f(x)=-f(-x) → **odd**

odd ± odd = odd

*) f(x) = $\frac{x}{x^3}$, x≠ 0 → f(-x) = $\frac{(-x)}{(-x)^3} = \frac{-x}{-x^3} = \frac{x}{x^3}$ → f(x) = f(-x) → **even**, $\frac{x}{x^3} = \frac{odd}{odd} = even$

*) f(x)=$x^3(x^5+x)$ → f(-x)=$(-x)^3((-x)^5+(-x)) = -x^3(-x^5-x) = x^3(x^5+x)$

Odd *Odd = even

f(x) = f(-x) → **even**

*) f(x)=$x^2(x^5-x)$ → f(-x)=$(-x)^2((-x)^5-(-x)) = x^2(-x^5+x) = -x^2(x^5-x)$

even *Odd = odd

f(x) = -f(-x) → **odd**

*)f(x) = $\frac{x^2}{x^3+x}$, x≠ 0 → f(-x) = $\frac{(-x)^2}{(-x)^3-x} = \frac{x^2}{-(x^3+x)}$ → f(x) = f(-x) → **odd**, $\frac{x^2}{x^3+x} = \frac{even}{odd} = odd$

So, we can say that:

even ± even =even , even * even = even, even ÷ even =even

odd ± odd = odd, odd * odd = even , odd ÷ odd = even

*) And when we got:

(Odd $\pm$ even)or (neither $\pm$ * $\div$ neither) or (odd $\pm$ * $\div$ neither)

we need to calculate f(-x).

*) f(x) = $x + x^2 \rightarrow$ f(-x) = $-x + (-x)^2 = -x + x^2 \rightarrow$ f(x) $\rightarrow$ **neither**

*) f(x) = (1+x) (1-x)→ f(-x) =(1-x) (1+x)→ f(x) → even

*) f(x)=$\ln\frac{x+4}{-x+4} \rightarrow$ f(-x) =$\ln\frac{-x+4}{x+4}$ =$\ln\left(\frac{x+4}{-x+4}\right)^{-1} = -\ln\left(\frac{x+4}{-x+4}\right) \rightarrow$ **odd**

---

*)show that f(x) =$\frac{x^2\cos x}{x^3+x} + \ln\frac{1+x}{1-x}$ is an odd function :

→ F(-x) = $\frac{(-x)^2\cos(-x)}{(-x)^3+-x} + \ln\frac{1-x}{1+x} = \frac{x^2\cos x}{-x^3-x} + \ln\frac{1+x}{1-x}^{-1} = \frac{-x^2\cos x}{x^3+x} - \ln\frac{1+x}{1-x}$

= - ($\frac{x^2\cos x}{x^3+x} + \ln\frac{1+x}{1-x}$) → f(x) = - f(-x) → **odd function**

*)the function f(x) = $\frac{x}{x^2+1}$ is symmetric about ?

f(-x) =$\frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1}$→f(x) = -f(-x)→odd function →symmetric about the **origin**

*) the function f(x) =$e^{(2+x)}$-$e^{(2-x)}$ is symmetric about ?

f(-x) =$e^{(2-x)} - e^{(2+x)}$ = - ($e^{(2+x)} - e^{(2-x)}$)→ f(x) = − f(−x)

odd function → symmetric about the **origin**

إضافي (سؤال تميز)

*) the function f(x) = $\frac{(\sqrt{x^3+1}-\sqrt{x+1})(x^3+x+2\sqrt{x^3+1}\sqrt{x+1}+2)}{\sqrt{x^3+1}+\sqrt{x+1}}$ is odd, even, neither on interval [-1,1]?

f(x)=$\frac{(\sqrt{x^3+1}-\sqrt{x+1})(x^3+1+2\sqrt{x^3+1}\sqrt{x+1}+\ x+1)}{\sqrt{x^3+1}+\sqrt{x+1}} = \frac{(\sqrt{x^3+1}-\sqrt{x+1})\left(\sqrt{x^3+1}^2\right)+2\sqrt{x^3+1}\sqrt{x+1}+\sqrt{x+1}^2)}{\sqrt{x^3+1}+\sqrt{x+1}}$

f(x)=$\frac{(\sqrt{x^3+1}-\sqrt{x+1})(\sqrt{x^3+1}+\sqrt{x+1})^2}{\sqrt{x^3+1}+\sqrt{x+1}}$ = $(\sqrt{x^3+1} - \sqrt{x+1})(\sqrt{x^3+1} + \sqrt{x+1})= x^3 + x$

f(x) = $x^3 - x \rightarrow$ f(-x) = $-x^3 + x \rightarrow$ f(x) = - f(-x) → **odd function**

*) Are the functions → $\ln(\sqrt{x^2+1}+|x|)$, $\ln(\sqrt{x^2+1}+x)$, $\ln\frac{x+1}{1-x}$, $\frac{\sin x^2}{x^3-2x}$

Even , odd , neither function ?

1] f(x)= $\ln(\sqrt{x^2+1}+|x|)$ → f(-x) = $\ln(\sqrt{x^2+1}+|x|)$ → **<u>even</u>**

2] f(x)= $\ln(\sqrt{x^2+1}+x)$ → f(-x) = $\ln(\sqrt{x^2+1}-x)$=$\ln(\sqrt{x^2+1}-x)*\frac{\ln(\sqrt{x^2+1}+x)}{\ln(\sqrt{x^2+1}+x)}$

f(-x) = $ln\frac{1}{\sqrt{x^2+1}+x}$ = $\ln(\sqrt{x^2+1}+x)^{-1}$ = - ln$(\sqrt{x^2+1}+x)$→ f(x)=- f(-x)→ **<u>odd</u>**

3]f(x) = $\ln\frac{x+1}{1-x}$ → f(-x) = $\ln\frac{1-x}{x+1}$ = $\ln(\frac{x+1}{1-x})^{-1}$= -$\ln\frac{x+1}{1-x}$ → f(x)= - f(-x)→ **<u>odd</u>**

4] f(x) = $\frac{\sin x^2}{x^3-2x}$ →f(-x) = $\frac{\sin(-x)^2}{(-x)^3-2(-x)}$= $\frac{\sin x^2}{-(x^3-2x)}$ → f(x)= - f(-x) →**<u>odd</u>**

---

*)Is the function f(x) = $\ln(4x+1)-\ln(1-4x)$ odd, even, neither function?

f(-x) =$\ln(1-4x)-\ln(4x-1)=-(\ln(4x+1)-\ln(1-4x))$→ f(x)=-f(-x)→ **<u>odd</u>**

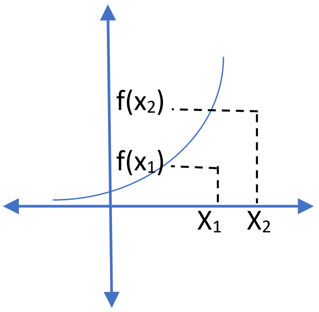
*) Is the function f(x) = $\sqrt{x^2-4}$ odd, even, neither function?

f(-x) = $\sqrt{(-x)^2-4}$ = $\sqrt{x^2-4}$ → f(x) = f(-x) → **<u>even function</u>**

*)Is the function f(x) =$\frac{1-e^x}{1+e^x}$ odd, even, neither function?

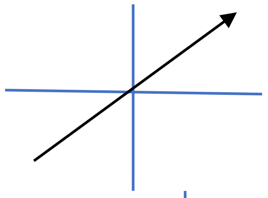
f(-x) =$\frac{1-\frac{1}{e^x}}{1+\frac{1}{e^x}}$ = $\frac{e^x-1}{1+e^x}$ = $\frac{-(1-e^x)}{1+e^x}$ → f(x)= -f(-x) →**<u>odd function</u>**

Increasing function



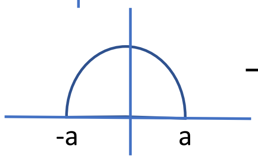
*)If $f(x_2) > f(x_1)$, $x_2 > x_1$ → it`s an increasing function

f(x) =



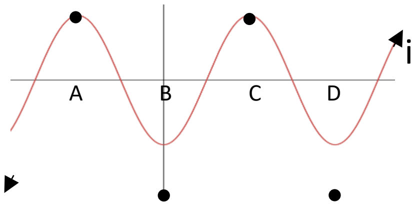
→ increasing function

f(x)=



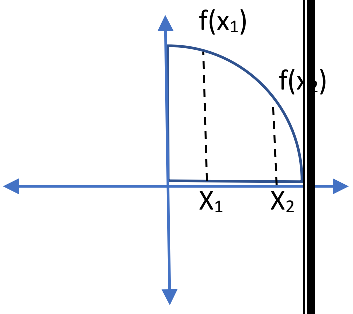
→ increasing on the interval [-a , 0]

f(x) =



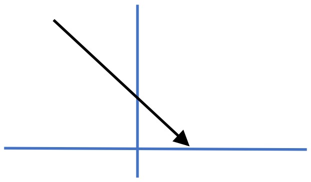
increasing → $(-\infty , A] \cup [B , C] \cup [D , \infty)$

Decreasing function



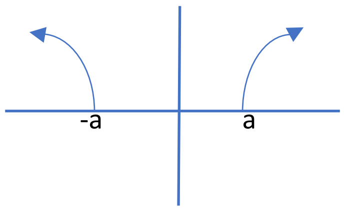
*)If $f(x_2) < f(x_1)$, $x_2 < x_1$ → it`s an Decreasing function

f(x) =



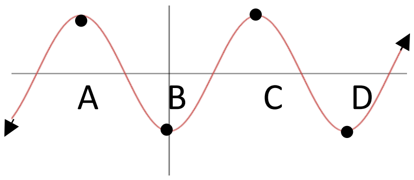
→ Decreasing function

f(x) =



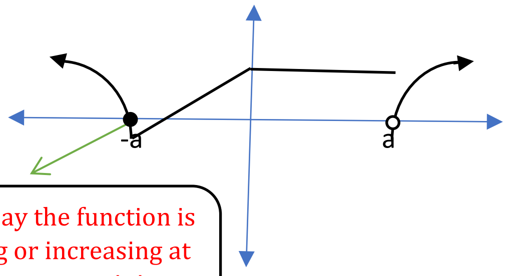
Decreasing function on interval (−∞ , -a]

f(x) =



decreasing at [A , B] ∪ [C , D]

f(x)



We can`t say the function is decreasing or increasing at this point so we put it in Poth intervals

Decreasing → (−∞ , -a]

increasing → [-a , 0] ∪ (a, ∞)

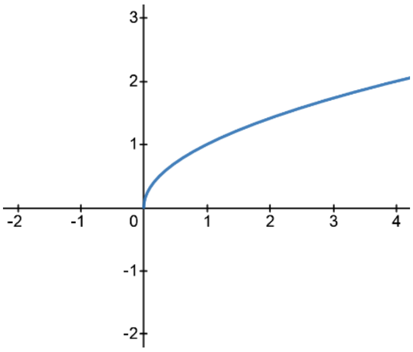
constant function → [0, a]

(Neither increasing nor decreasing)

New functions from old functions

→ Horizontal and vertical shifts

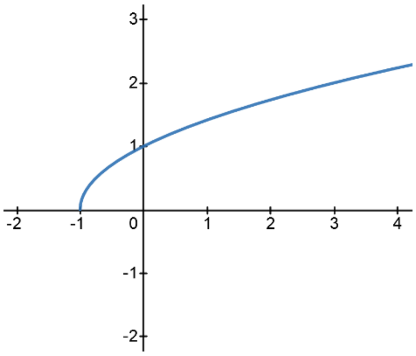
If $f(x) = \sqrt{x}$



ثبت قيم y

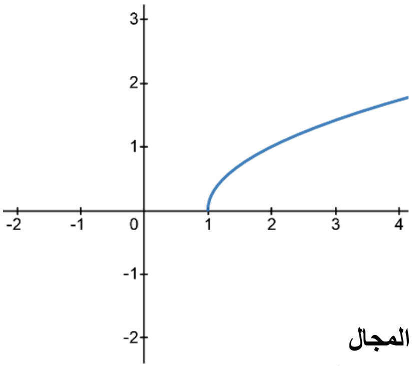
y	0	1	2
x	0	1	4

$f(x) = \sqrt{x+1}$



y	0	1	2
x	-1	0	3

$f(x) = \sqrt{x-1}$



y	0	1	2
x	1	2	5

المجال

Domain

Horizontal shifts

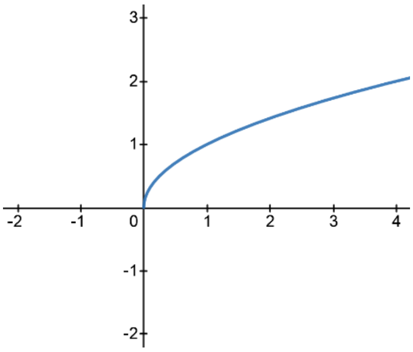
إزاحة أفقية

← نلاحظ أن $f(x+1)$ أدت إلى إزاحة الإقتران و قيم x بمقدار 1 إلى اليسار

← و أن $f(x-1)$ أدت إلى إزاحة الإقتران و قيم x بمقدار 1 إلى اليمين

- To draw the graph of $f(x-c)$ → We shift the graph of $f(x)$ to the right by a distance of c units
- To draw the graph of $f(x+c)$ → We shift the graph of $f(x)$ to the left by a distance of c units

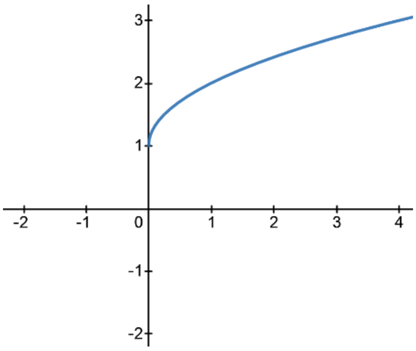
$If \;\; f(x) = \sqrt{x}$



y	0	$\sqrt{2}$	$\sqrt{3}$
x	0	2	3

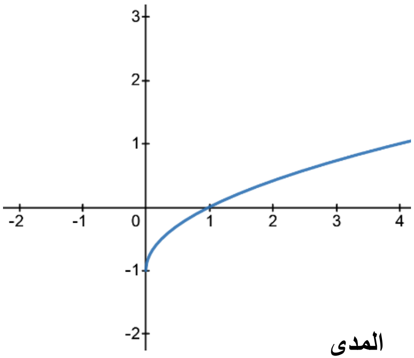
ثبت قيم x

$f(x) + 1 = \sqrt{x} + 1$



y	0+1	$\sqrt{2}+1$	$\sqrt{3}+1$
x	0	2	3

$f(x) - 1 = \sqrt{x} - 1$



y	0-1	$\sqrt{2}-1$	$\sqrt{3}-1$
x	0	2	3

المدى

Range

Vertical shifts

إزاحة عمودية

← نلاحظ أن $f(x) + 1$ أدت إلى إزاحة الإقتران و قيم y بمقدار 1 إلى الأعلى

← و أن $f(x) - 1$ أدت إلى إزاحة الإقتران و قيم y بمقدار 1 إلى الأسفل

- To draw the graph of $f(x) + c$ → We shift the graph of $f(x)$ upwards by a distance of c units

- To draw the graph of $f(x) - c$ → We shift the graph of $f(x)$ downwards by a distance of c units

Ex. If $f(x) = x^2$, sketch the graph of $g(x) = x^2 + 2x + 4$

$$g(x) = x^2 + 2x + 1 + 3$$

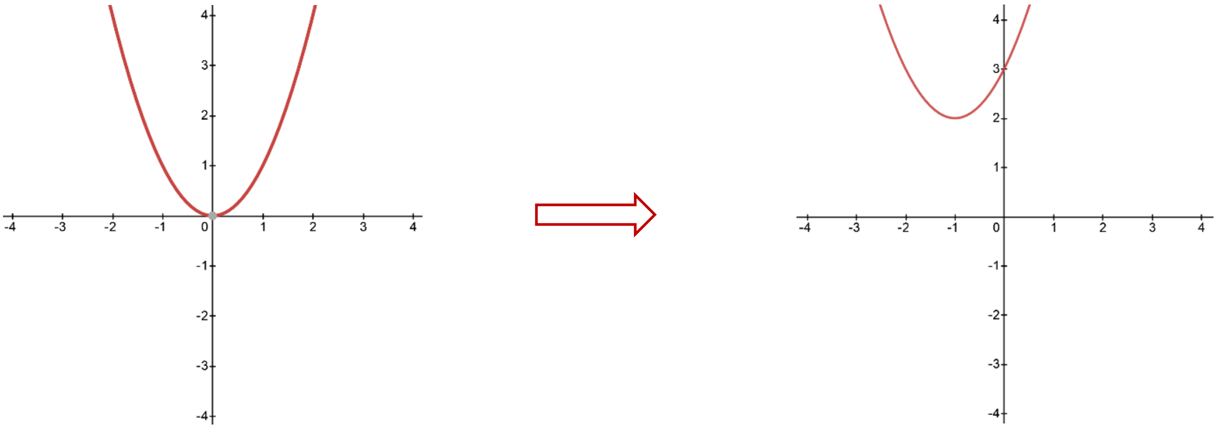
$$g(x) = \boxed{(x+1)^2} + 3$$

$x \rightarrow x + 1$

Shift to the left 1 unit

$f(x) \rightarrow f(x) + 3$

Shift upwards 3 units

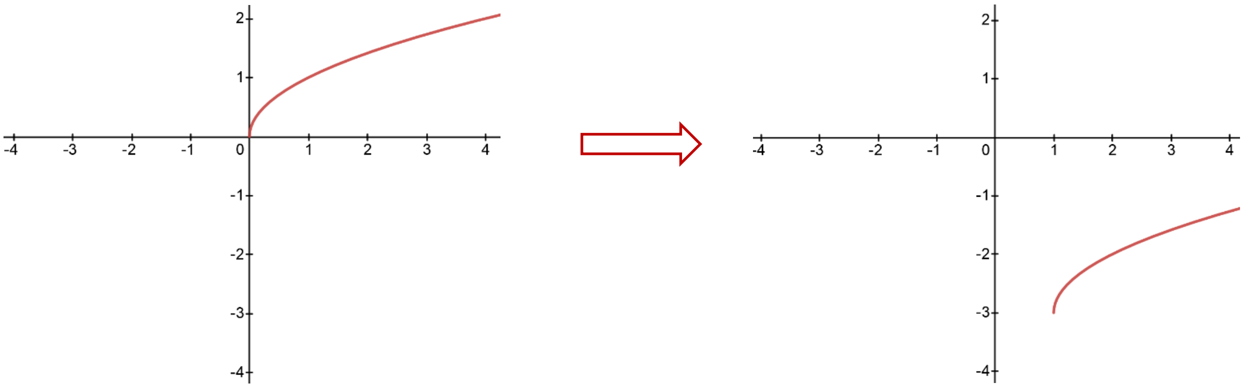


Ex. If $f(x) = \sqrt{x-1} - 3$, sketch it.

$$f(x) = \sqrt{x} \rightarrow \boxed{f(x-1)} = \sqrt{x-1} \rightarrow f(x-1) - 3 = \sqrt{x-1} - 3$$

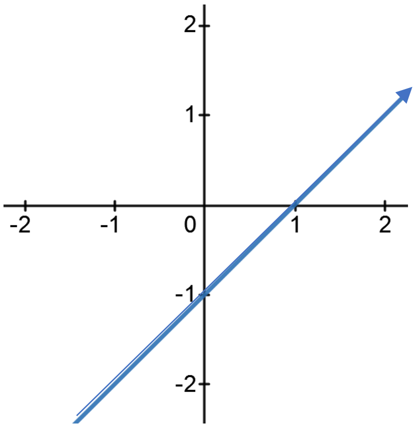
Shift to the right 1 unit

Shift downwards 3 units



→ Reflections

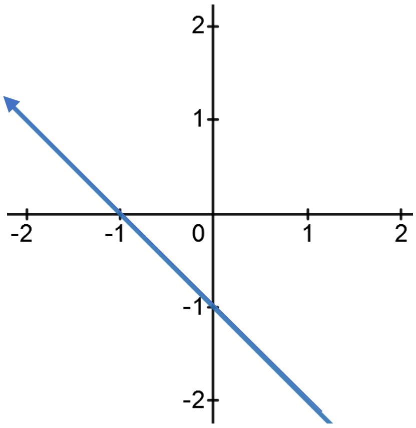
$If\ \ f(x) = x - 1$



ثبت قيم y

y	0	1	2
x	1	2	3

$f(-x) = -x - 1$



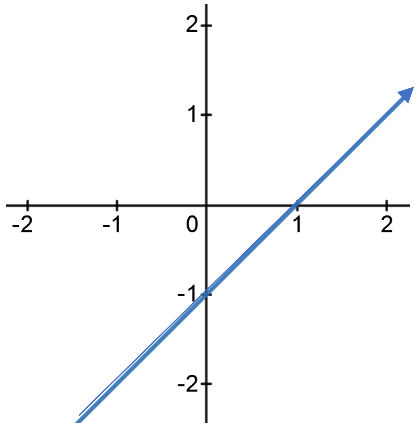
y	0	1	2
x	-1	-2	-3

← نلاحظ أن $f(-x)$ أدت إلى دوران الإقتران حول محور y و انعكس ال Domain مع بقاء ال Range ثابتاً

$$f(x) \rightarrow f(-x) \rightarrow reflection\ about\ y - axis$$

كلام صحيح لكن له استثناءات مثل $\ln(x \pm a)$

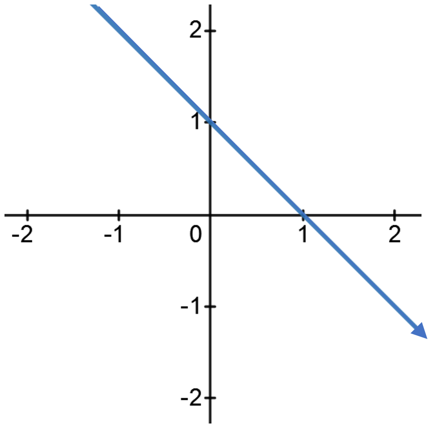
$If\ \ f(x) = x - 1$



y	0	1	2
x	1	2	3

ثبت قيم x

$-f(x) = 1 - x$

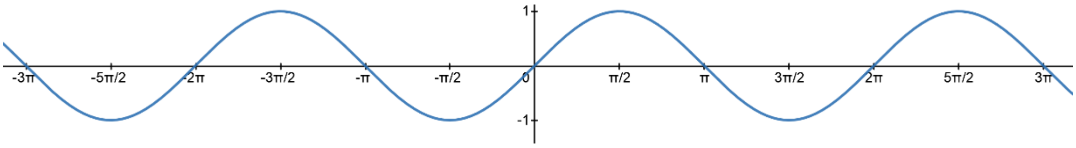


y	0	-1	-2
x	1	2	3

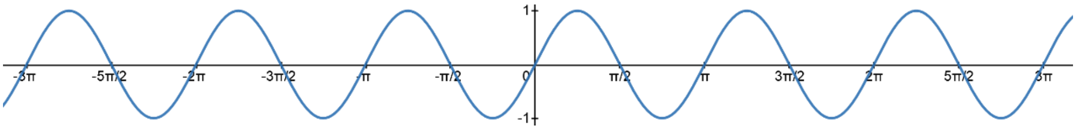
← نلاحظ أن $-f(x)$ أدت إلى دوران الإقتران حول محور x و انعكس ال Range مع بقاء ال Domain ثابتاً

$$f(x) \rightarrow -f(x) \rightarrow reflection\ about\ x - axis$$

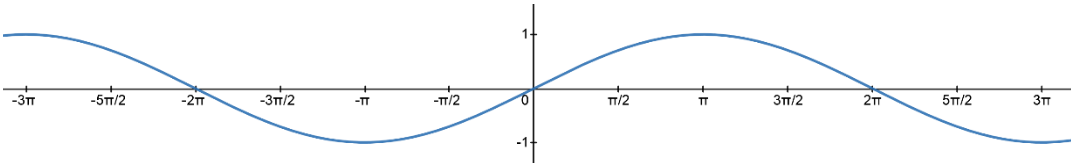
- $\sin(x)$



- $\sin(2x)$

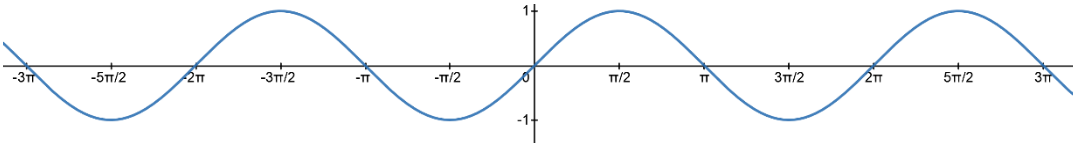


- $\sin\left(\frac{1}{2}x\right)$

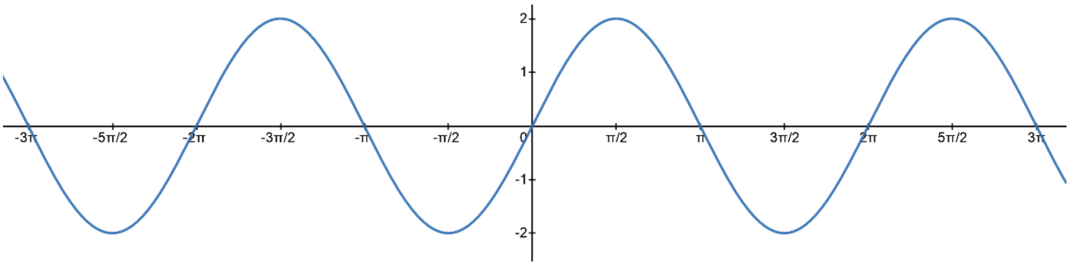


← لاحظ أن قيم ال Range لم تتغير أبداً و لكن قيم x (Domain) هي التي تتغير

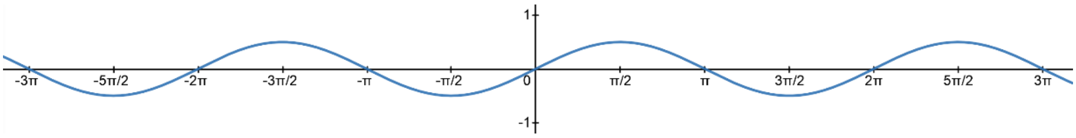
- $\sin(x)$



- $2\sin(x)$



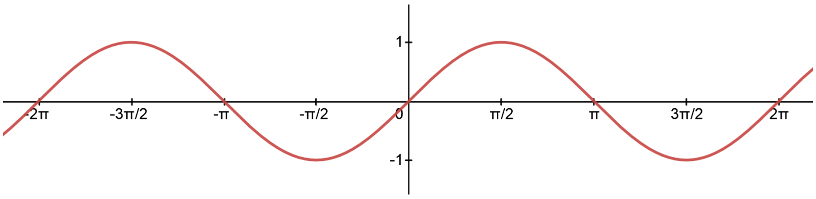
- $\frac{1}{2}\sin(x)$



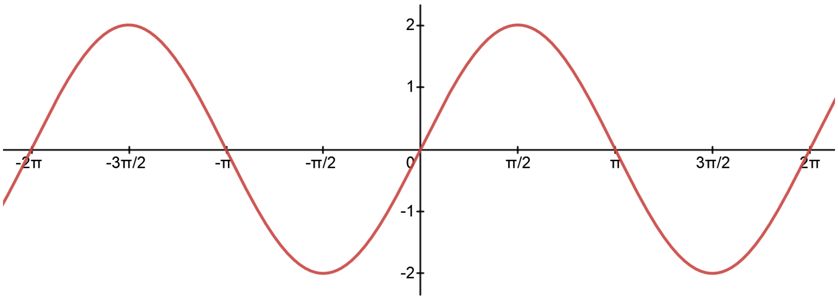
← لاحظ أن قيم ال Domain لم تتغير أبداً و لكن قيم y (Range) هي التي تتغير

Ex: If you know that $f(x) = sin(x)$, sketch $g(x) = 1 - 2sin(x)$

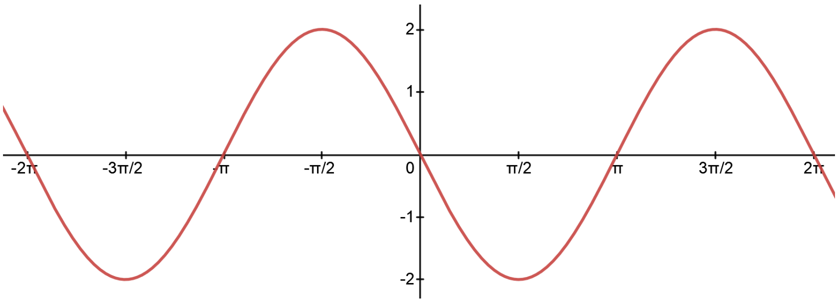
$f(x) = \sin(x)$



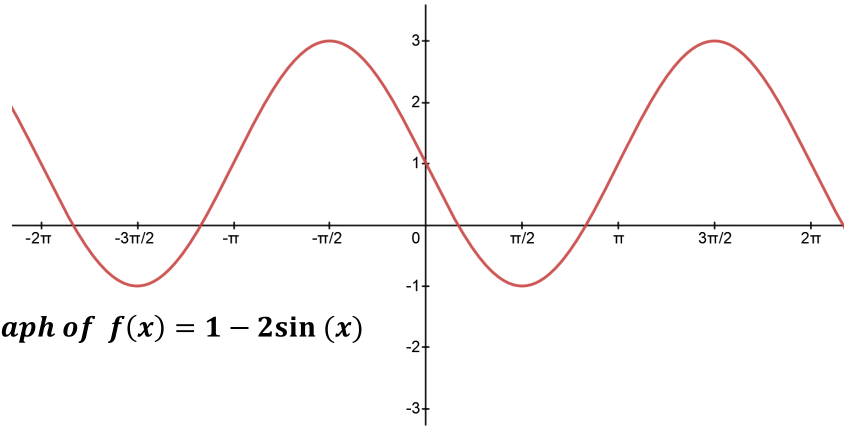
$2f(x) = 2\sin(x)$ → Stretch vertically by a factor of 2



$-2f(x) = -2\sin(x)$ → Reflect about x-axis



$Then, -2\sin(x) + 1\ means\ shift\ the\ graph\ upwards\ by\ 1\ unit$

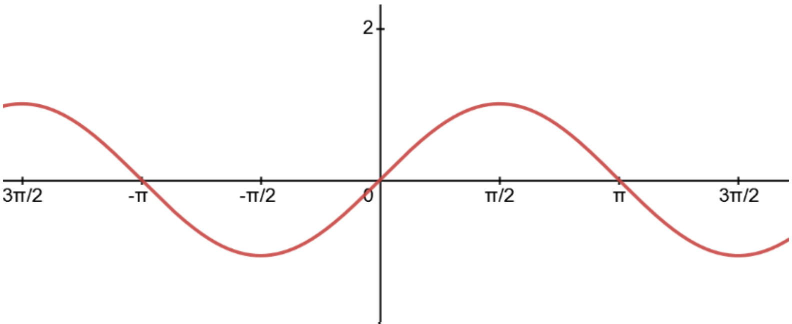


Graph of $\boldsymbol{f(x) = 1 - 2\sin(x)}$

*إنتبه للإزاحات في حالة وجود ضرب أو قسمة

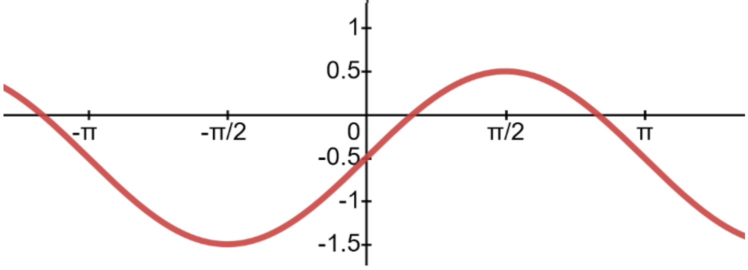
$f(x) = 1 - 2\sin(x)$

$f(x) = \sin(x)$

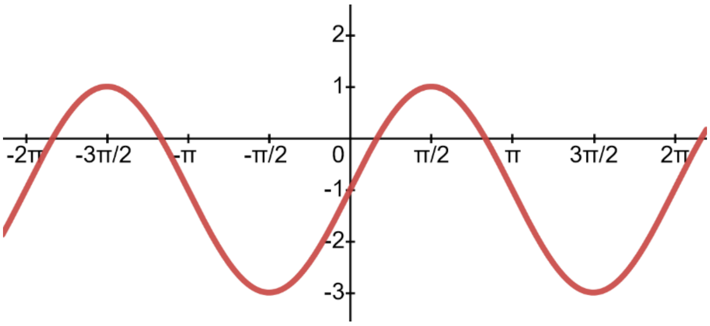


$f(x) = sin(x) - \frac{1}{2}$

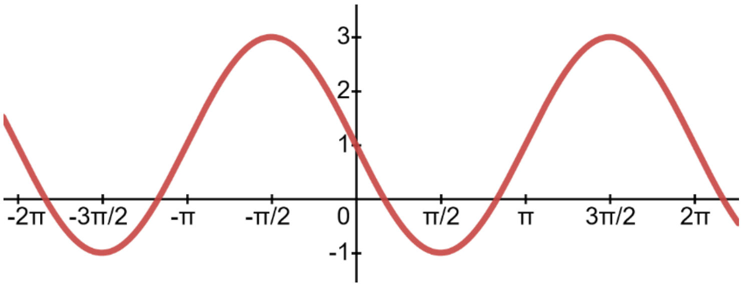
*Shift down by ½ unit



$f(x) = 2 * \left(sin(x) - \frac{1}{2}\right)$



$f(x) = -2 * \left(sin(x) - \frac{1}{2}\right)$



Shift → ½ units downward → 1 unit downward → 1 unit upward

لما تمدد الضعف عاموديا

لما انعكس حول محور X صار للأعلى بدل للأسفل

Ex: Explain how the graph of $f(x) = x^2 - 4x + 1$ is obtained from the graph of $g(x) = x^2$.

$f(x) = (x-2)^2 - 3$

② Shifted 3 units downward

① Shifted 2 units to the right

Ex: Let $f(x) = x^2 + 2x$, if $f(x)$ is shifted 2 units right, 3 units up then reflected about the Y-axis, what is the obtained function?

a) $x^2 + 2x + 3$
b) $x^2 - 2x + 3$
c) $x^2 + 2x - 3$
d) $x^2 - 2x - 3$
e) $x^2 - 6x + 3$

Solution:

reflected about the Y-axis → replace every x by $-x$

shifted 3 units up → add 3 to the function

shifted 2 units right → replace every x by $(x-2)$

$g(x) = x^2 + 2x + 3$

Ex: Explain how the graph of $f(x) = x^2 + 6x + 4$ obtained from the graph of $g(x) = x^2$?

Solution:

$f(x) = (x+3)^2 - 5$

Shifted 5 units downward

Shifted 3 units to the left

Ex: the graph of $f(x) = sinx$ was shifted horizontally $\frac{\pi}{2}$ units to the right to get $g(x)$, find $g(x)$.

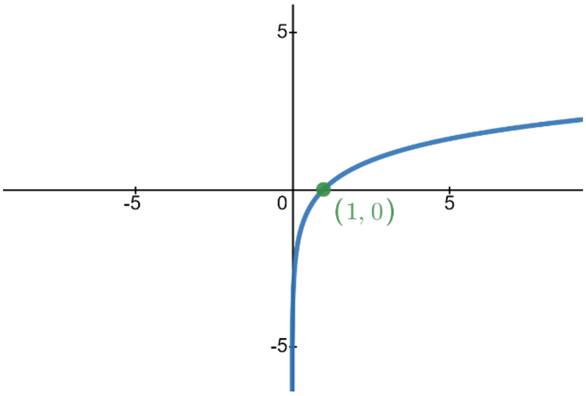
a) $g(x) = -cosx$
b) $g(x) = cosx$
c) $g(x) = \frac{\pi}{2} + sinx$
d) $g(x) = sinx - \frac{\pi}{2}$

Solution: $g(x) = f\left(x - \frac{\pi}{2}\right)$

$g(x) = sin\left(x - \frac{\pi}{2}\right) = -cosx$

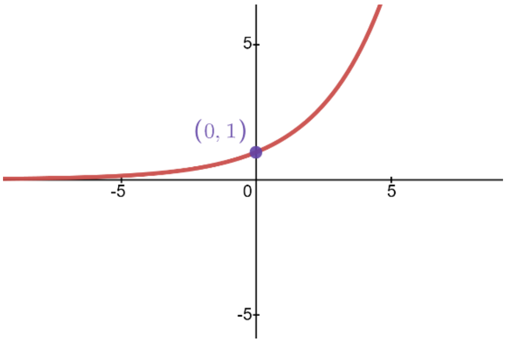
$^*\sin\left(x - \frac{\pi}{2}\right) = sinx.cos\frac{\pi}{2} - sin\frac{\pi}{2}.cosx = -cosx$

The graph of $\ln x$ →

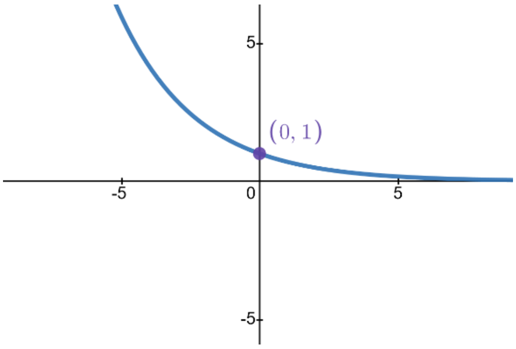


The graph of a^x, where $a > 0$

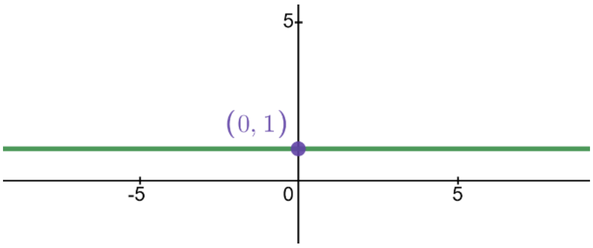
- when $a > 1$



- when $0 < a < 1$

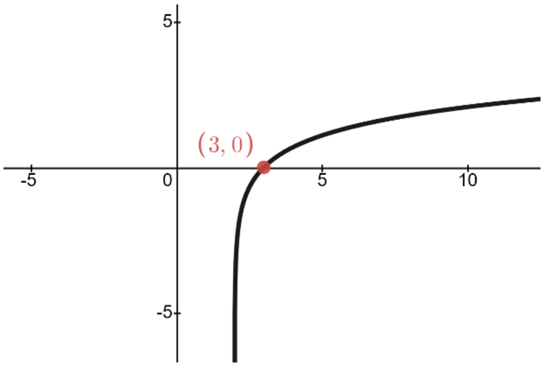


- when $a = 1$

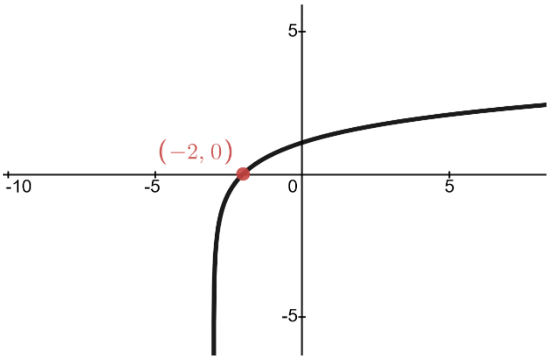


How to sketch:

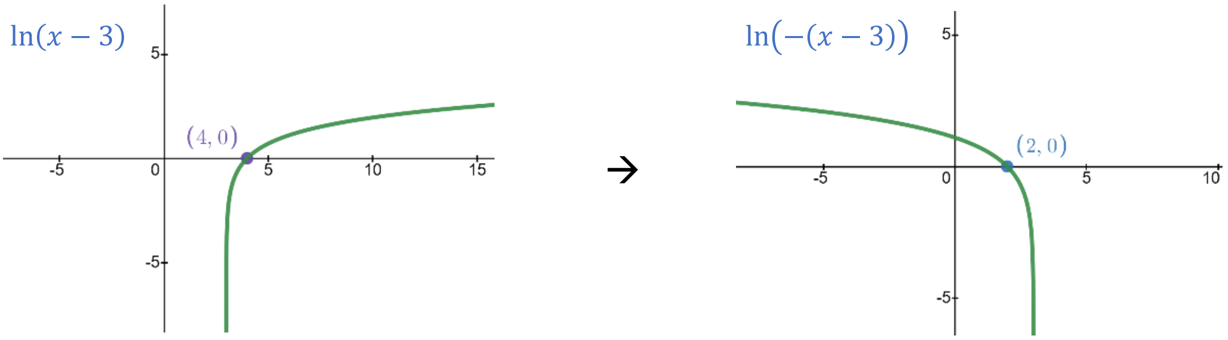
1) $\ln(x-2)$



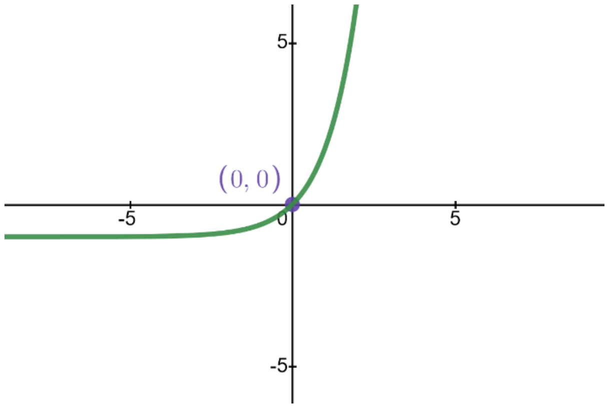
2) $\ln(x+3)$



3) $\ln(-x+3) = \ln\big(-(x-3)\big)$

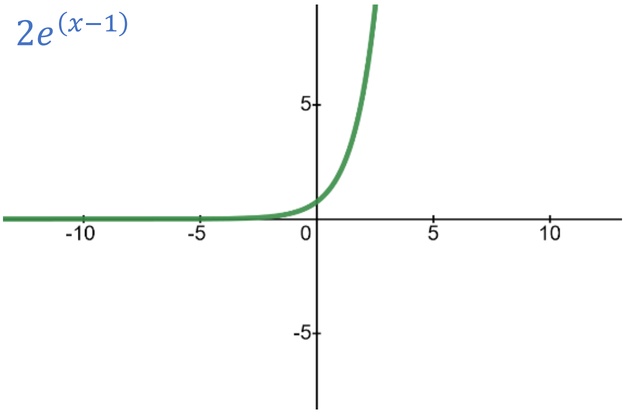


4) $e^x - 1$



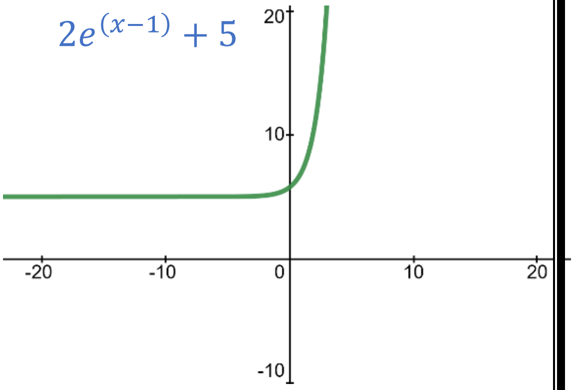
5) $2e^{(x-1)} + 5$

$2e^{(x-1)}$



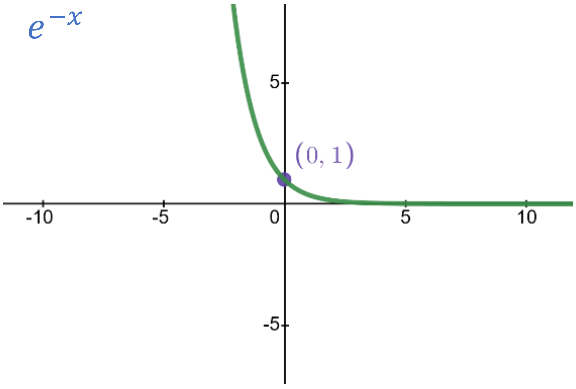
→

$2e^{(x-1)} + 5$



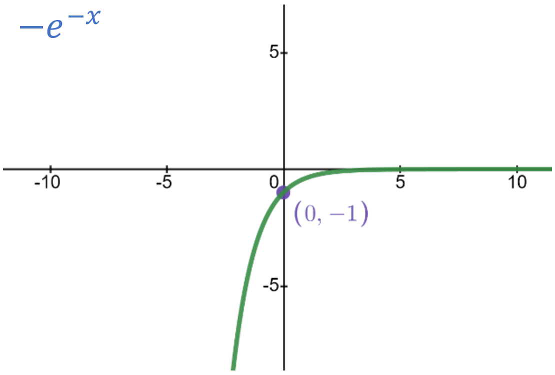
6) $-e^{-x} + 5$

e^{-x}

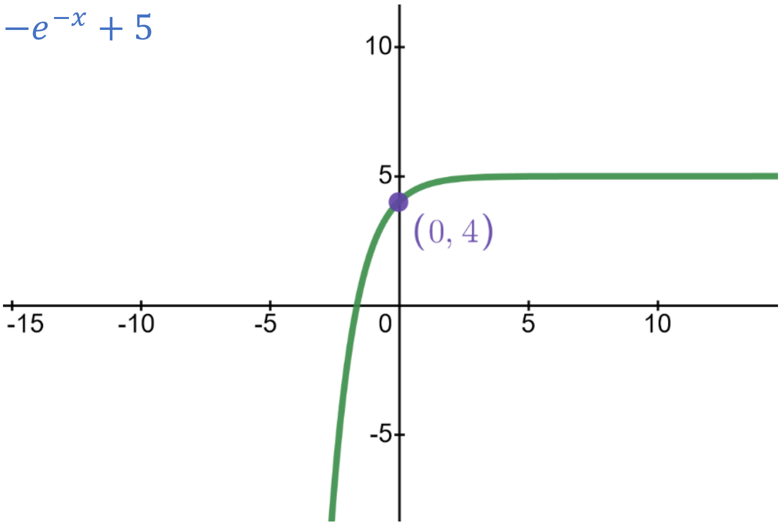


→

$-e^{-x}$



$-e^{-x} + 5$



Ex: If the domain and range of $f(x)$ are $[-1,3]$ and $[0,4]$, find the range of $1-4f^{-1}(x)$.

Solution:

Range of $f^{-1}(x)$ is the domain of $f(x)$

$$3 \geq f^{-1}(x) \geq -1$$

$$4 \geq -4\, f^{-1}(x) \geq -12$$

$$5 \geq 1-4\, f^{-1}(x) \geq -12$$

Range of $f^{-1}(x)$ is $[-12,5]$

Ex: If the range of $f(x)$ is $[-3,5]$ and $g(x)=4-f(3x-1)$, find the range of $g(x)$.

Solution:

Range of $f(3x-1)$ = Range of $f(x)$, because the shifting and stretching is at values of the domain (x).

$$5 \geq f(x) \geq -3$$

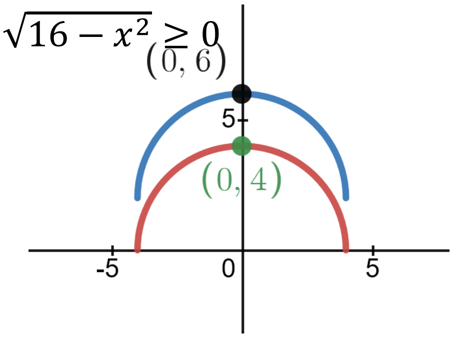
$$3 \geq f(3x-1) \geq -5$$

$$7 \geq 4-f(3x-1) \geq -1$$

Ex: The range of $f(x)=\sqrt{16-x^2}+2$ is?

$\sqrt{16-x^2}$ $4 \geq \sqrt{16-x^2} \geq 0$

$\sqrt{16-x^2}+2$



$$6 \geq \sqrt{16-x^2}+2 \geq 2$$

The range of $f(x)$ is $[2,6]$

Ex: Find the range of the function $f(x)=|x|+3|x-2|$

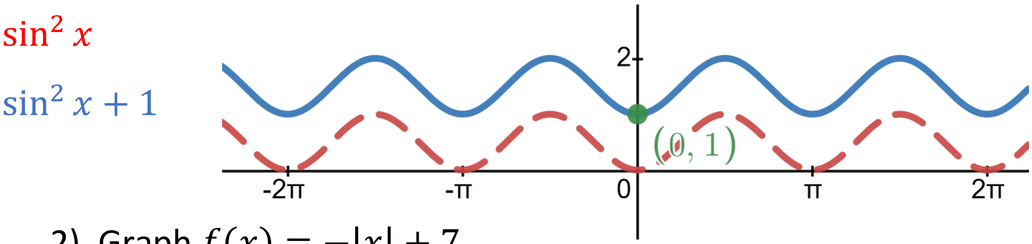
* The value of $|x|$ is ≥ 0 & the value of $3|x-2|$ is ≥ 0, so the sum of them is ≥ 0

The least value of $|x|$ or $|3x-2|$ is 0 so the least value of the function is when $x=0$ or $x=2$.

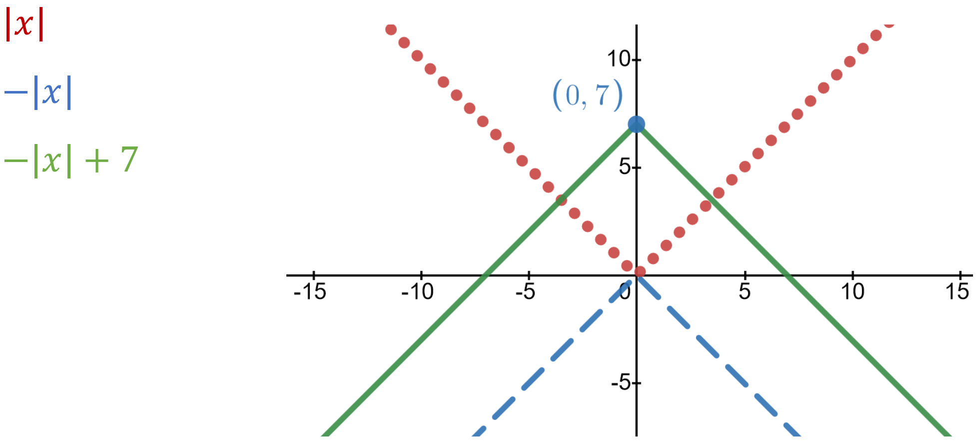
$f(0)=6$ and $f(2)=2$, so the least value of the function is 2, Range of $f(x)$ is $[2,\infty]$

Extra exercises:

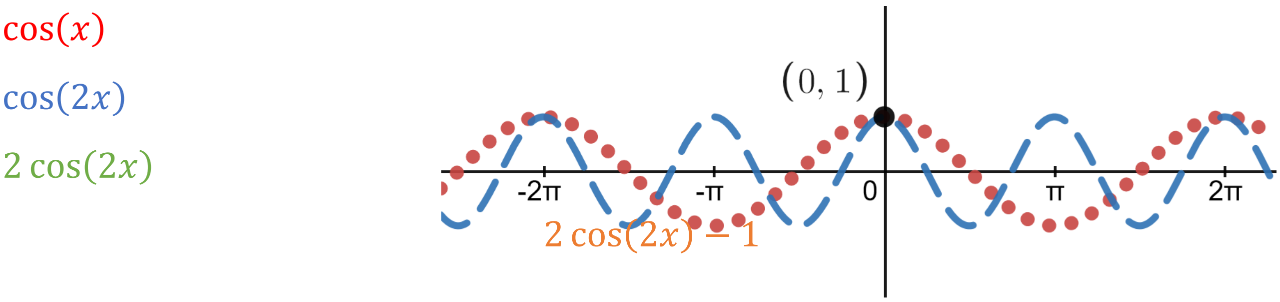
1) Graph $f(x) = \sin^2 x + 1$

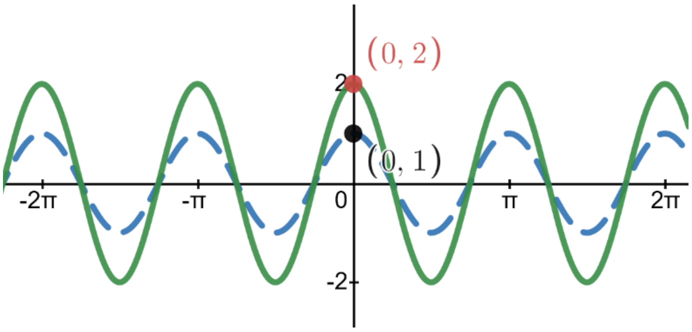


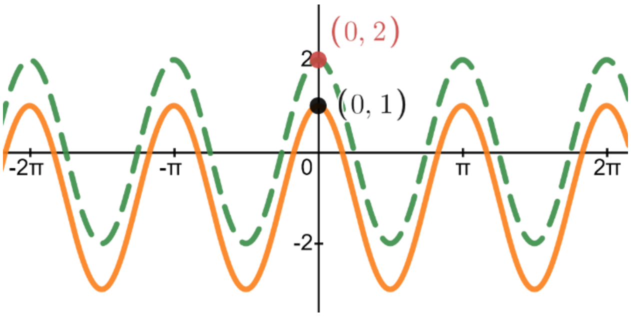
2) Graph $f(x) = -|x| + 7$



3) Graph $f(x) = 2\cos(2x) - 1$



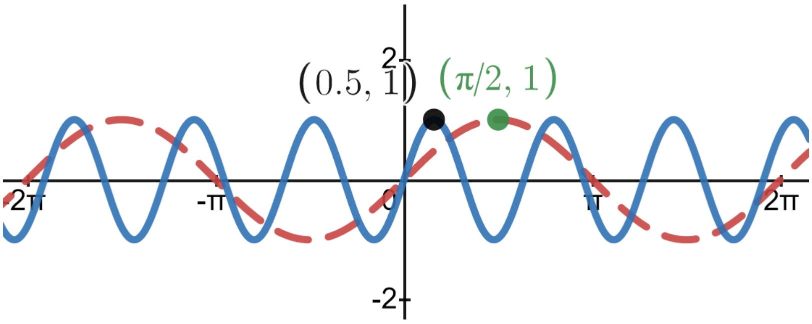




4) Graph $f(x) = \sin(\pi x)$

$\sin(x)$

$\sin(\pi x)$

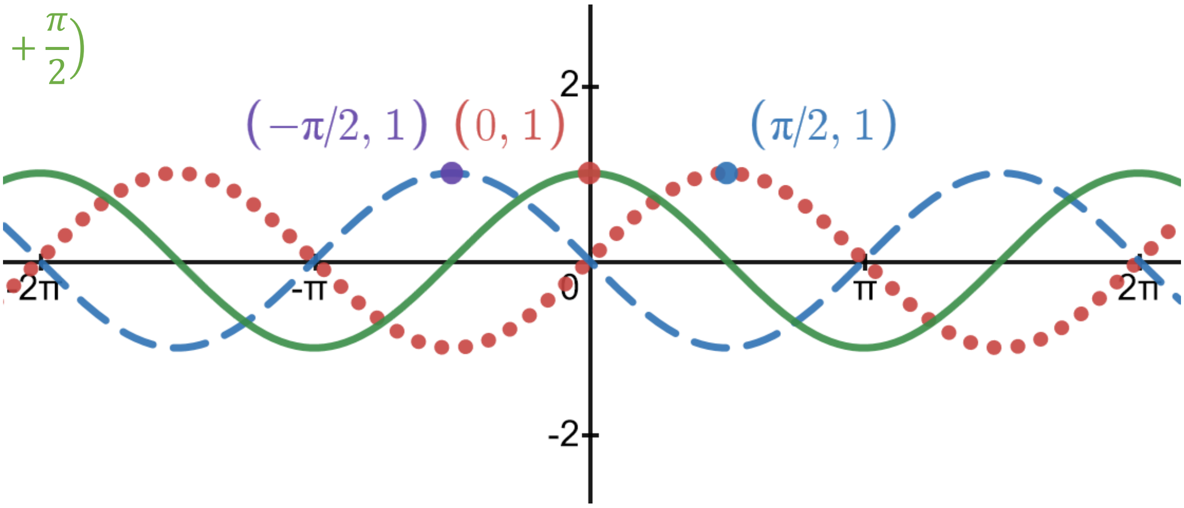


5) Graph $f(x) = \sin\left(\frac{\pi}{2} - x\right)$

$\sin(x)$

$\sin(-x)$

$\sin\left(-x + \frac{\pi}{2}\right)$



Composition of functions (تركيب الإقترانات)

Definition: Given two functions $f(x)$ & $g(x)$, the composition function $(f \circ g)$ is defined by $(f \circ g)(x) = f\big(g(x)\big)$

- $(f \circ g)(x) = f\big(g(x)\big)$
- $(g \circ f)(x) = g\big(f(x)\big)$
- $(f \circ f)(x) = f\big(f(x)\big)$
- $(g \circ g)(x) = g\big(g(x)\big)$

Ex: If $\boldsymbol{f(x) = x^2}$ and $\boldsymbol{g(x) = x - 3}$, find $\boldsymbol{(f \circ g), (g \circ f), (f \circ f), (g \circ g)}$.

1) $(f \circ g)$

$f \circ g = f(x-3)$

*بمسك كل x موجودة بـ$f(x)$ وبحط بدالها $(x-3)$

$f \circ g = (x-3)^2$

2) $(g \circ f)$

$g \circ f = g(x^2)$

*بمسك كل x موجودة بـ$g(x)$ وبحط بدالها (x^2)

$g \circ f = x^2 - 3$

3) $(f \circ f)$

$f \circ f = f(x^2)$

$f \circ f = x^4$

4) $(g \circ g)$

$g \circ g = g(x-3)$

$g \circ g = (x-3) - 3$

***$(f \circ g) \neq (g \circ f)$**

$g \circ g = x - 6$

Ex: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find $(f \circ g), (g \circ f), (f \circ g \circ g)$.

1) $(f \circ g)$

$$f \circ g = f\left(\sqrt{2-x}\right)$$

$$f \circ g = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

2) $(g \circ f)$

$$g \circ f = g(\sqrt{x}\,)$$

$$g \circ f = \sqrt{2-\sqrt{x}}$$

3) $(f \circ g \circ g)$

$$f \circ g \circ g = f\left(g\left(g(x)\right)\right)$$

$$f \circ g \circ g = f\left(g\left(\sqrt{2-x}\right)\right)$$

$$f \circ g \circ g = f\left(\sqrt{2-\sqrt{2-x}}\right)$$

$$f \circ g \circ g = \sqrt{\sqrt{2-\sqrt{2-x}}}$$

$$f \circ g \circ g = \sqrt[4]{2-\sqrt{2-x}}$$

Domain of Composition function

$(D_{f\circ g})$ Domain of $(f\circ g) = (D_{ans})$ Domain of (ans) $\cap$ (D_g) Domain of $g(x)$

الاقتران المركب   الاقتران الثاني

$(D_{g\circ f})$ Domain of $(g\circ f) = (D_{ans})$ Domain of (ans) $\cap$ (D_f) Domain of $f(x)$

الاقتران المركب   الاقتران الثاني

Ex: $f(x) = x^2 + 3,\ g(x) = \sqrt{x}$, find $D_{f\circ g}$ & $D_{g\circ f}$.

1) $D_{f\circ g} = D_{ans} \cap \underline{D_g} \longrightarrow x \geq 0$

$ans = f\circ g = x + 3$

$D_{ans} = [-\infty, \infty] = R$

$D_g = [0, \infty]$

$D_{f\circ g} = R \cap [0, \infty]$

$D_{f\circ g} = [0, \infty]$

2) $D_{g\circ f} = D_{ans} \cap D_f$

$ans = g\circ f = \sqrt{x^2 + 3}$

$D_{ans} = [-\infty, \infty] = R$

$D_f = [-\infty, \infty] = R$

$D_{g\circ f} = R \cap R$

$D_{g\circ f} = R$

Ex: $\boldsymbol{f(x) = \sqrt{x}, g(x) = \sqrt{2-x}}$**, find** $\boldsymbol{D_{f\circ g}}$**&** $\boldsymbol{D_{g\circ f}}$ **.**

1) $D_{f\circ g} = D_{ans} \cap D_g$

$ans = f \circ g = \sqrt[4]{2-x}$

$D_{ans}: 2 - x \geq 0$

$DansD_{ans}: 2 \geq \mathrm{x}$

$D_{ans} = [-\infty, 2]$

$D_g = [-\infty, 2]$

$D_{f\circ g} = [-\infty, 2] \cap [-\infty, 2]$

$D_{f\circ g} = [-\infty, 2]$

2) $D_{g\circ f} = D_{ans} \cap D_f$

$ans = g \circ f = \sqrt{2 - \sqrt{x}}$

$D_{ans}: 2 - \sqrt{x} \geq 0$

$D_{ans}: 2 \geq \sqrt{x}$

$D_{ans}: 4 \geq x$

$D_{ans} = [-\infty, 4]$

$D_f = [0, \infty]$

$D_{g\circ f} = [-\infty, 4] \cap [0, \infty]$

$D_{g\circ f} = [0,4]$

Decomposition (فك التركيب)

Ex: Given that $h(x) = (f \circ g)(x)$, find $f(x)$ & $g(x)$.

1) $h(x) = (x-2)^3$

$g(x) = x - 2$, (Inside term)

$f(x) = x^3$, (Outside term)

2) $h(x) = \sqrt{1+5x}$

$g(x) = 1 + 5x$, (Inside term)

$f(x) = \sqrt{x}$, (Outside term)

3) $h(x) = sin(x^3)$

$g(x) = x^3$, (Inside term)

$f(x) = \sin(x)$, (Outside term)

4) $h(x) = \sin^3 x$

$g(x) = \sin(x)$, (Inside term)

$f(x) = x^3$, (Outside term)

5) $h(x) = \frac{1}{2+3x}$

$g(x) = 2 + 3x$, (Inside term) OR $g(x) = 3x$

$f(x) = \frac{1}{x}$, (Outside term) $f(x) = \frac{1}{2+x}$

Ex: Let $\mathbf{F(x) = \cos^2(x - 9)}$, find $\mathbf{f(x), g(x), h(x)}$, such that $\mathbf{f \circ g \circ h = F}$

$f(x) = \cos^2 x$

$g(x) = x - 9$

$h(x) = x$

---

*Very important example

Ex: If $\boldsymbol{g(x) = 2x + 1, h(x) = 4x^2 + 4x + 7}$, find a function $\boldsymbol{f(x)}$ such that $\boldsymbol{f \circ g = h}$.

$f\big(g(x)\big) = h(x)$

$f(2x + 1) = 4x^2 + 4x + 7$

↳ we can say that $f(x) = ax^2 + bx + c$, but $f(2x + 1) = a(2x + 1)^2 + b(2x + 1) + c$

$4ax^2 + 4ax + a + 2bx + b + c = 4x^2 + 4x + 7$

لازم مجموع الحدود لx^2 على الطرفين يتساوى

$4ax^2 = 4x^2$

$\boxed{a = 1}$

لازم مجموع الحدود لx على الطرفين يتساوى

$4ax + 2bx = 4x$

$4x + 2bx = 4x$

$2bx = 0$

$\boxed{b = 0}$

$c + b + a = 7$

$c + 0 + 1 = 7$

$\boxed{c = 6}$

$f(x) = x^2 + 6$

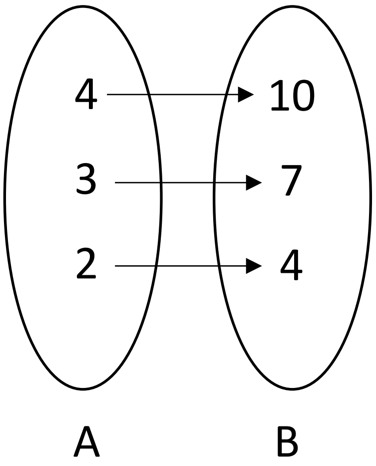
Inverse function (الاقترانات العكسية)

Before talking about inverse we must talk about one-to-one function.

Def: A function is called one to one function if it never takes on the same value twice, that is:

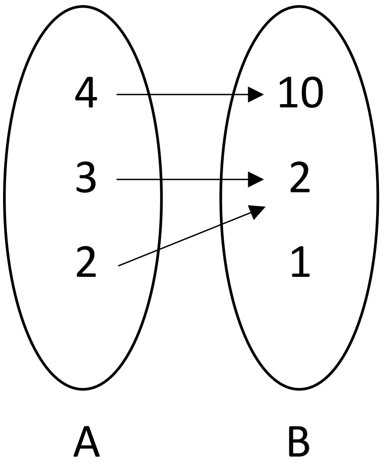
$$f\,(x_1) \neq f(\mathrm{x}_2\,) \quad whenever\; x_1 \neq x_2$$

بالمختصر ..... الإقتران one to one .... كان لأي نقطتين في المجال نفس الصورة .



$f\,(x\,)$

One to one function

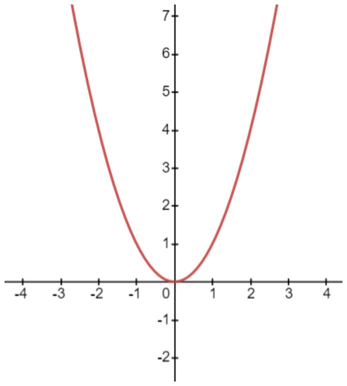


$f\,(x\,)$

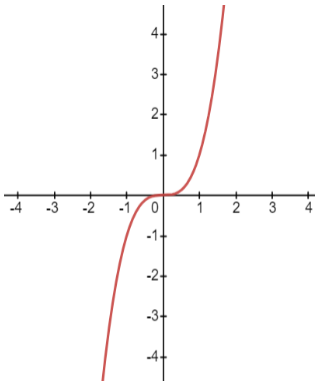
Not one to one function

لإنه 3,2 الهم نفس الصور

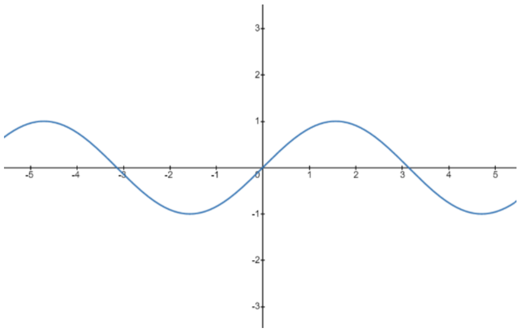
Horizontal line test: A function is one to one if and only if no horizontal line interest it`s graph more than once.



$y = x^2$



$y = x^3$



$y = \sin\,(x)$

*الملخص اذا خط افقي قطع الرسمة بأكثر من نقطة معناته مش one to one

* If f (x) is one to one $\longrightarrow$ Inverse

* If f (x) is not one to one $\longrightarrow$ Inverse ✕

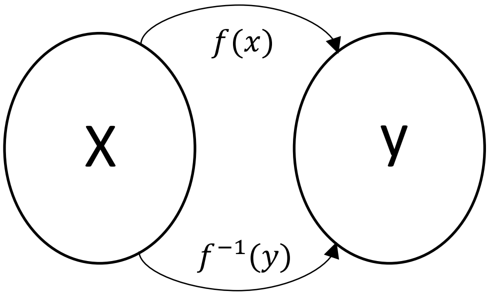
ما بقدر اعمله inverse

*بالمختصر اذا الإقتران مش one to one ما بقدر اعمله inverse .

Inverse:

Def : let *f* be a one to one function with domain A and angle B and range A and is define by

$$f(x) = y \longleftrightarrow f^{-1}(y) = x$$



Note:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

*Note: $domine\ of\ \ f^{-1}(x) \quad = \quad range\ of\ f$

$range\ of\ f^{-1}\ (x) \quad = \quad domine\ of\ f$

$D_{f^{-1}} \quad = R_f$

$R_{f^{-1}} \quad = \ D_f$

Ex: $\boldsymbol{f\ (x)\ =\ e^x + 1\ , find\ f^{-1}(2)}$?

شو العدد الي لو عوضته
بدل x بعطيني الجواب
2
يلي هو الصفر

if $x = 0$

$f\ (0) = \ e^0 + 1 = \quad 1 + 1 = 2$

$f^{-1}(2) = 0 \longleftrightarrow f(0) = 2$

*One of the most common *f* Inverse function are:

$$e^x \, , \ln x$$

$$\begin{tikzcd} & e^x \arrow[r, leftrightarrow] & \ln x \\ \text{D:} & R \arrow[dr, dash] & (0,\infty) \arrow[dl, dash] \\ \text{R:} & (0,\infty) & R \end{tikzcd}$$

$$\begin{tikzcd} \begin{matrix} D_f = R_{f-1} \\ R_f = D_{f^{-1}} \end{matrix} \arrow[r] & \begin{matrix} D_{e^x} = R_{\ln x} \\ R_{e^x} = D_{\ln x} \end{matrix} \end{tikzcd}$$

وصلت ؟!

$$\boxed{\begin{matrix} D_f = R_{f^{-1}} \\ R_f = D_{f^{-1}} \end{matrix}}$$

مهم
جداً

How to find inverse:

1) write the equation as *y* = ... *y* = بكتب الاقتران على شكل
2) we replace between *x* a ببدل كل x ب y والعكس
3) we make y along side
4) at last, $y = f^{-1}(x)$

Ex: Find $f^{-1}(x)$ if:

1) $f(x) = (4x-2)^3 + 3$ **2) $f(x) = x^3 + 1$**

ببدل بينهم

$$y = (4x-2)^3 + 3$$

$$x = (4y-2)^3 + 3$$

$$x - 3 = (4y-2)^3$$

$$\sqrt[3]{x-3} = 4y - 2$$

$$\sqrt[3]{x-3} + 2 = 4y$$

$$y = \frac{\sqrt[3]{x-3}+2}{4}$$

$$f^{-1}(x) = \frac{\sqrt[3]{x-3}+2}{4}$$

$$y = x^3 + 1$$

$$x = y^3 + 1$$

$$y^3 = x - 1$$

$$y = \sqrt[3]{x-1}$$

$$f^{-1}(x) = \sqrt[3]{x-1}$$

3) $f(x) = \frac{2x+1}{x-1}$

$$y = \frac{2x+1}{x-1}$$

$$\frac{x}{1} = \frac{2y+1}{y-1}$$

$$xy - x = 2y + 1$$
$$xy - 2y = 1 + x$$
$$y(x-2) = 1 + x$$

$$y = \frac{1+x}{x-2} \longrightarrow f^{-1}(x) = \frac{1+x}{x-2}$$

***Find $f^{-1}(4)$** بالتجربة

$$f^{-1}(4) = 3$$

4) $\boldsymbol{f\,(\,x\,) = \frac{e^x}{3-e^x}}$

$$y = \frac{e^x}{3-e^x} \xrightarrow{\qquad\qquad} \frac{1}{x} = \frac{e^x}{3-e^x}$$

$$3x - xe^y = e^y$$
$$e^y + xe^y = 3y$$
$$e^y\,(\,1 + x\,) = 3y$$
$$e^y = \frac{3y}{1+x}$$
$$y = \ln\left(\frac{3x}{1+x}\right)$$
$$f^{-1}(\,x\,) = \ln\left(\frac{3x}{1+x}\right)$$

5) $\boldsymbol{f(\,x\,) = x^3 + 3x - 2}$

$$y = x^2 + 3x - 2$$
$$x = y^2 + 3y - 2$$

ضل علي اخلي ال y لحال
فبلجأ لإكمال المربع

Complete square:

$$y = ax^2 + bx + c$$

$$*d = (\tfrac{b}{2})^2$$

$$y = ax^2 + bx + d - d + c$$

⇨ $x = y^2 + 3y - 2$ $\qquad$ $(\frac{3}{2})^2 = \frac{9}{4}$

$$x = y^2 + 3y + \frac{9}{4} - \frac{9}{4} - 2$$

$$x = (\,y + \frac{3}{2})^2 - \frac{17}{4}$$

$$\left(y + \frac{3}{2}\right)^2 = x + \frac{17}{4}$$

$$y + \frac{3}{2} = \sqrt{x + \frac{17}{4}}$$

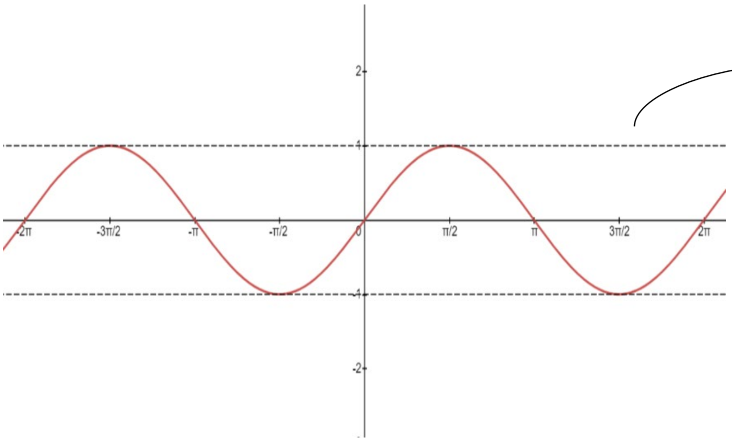
$$y = \sqrt{x + \frac{17}{4}} - \frac{3}{2} \xrightarrow{\qquad\qquad} f^{-1}(\,x\,) = \sqrt{x + \frac{17}{4}} - \frac{3}{2}$$

*Inverse trigonometric functions:

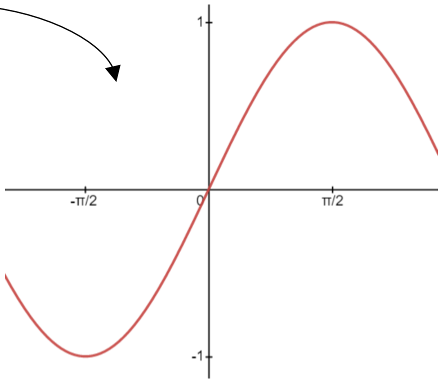
1) $\sin(x)$

*ال sin مش one to one

فقصينا منه فترة عشان يصير 1 to 1



isn`t one to one



one to one

	$\sin x$	$\sin^{-1} x$
D	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[\,1,1\,]$
R	$[\,1,1\,]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

D = $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

R = $[\,1,1\,]$

*Rule:

$$\boxed{\begin{array}{ll} f^{-1}(f(x)) = x \quad , & x \in D_f \\ f(f^{-1}(x)) = x \quad , & x \in D_{f^{-1}} \end{array}}$$

تذكر :

$$D_f = R_{f^{-1}}$$

$$R_f = D_{f^{-1}}$$

*قانون مهم جداً المختصر انه لما يجي إقتران و عكسه مركبين ببعض بشطبوا بعض بس بشرط مهم :

إنه قيمة ال y تكون داخلة ضمن مجال الإقتران الداخلي و لقدام رح نحل أمثلة عليها و نوضحها + شرحها موجود بالفيديو هات

Ex: $f(x) = e^x \quad , \quad g(x) = \ln x$

1) $f(g(x)) = e^{\ln x} = x \, , x \in \underset{D_g}{(0,\infty)}$

2) $g(f(x)) = \ln e^x = x \, , x \in \underset{D_f}{R}$

$* \ \sin^{-1}\sin X = X \qquad X \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

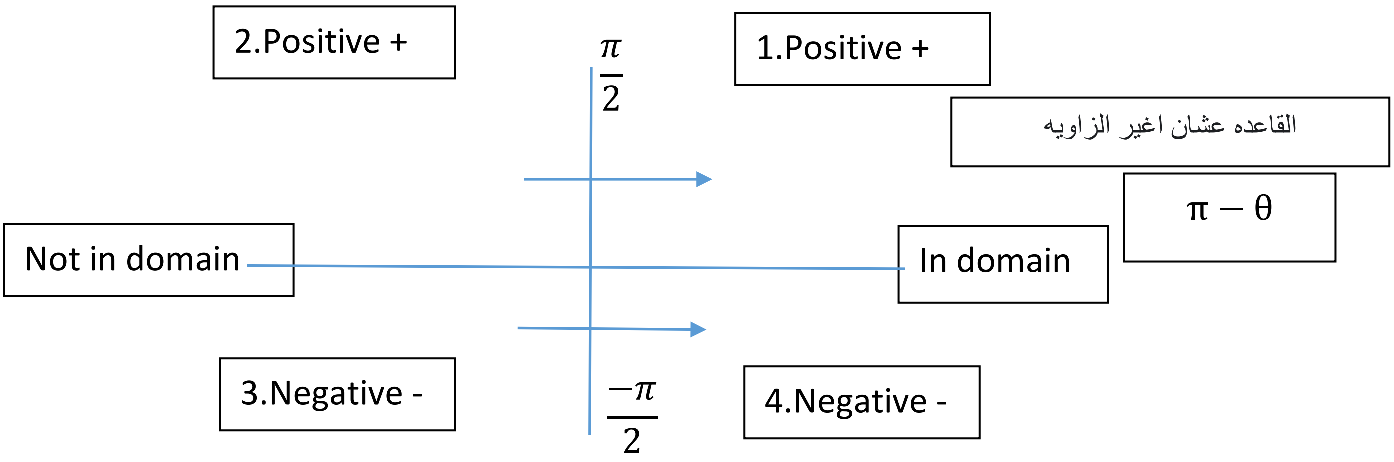
داخل مجال

$\sin X$

$* \ \sin\sin^{-1} X = X \qquad X \in [-1,1]$

داخل مجال

$\sin^{-1} X$



ملاحظات على الرسمة:

1-ال sin موجب بالربع الاول و الثاني.

2-كل الزوايا لازم تكون داخل مجال ال sin الي هو بين الربع الاول و الرابع.

3-اي زاويه خارجهم برجعها للمجال بعمليه π − θ

Examples:

1.$\sin^{-1}\sin\frac{5\pi}{6} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$

هون الزاوية $\frac{5\pi}{6}$ بالربع الثاني فبرجعها للاول

2.$\sin^{-1}\sin\frac{\pi}{4} = \frac{\pi}{4}$

داخل المجال

3.$\sin^{-1}\sin\frac{3\pi}{2} = \pi - \frac{3\pi}{2} = \frac{-\pi}{2}$

$4.\sin^{-1}\sin\frac{7\pi}{6} = \pi - \frac{7\pi}{6} = \frac{-\pi}{6}$

موجوده ضمن المجال $[\frac{-\pi}{2},\frac{\pi}{2}]$

$5.\sin^{-1}\sin\frac{11\pi}{6} = \frac{11\pi}{6} - 2\pi = \frac{-\pi}{6}$

بالربع الرابع

هي فعليا داخل المجال بس بدي اغير بصورتها عشان تصير جوا

$[\frac{-\pi}{2},\frac{\pi}{2}]$

$6.\sin^{-1}\sin\frac{8\pi}{5} = \frac{8\pi}{5} - 2\pi = \frac{-2\pi}{5}$

نفس الي قبلها

$7.\sin^{-1}\sin\frac{13\pi}{4}$

الزاويه اكبر من 2π, فبضلني اطرح منها 2π, عبين ما تصير ضمن الدوره الواحده, بعدين بطبق عليها القانون π − θ

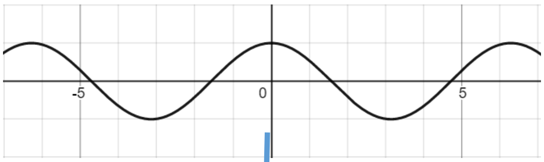
$$* \frac{13\pi}{4} > 2\pi$$

$$\frac{13\pi}{4} - 2\pi = \frac{5\pi}{4} < 2\pi$$

لو كانت اكبر من 2π, بضل اطرح

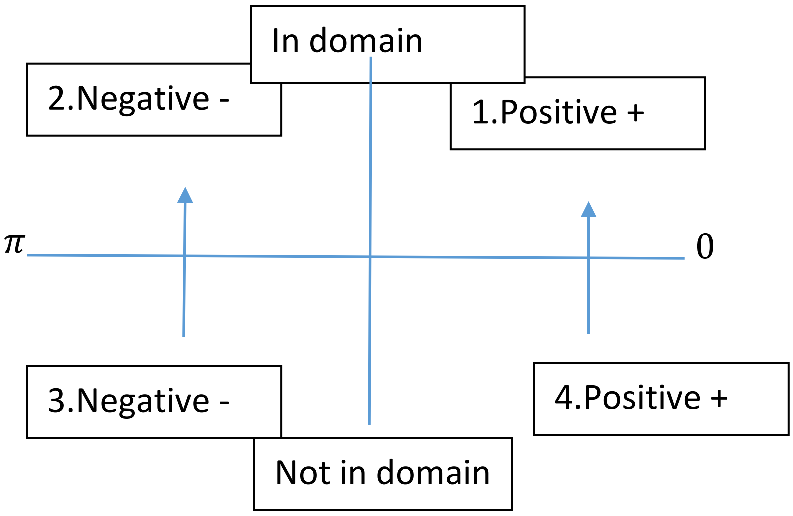
$$\pi - \frac{5\pi}{4} = \frac{-\pi}{4}$$

2. $\cos X$



Not one to one

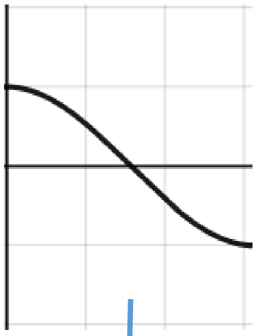
	$\cos X$	$\cos^{-1} X$
Domain	[0,π]	[-1,1]
Range	[-1,1]	[0,π]



$$\cos^{-1} \cos X = X \,, X \in [0, \pi]$$

$$\cos \cos^{-1} X = X \,, X \in [-1,1]$$

ال $\cos$ مش one-to-one , فقصينا منه لعبين ما صار one-to-one.



One to one

D=[0,π]

R=[-1,1]

القاعده عشان اغير الزاويه

$2\pi - \theta$

ملاحظات على الرسمة:

1-ال $\cos$ موجب بالربع الاول و الرابع.

2-كل الزوايا لازم تكون داخل مجال ال $\cos$ الي هو بين الربع الاول و الثاني.

3-اي زاويه خارجهم برجعها للمجال بعمليه $2\pi - \theta$

$1. \cos^{-1}\cos\frac{\pi}{3}=\frac{\pi}{3}$

بالربع الاول

$2. \cos^{-1}\cos\frac{5\pi}{6}=\frac{5\pi}{6}$

بالربع الثاني

$3. \cos^{-1}\cos\frac{7\pi}{6}=2\pi-\frac{7\pi}{6}=\frac{5\pi}{6}$

بالربع الثالث

$4. \cos^{-1}\cos\frac{5\pi}{3}=2\pi-\frac{5\pi}{3}=\frac{\pi}{3}$

بالربع الثالث

$5. \cos^{-1}\cos\frac{3\pi}{2}=2\pi-\frac{3\pi}{2}=\frac{\pi}{2}$

$6. \cos^{-1}\cos\frac{13\pi}{5}=2\pi-\frac{13\pi}{5}=\frac{3\pi}{5}$

عشان ارجع ضمن الدورة الاولى بطبق القاعده

اكبر من 2π

$\cos^{-1}\cos\frac{3\pi}{5}=\frac{3\pi}{5}$

$7. \cos\cos^{-1}\pi\neq\pi$

لان الπ خارج مجال الاقتران الداخلي ($\cos^{-1}$)

Domain($\cos^{-1}$) = [-1,1]

Π > 1

$\cos\cos^{-1}\pi\ is\ undefined$

$7. \cos\cos^{-1}5\neq 5$

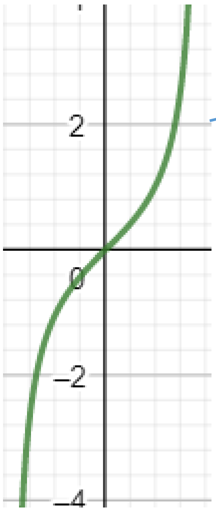
لان ال5 خارج مجال الاقتران الداخلي ($\cos^{-1}$)

Domain($\cos^{-1}$) = [-1,1]

5 > 1

$\cos\cos^{-1}5\ is\ undefined$

3. $\tan X$



One-to-one

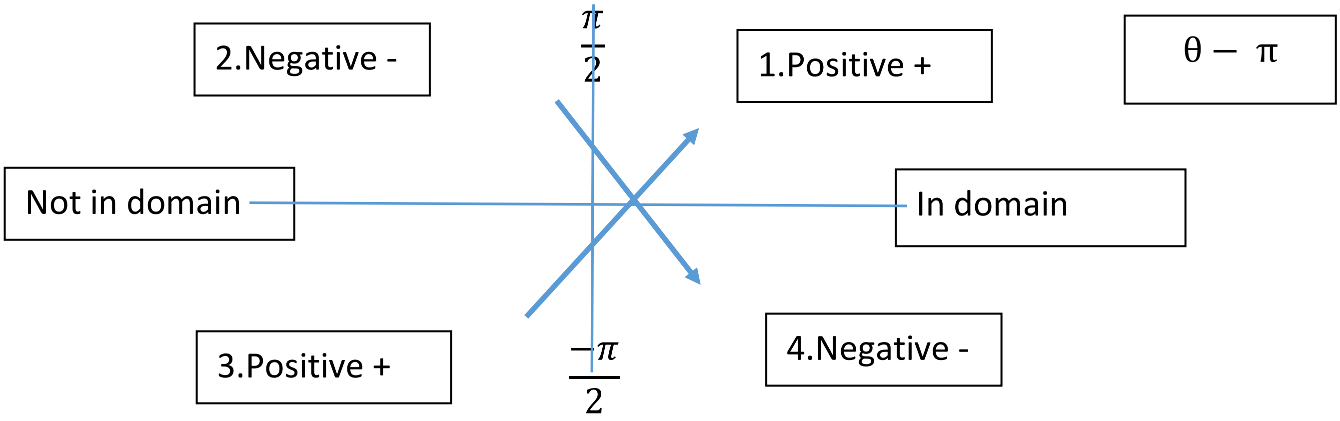
(-∞,∞)	$\tan X$	$\tan^{-1} X$
Domain	$(-\frac{\pi}{2},\frac{\pi}{2})$	(-∞,∞)
Range	(-∞,∞)	$(-\frac{\pi}{2},\frac{\pi}{2})$

كلهم فترات مفتوحه

$$\tan^{-1} \tan X, X \in (-\frac{\pi}{2},\frac{\pi}{2})$$

$$\tan \tan^{-1} X = X, X \in (-\infty,\infty)$$

القاعده عشان اغير الزاويه



ملاحظات على الرسمه:

1-ال tan موجب بالربع الاول و الثالث.

2-كل الزوايا لازم تكون داخل مجال ال tan الي هو بين الربع الاول و الرابع.

3-اي زاويه خارجهم برجعها للمجال بعمليه $\theta - \pi$

$$1.\ \tan^{-1}\tan\frac{\pi}{7}=\frac{\pi}{7}$$

$$2.\ \tan^{-1}\tan\frac{5\pi}{6}=\frac{5\pi}{6}-\pi=\frac{-\pi}{6}$$

$$3.\ \tan^{-1}\tan\frac{4\pi}{4}=\frac{4\pi}{3}-\pi=\frac{\pi}{3}$$

$$4.\ \tan^{-1}\tan\frac{11\pi}{6}=\frac{11\pi}{6}-2\pi=\frac{-\pi}{6}$$

موجوده بالربع الرابع و لكن مش داخل $(-\frac{\pi}{2},\frac{\pi}{2})$, فلازم ارجعها دوره كامله.

$$5.\ \tan^{-1}\tan\frac{11\pi}{6}=\frac{11\pi}{6}-2\pi=\frac{-\pi}{6}$$

$$6.\ \tan^{-1}\tan\frac{11\pi}{4}=\frac{11\pi}{4}-2\pi=\frac{3\pi}{4}-\pi=\frac{-\pi}{4}$$

بنرجعها للدوره الاولى, بعدين بنستخدم متطابقه

$$7.\ \tan^{-1}\tan 3=3, 3\ \in Domain(\tan^{-1}x)$$

Notes:

1.$\sin^{-1}X$ = arcsinc(X)

2.$\cos^{-1}X$ = arccos(X)

3.$\tan^{-1}X$ = arctan(X)

Examples:

Evaluate:

1.$\tan^{-1}(-1) = \frac{3\pi}{4}$

2.$\sin \sin^{-1}\frac{1}{2} = \frac{1}{2}$

3. $\sin \sin^{-1} 3 = undefined$

4.$\tan \tan^{-1} 10 = 10$

-In the previous cases we solved the composite of a certain function and its inverse($\sin \sin^{-1} x, for\ example$), but what if we had the composite of a function and the inverse of another function ($\cos^{-1} \sin \theta\ , for\ example$).

It has a different way of solving.

بفرض الداخل ب y
بدخل معكوس الاقتران على الجهتين
برسم مثلث قائم وبحدد فيه الزاوية y
بحدد قيم أضلاع المثلث(فيثاغورس)
برجع للأصل وبطلع جواب مباشرة من المثلث

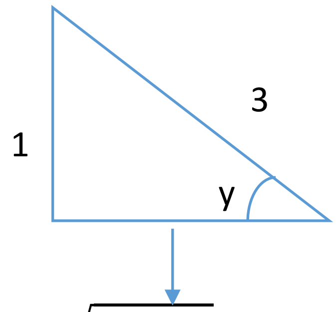
Example:
Evaluate:

1.$\tan \sin^{-1}\frac{1}{3}$

$y = \sin^{-1}\frac{1}{3}$

$\sin y = \frac{1}{3}$

$\tan y = \frac{1}{\sqrt{8}}$



$\sqrt{3^2 - 1^2}$

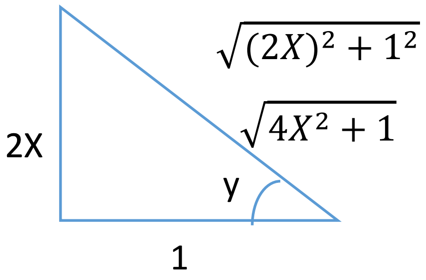
$\sqrt{8}$

2. $\cos \tan^{-1} 2X$

$y = \tan^{-1} 2X$

$\tan y = \frac{2X}{1}$

$\cos y = \frac{1}{\sqrt{4X^2+1}}$



3. $\sin 2 \sin^{-1} \frac{X}{2}$

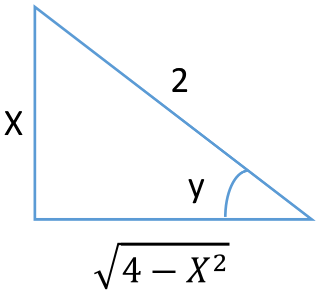
$y = \sin^{-1} \frac{X}{2}$

$\sin 2y$

$2 \sin y \cos y$

$2(\frac{X}{2})(\frac{\sqrt{4-X^2}}{2})$

$\frac{X\sqrt{4 - X^2}}{2}$



***Extra Examples**

1. Find domain of $f(X) = \frac{\sin\frac{1}{\pi}}{\sin X}$ $\longrightarrow$ Constant

Domain(f(X)= R – (اصفار المقام))

D(f(X))=R-(sinX=0)

D(f(X))= R- (nπ, n = 0,1,2,3)

2. Find range of f(X)= $1 - 2 \sin \frac{X^2-1}{e^{\cos x}}$

$-1 \leq \sin(anything) \leq 1$

$2 \geq -2\sin(anything) \geq -2$

$3 \geq 1 - 2\sin(anything) \geq -1$

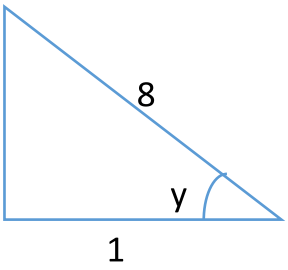
$3 \geq f(X) \geq -1$

Range(f(X)) = [3,-1]

3. $\sec \cos^{-1} \frac{1}{8}$

$y = \cos^{-1} \frac{1}{8}$

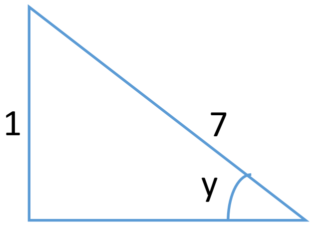
$\sec y = \frac{8}{1}$



4.$\csc \sin^{-1} \frac{1}{7}$

$y = \sin^{-1} \frac{1}{7}$

$\csc y = \frac{7}{1}$



CHAPTER ‘2’

Limits and Derivatives

Limits and derivatives

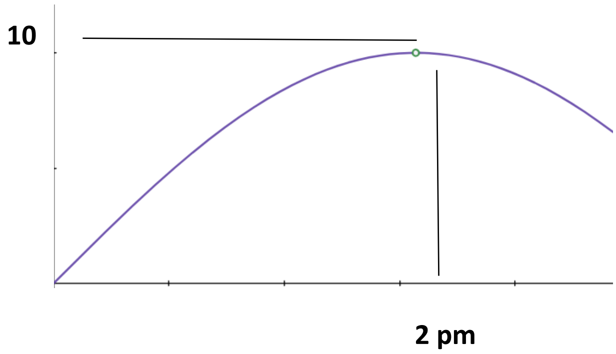
Def:- $\lim_{x \to a} f(x)$ **= L , means:**

As x approaches a, then f(x) approaches L

ال Lim هي النهاية التي نعرفها ونعرف أنها مختصة بالاقترانات وأنها لا تعطيني الجواب الصحيح 100% ولكن تقترب من العدد لدرجة كبيرة جدا (بمعنى أن الجواب تقريبي)

***from where limits come? (من أين جاءت النهايات؟)**

سوف نضع مثال توضيحي بسيط في حياتنا العملية



لنفترض أن الاقتران أعلاه هو سلوك تغير سعر سهم لشركة معينة في البورصة

كلنا يعرف أن السعر يزيد تارة وينقص تارة .

في يوم من الأيام عند الساعة 2 عصرا انقطعت الكهرباء لمدة دقيقة ثم عادت بعدها

فأصبح لدينا هنا مشكلة في ذلك الوقت, أما السؤال الان كم كان سعر السهم عند الساعة 2؟

صدق أو لا تصدق ولكن نعم سيكون الحل بالنهايات

ال 2من من تقريبي جواب بأخذ فنقوم

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 10\,JD$$

Example: - consider the function $f(x) = \frac{x}{\sqrt{(x+4)}-2}$ examine the values of the function when x is near 0?

$X \longrightarrow 0^-$ $\qquad\qquad$ $X \longrightarrow 0^+$

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
F(x)	3.9975	3.9998	3.99998	undefined	4.000025	4.00025	4.0025

$F(x) \longrightarrow 4$ $\qquad\qquad$ $F(x) \longrightarrow 4$

نلاحظ أنه عند الاقتراب من الصفر من اليمين واليسار كل ما اقتربنا من الجواب 4

$$\left.\begin{array}{l}\lim_{x\to 0^+} f(x) = 4\\[1em] \lim_{x\to 0^-} f(x) = 4\end{array}\right\} \longrightarrow$$

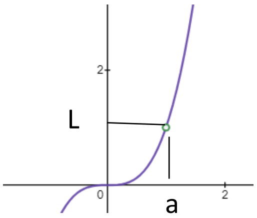
The limit equals 4

Remark: note that f(0) is undefined $\frac{0}{0}$ when dealing with limits, we are examining values as x approaches a, but not equal to a

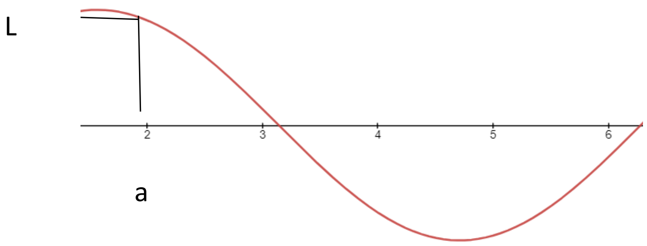
***if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \rightarrow \lim_{x\to a} f(x) = L\ (exists)$**

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x) \rightarrow \lim_{x\to a} f(x)\ doesn't\ exist$$

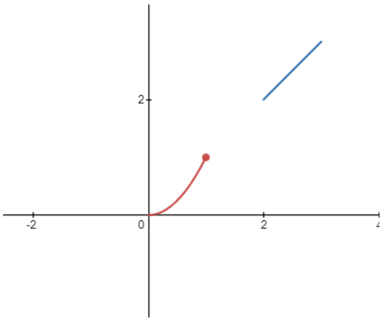
Limit form graph



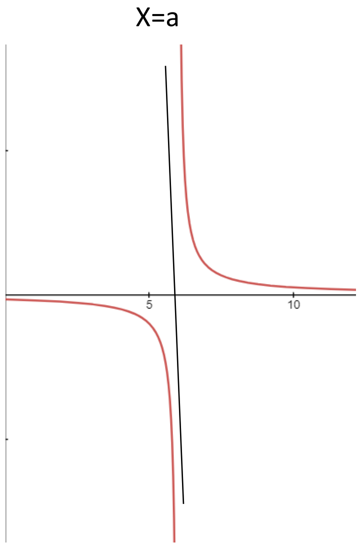
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \quad \lim_{x\to a} f(x) = L\ (exists)$$



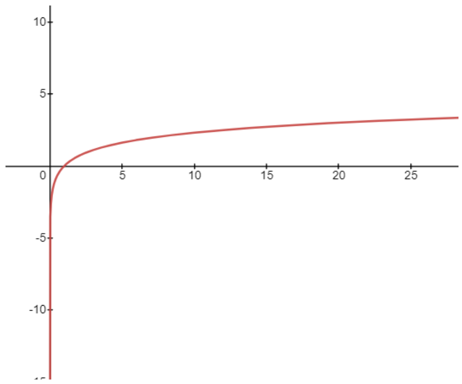
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \quad \lim_{x\to a} f(x) = L\ (exists)$$



$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x) => the\ limit\ D.N.E$$



$$\left.\begin{array}{l}\lim\limits_{x\to a^{+}} h(x) = \infty \\ \lim\limits_{x\to a^{-}} h(x) = -\infty\end{array}\right\} \text{the limit D.N.E}$$



$$\lim_{x\to 0^{+}} f(x) = -\infty$$

$$\lim_{x\to 0^{-}} f(x) = D.N.E$$

***أحبائي طلاب كالكولاس 1, أطلب من حضرتكم أن تفتحوا كتبكم على السؤال الخامس في صفحة 92 لكي نجيب على الاسئلة المجاورة للرسمة

Q5:

Ans:

a) $\lim_{x \to 3^-} f(x) = 1$

b) $\lim_{x \to 3^+} f(x) = 4$

c) $\lim_{x \to 3} f(x)\ D.N.E$

d) F(3)=3

Rules of limits

1) $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2) $\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$

3) $\lim_{x \to a}[f(x) * g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$

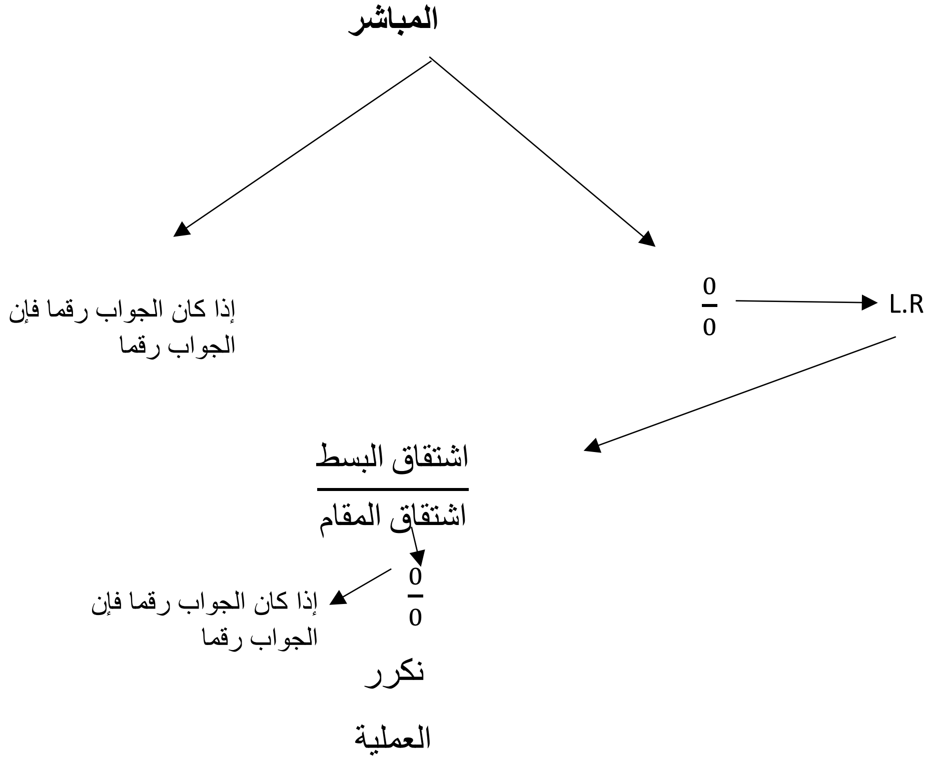
4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $g(x) \neq 0$

5) $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n, n\ is\ a\ positive\ integer$

المختصر:-

النهاية تدخل كل مكان دون أن تأثر على شيء حتى أنها لا تؤثر على الجذور والزوايا

For all limits → **الاساس بالنهاية هو التعويض المباشر**



Evaluate:

1) $\lim_{x\to1}\frac{x^2-4}{x-2}=\frac{-3}{2}$

2) $\lim_{x\to2}\frac{x^2-4}{x-2}=\frac{0}{0}\to L.R\to\frac{2x}{1}\to4$

Or solving it with tawjihi methods if you know what I mean 😉

3) $\lim_{x\to0}\frac{(3+x)^2-9}{x}=\frac{0}{0}\to L.R\to6$

Or solving it with tawjihi methods if you know what I mean 😉

4) $\lim\limits_{x\to 2}\frac{\sqrt{(6-x)}-2}{\sqrt{(3-x)}-1}$ ينصح بعدم استعمال لوبيتال في حالة الجذور

$$\lim_{x\to 2}\frac{\sqrt{(6-x)}-2}{\sqrt{3-x}-1}*\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}*\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}=\frac{1}{2}$$

5) $\lim\limits_{x\to 0}|x| \longrightarrow f(x)\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$

$$\lim_{x\to 0^+} x = 0,\ \lim_{x\to 0^-} -x = 0 \to \lim_{x\to 0}|x| = 0$$

6) $\lim\limits_{x\to 4} f(x) \quad , f(x)\begin{cases} \sqrt{x-4} & x>4 \\ 8-2x & x<4 \end{cases}$

$$\lim_{x\to 4^+}\sqrt{x-4}=0$$

$$\lim_{x\to 4^-} 8-2x = 0$$

$$\lim_{x\to 4} f(x) = 0$$

7) $\lim\limits_{x\to 0}\frac{|x|}{x} \qquad |x|\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$

$$\lim_{x\to 0^+}\frac{x}{x}=1$$

$$\lim_{x\to 0^-}\frac{-x}{x}=-1$$

D.N.E

8) $\lim\limits_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\frac{0}{0}\to L.R$

Answer is $\frac{-1}{9}$ او حلها بتوحيد المقامات 😉

9) $\lim_{x\to -6} \frac{2x+12}{|x+6|} = \frac{0}{0}$

لا تقوم بإعادة التعريف إلا عندما يكون جواب ما داخل المطلق صفر

$\lim_{x\to -6^+} \frac{2x+12}{x+6} = 2$ **the limit D.N.E**

$\lim_{x\to -6^-} \frac{2x+12}{-(x+6)} = -2$

10) $g(x)\{\begin{matrix} x^3 & x \geq 0 \\ 2x-1 & x < 0 \end{matrix}$

$a) \lim_{x\to 2} g(x) = 8$

$b) \lim_{x\to 0} g(x)\{\begin{matrix} \lim_{x\to 0^+} x^3 = 0 \\ \lim_{x\to 0^-} 2x-1 = -1 \end{matrix}$ **(D.N.E)**

11) $t(x)\{\begin{matrix} \frac{x^3+8}{x+2} & x \neq 2 \\ 1 & x = 2 \end{matrix}$

- **a) t (2) =1**
- **b) the limit equals 12**

*) The infinity (∞) $\longrightarrow$ مالا نهاية

$$\frac{\text{any number}}{0} = \pm\infty \quad , \quad \frac{2}{0} = \infty \quad , \quad \frac{-2}{0} = -\infty$$

$$\frac{\sqrt{3}}{0} = \infty \quad , \quad \frac{-\sqrt{2}}{0} = -\infty$$

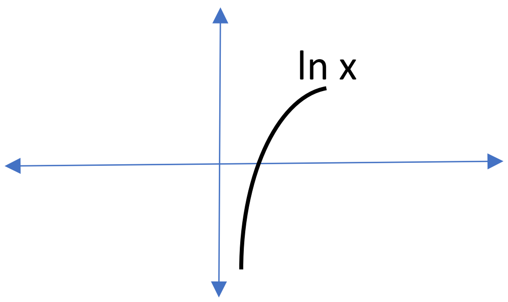
➡ $\frac{c}{0^+} = \infty \quad , \quad \frac{c}{0^-} = -\infty$

EX : Evaluate :

1) $\lim\limits_{x \to 1^+} \frac{2}{x-1} = \frac{2}{0^+} = +\infty$

2) $\lim\limits_{x \to 1^-} \frac{2}{x-1} = \frac{2}{0^-} = -\infty$

3) $\lim\limits_{x \to 1^+} \frac{-2}{x-1} = \frac{-2}{0^+} = -\infty$



4) $\lim\limits_{x \to 0^+} \ln x = \ln 0^+ = -\infty$

5) $\lim\limits_{x \to 0^-} \ln x = \text{undefined}$

6) $\lim\limits_{x \to -2} \frac{2x-12}{|x+2|} = \frac{-16}{0} = -\infty$

<u>Limit at infinity</u> هذه فقط مقدمة والتفاصيل في درس

*) Indeterminate forms : $\longrightarrow$ المقادير الي بواجه عندها مشكلة

$$\frac{\mathbf{0}}{\mathbf{0}}\ ,\ \frac{\infty}{\infty}\ ,\ \infty-\infty\ ,\ \mathbf{0}\ .\ \infty\ ,\ \mathbf{1}^{\infty}\ ,\ \mathbf{0}^{\mathbf{0}}\ ,\ \infty^{\mathbf{0}}$$

<u>but :</u>

*) $\frac{\infty}{0}=\infty$

*) $\frac{0}{\infty}=0$

*) $\frac{2}{\infty}=0$

*) $\frac{\infty}{2}=\infty$

*) $\infty+\infty=\infty$

*) $-\infty-\infty=-\infty$

*) $-\infty+\infty$? ? **مشكلة**

*) $\infty-\infty$? ? مشكلة

*) The squeeze theorem : (Sandwich)

If g(x) ≤ f(x) ≤ h(x)

and $\lim_{x\to a} \mathrm{g}(x) = L$ **and** $\lim_{x\to a} \mathrm{h}(x) = L$

so ⟶ $\lim_{x\to a} \mathrm{f}(x) = L$

*) إذا عندي اقتران محصور بين اقترانين وكانت النهايات للإقترانات موجودة ونفس الرقم يعني نهاية الرقم الأوسط موجودة

*) Note: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

*) Note : $\lim_{x\to 0} \frac{\tan x}{x} = 1$

*) معظم حالات هاي الطريقة تستخدم على الإقترانات المثلثية لما أحصرها بين [-1,1]

*) Note : $\sin\infty,\ \cos\infty = D.N.E$

Ex : $4x - 9 \le g(x) \le x^2 - 4x + 7$, using sandwich theorem to find $\lim_{x\to 4} g(x)$:

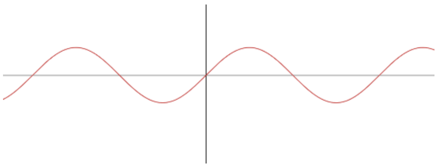
1)$\lim_{x\to 4} 4x - 9 = 16 - 9 = 7$

2)$\lim_{x\to 4} x^2 - 4x + 7 = 16 - 16 + 7 = 7$

$\lim_{x\to 4} g(x) = 7$

Ex : Evaluate the following

1) $\lim_{x\to 0} \sin\frac{1}{x} = \sin\frac{1}{0} = \sin\infty = \mathrm{D.N.E}$ Why ?



Sin عند المالانهاية ما بعرف كم قيمته ممكن يكون 1 / 1- / 0 / $\frac{-1}{2}$ / $\frac{1}{2}$ فالجواب D.N.E

2) $\lim_{x\to 0} x^2 .\ \sin(\frac{1}{x})$ = 0.∞ ??

$*\ x^2 \quad (1 \geq \sin(\frac{1}{x}) \geq -1)$

مهما كانت زاوية sin, cos دائما بحصرها بين [-1,1]

$x^2 \geq \mathrm{x}^2 \sin(\frac{1}{x}) \geq -x^2$ squeeze theorem

$$\left.\begin{array}{l} \to \lim_{x\to 0} x^2 \ = 0 \\ \to \lim_{x\to 0} -x^2 \ = 0 \end{array}\right] \longrightarrow \lim_{x\to 0} x^2 .\ \sin(\frac{1}{x}) = 0$$

by sandwich theorem

3) $\lim_{x\to 0} x^4 \cos\frac{2}{x}$ H.W (Ans = 0)

4) $\lim_{x\to 0} \ x^2 \sin^2\frac{1}{x}$ = 0.∞ ??

$1 \geq \sin\frac{1}{x} \ \geq -1$

$1 \geq \sin^2\frac{1}{x} \geq 0$

$x^2 \geq \sin^2\frac{1}{x} \geq 0$

$$\left.\begin{array}{l} \to \lim_{x\to 0} x^2 \ = 0 \\ \to \lim_{x\to 0} 0 \ \ = 0 \end{array}\right] \longrightarrow \lim_{x\to 0} \ x^2 \sin^2\frac{1}{x} = 0$$

by sandwich theorem

`

5) $\lim\limits_{x\to\infty} \frac{\sin x}{x}$

أول مرة بتشوفها عليها درس كامل لقدام بس عاملها كأنها تؤول الى عدد طبيعي

$$\lim_{x\to\infty} \frac{\sin x}{x} = \frac{\sin \infty}{\infty} \quad ??$$

$$1 \geq \sin x \geq -1$$

$$\frac{1}{x}$$

$$\frac{1}{x} \geq \frac{\sin x}{x} \geq \frac{-1}{x}$$

$$\left.\begin{aligned} &\to \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0 \\ &\to \lim_{x\to\infty} \frac{-1}{x} = \frac{-1}{\infty} = 0 \end{aligned}\right] \to \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Ex: $1 - \frac{x^2}{2} \leq f(x) \leq \cos x$, find $\lim\limits_{x\to 0} f(x)$

$$\left.\begin{aligned} &\to \lim_{x\to 0} 1 - \frac{x^2}{2} = 1 \\ &\to \lim_{x\to 0} \cos x = 1 \end{aligned}\right] \to \lim_{x\to\infty} f(x) = 1$$

Extra Example: مهم جدًا

$2|x| \leq g(x) \leq x^2 + 1$, find $\lim\limits_{x \to a} g(x)$, $\underline{\underline{\mathbf{x} > \mathbf{0}}}$ in which verifies the sandwich theorem

Sol:

*) النظرية تتحقق لما بالرسم الإقترانات تكون النهاية عند عدد معين الهم متساوية بمعنى آخر (لما يتقاطعوا).

$2|x| = x^2+1$

X = + or ─ , x > 0 (from question)

$2x = x^2+1$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$\boxed{X=1}$

عرفت وين النهاية تؤول

,so $2|x| \leq g(x) \leq x^2 + 1$

$$\lim_{x \to 1} 2|x| \leq \lim_{x \to 1} g(x) \leq \lim_{x \to 1} x^2 + 1$$

$$1 \leq \lim_{x \to 1} g(x) \leq 1$$

,so $\lim\limits_{x \to 1} g(x) = 1$

Limit of infinity

$$\lim_{x\to 3^+}\frac{2x}{x-3}$$

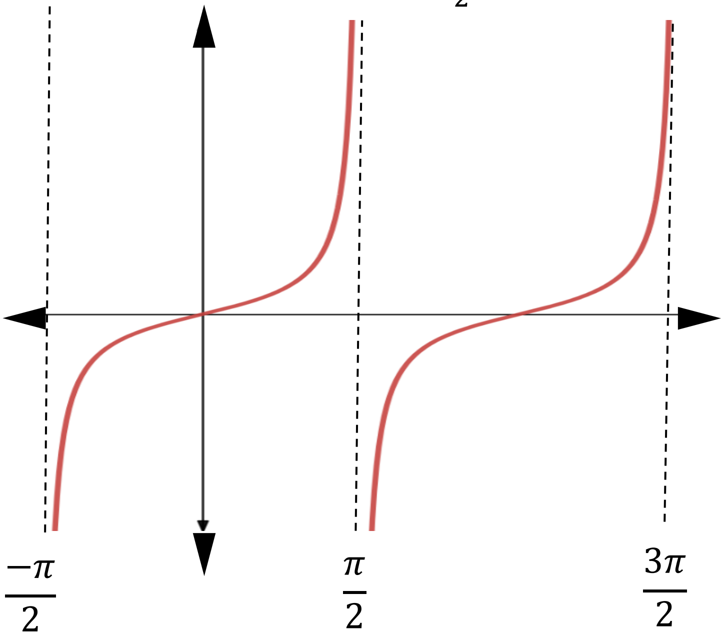
$$=\frac{6}{3.0001-3}=\frac{6}{0.0001}=\infty$$

$$\lim_{x\to 3^-}\frac{2x}{x-3}$$

$$=\frac{6}{2.9999-3}=\frac{6}{-0.0001}=-\infty$$

*) $\lim_{x\to \frac{\pi}{2}^+}\tan x=-\infty$

*) $\lim_{x\to \frac{\pi}{2}^-}\tan x=\infty$



*) New type of limits when $\boxed{x\to\infty}$ بعاملها كأنها نهاية عادية لكن لها طرق حل خاصة

Ex: Evaluate :

$$1)\lim_{x\to\infty}\frac{\sqrt{3x^4+2x^2-1}}{\underbrace{x(x-1)}_{x^2-x}}=\frac{\infty}{\infty}$$

في حالة البسط على المقام بأخذ أكبر قوة بالبسط مع أكبر قوة بالمقام

$$\to\lim_{x\to\infty}\frac{\sqrt{3x^4}}{x^2}\to\lim_{x\to\infty}\frac{\sqrt{3}\,x^2}{x^2}=\sqrt{3}$$

ما في داعي للمطلق عشان التربيع

2) $\lim\limits_{x\to\infty} \sqrt{x^2+3x} - x = \infty - \infty$?? هيك طبيعة أسئلة بضرب

بالمرافق

$$\lim_{x\to\infty} \sqrt{x^2+3x} - x \;*\; \frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}$$

$$\lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x} \qquad \frac{\text{أعلى قوة}}{\text{أعلى قوة}}$$

$$\lim_{x\to\infty} \frac{3x}{\underline{|x|}+x} \longrightarrow$$ بما أن النهاية تؤول الى موجب ∞ $$\longrightarrow \lim_{x\to\infty} \frac{3x}{+2x} = \frac{3}{2}$$

أوعك تنساه

3) $\lim\limits_{x\to\infty} \dfrac{7x^2+4x-3}{2x^2-12x+13} = \dfrac{\infty}{\infty}$ $\quad \dfrac{\text{أعلى قوة}}{\text{أعلى قوة}} \quad \lim\limits_{x\to\infty} \dfrac{7\cancel{x^2}}{2\cancel{x^2}} = \dfrac{7}{2}$

4) $\lim\limits_{x\to\infty} \sqrt{x^2+4x} + x \longrightarrow$ ما بصير أعمل مرافق قبل ما أعوض $\longrightarrow = \infty + \infty = \infty$

$\therefore \lim\limits_{x\to\pm\infty} f(x)$ called horizontal asymptote we will discuss later

*)Another type of infinity limits is :

$$\boxed{\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a}$$

Ex : Evaluate

1) $\lim\limits_{x\to\infty} \left(1+\frac{2}{x}\right)^x = e^2$

2) $\lim\limits_{x\to\infty} \left(1-\frac{3}{x}\right)^x = \left(1+\frac{-3}{x}\right)^x = e^{-3}$

3) $\lim_{x\to\infty}\left(1+\frac{-3n}{4x}\right)^{x}$

$$=\lim_{x\to\infty}\left(1+\frac{\frac{-3n}{4}}{x}\right)^{x} = e^{\frac{-3n}{4}}$$

4) $\lim_{x\to\infty}\left(\frac{x-ln2}{x}\right)^{x}$

أي اشي قوة x بنحل بالقاعدة ∴

يوزع المقام على البسط

$$\lim_{x\to\infty}\left(1-\frac{ln2}{x}\right)^{x} = e^{-ln2} = e^{ln2^{-1}} = \cancel{e}^{\cancel{ln}\,2^{-1}}=2^{-1} = \frac{1}{2}$$

5) $\lim_{x\to\infty}\left(\frac{1-\frac{2}{3x}}{1+\frac{2}{-x}}\right)^{x}$

remember: $\left(\frac{9}{6}\right)^{c} = \frac{9^{c}}{6^{c}}$

$$\lim_{x\to\infty}\frac{\left(1+\frac{\frac{-2}{3}}{x}\right)^{x}}{\left(1+\frac{-2}{x}\right)^{x}} = \frac{e^{-\frac{2}{3}}}{e^{-2}} = e^{\frac{4}{3}}$$

6) $\lim_{x\to\infty}\left(1-\frac{3}{2x}\right)^{6x+1}$? ?

$\lim_{x\to\infty}\left(1-\frac{3}{2x}\right)^{6x} . \lim_{x\to\infty}\left(1-\frac{3}{2x}\right)^{1}$, remember: $a^{b.c} = [a^{b}]^{c}$

$$\lim_{x\to\infty}\left(\left(1-\frac{\frac{3}{2}}{x}\right)^{x}\right)^{6} . \underline{\lim_{x\to\infty}\left(1-\frac{3}{2x}\right)}$$

$\frac{1}{\infty} = 0$

تعويض مباشر

$$=\left(e^{\frac{-3}{2}}\right)^{6} . \ (1-0) = e^{-9}$$

7) $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x}$

$$= \lim_{x\to\infty}\left(\left(\frac{\frac{2x-3}{2x}}{\frac{2x+5}{2x}}\right)^{x}\right)^{2}$$

$$= \lim_{x\to\infty}\left(\frac{\left(1-\frac{\frac{3}{2}}{x}\right)^{x}}{\left(1+\frac{\frac{5}{2}}{x}\right)^{x}}\right)^{2} = \left(\frac{e^{-\frac{3}{2}}}{e^{\frac{5}{2}}}\right)^{2} = \left(e^{-\frac{8}{2}}\right)^{2} = e^{-8}$$

*Remark

1) $f(x)$ is continuous $\Rightarrow f^{-1}(x)$ is continuous
2) $\sin(x)\,,\cos(x)$ are cont. $\Rightarrow \sin^{-1}(x)\,,\cos^{-1}(x)$ are cont.
3) If $f(x)$ is continuous then $\lim\limits_{x\to a}(f\circ g)\,(x) = \lim\limits_{x\to a} f\big(g(x)\big) = f\left(\lim\limits_{x\to a} g(x)\right)$

Ex: Evaluate $\lim\limits_{x\to 1}\sin^{-1}\left(\frac{\sqrt{x}-1}{x-1}\right)$.

$$= \sin^{-1}\left(\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)\right)$$

مرافق

$x-1 = (\sqrt{x}-1)\cdot(\sqrt{x}+1)$

$$= \sin^{-1}\left(\lim_{x\to 1}\left(\frac{\cancel{\sqrt{x}-1}}{\cancel{(\sqrt{x}-1)}\cdot(\sqrt{x}+1)}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Examples for trigonometric limits:

Evaluate the following limits:

1) $\lim\limits_{x\to 0}\frac{sinx}{sin\pi x}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{\frac{sinx}{x}}{\frac{sin\pi x}{x}}=\frac{1}{\pi}$$

---

2) $\lim\limits_{x\to -3}\frac{\sin(\mathrm{x}^2+4x+3)}{x+3}=\frac{0}{0}$

$$\lim_{x\to -3}\frac{\sin\left((\mathrm{x}+3)\cdot(\mathrm{x}+1)\right)}{x+3}\cdot\frac{x+1}{x+1}$$

الهدف الوصول لنفس الزاوية

$$\lim_{x\to -3}(x+1)\cdot\lim_{x\to -3}\frac{\sin(\mathrm{x}^2+4x+3)}{(\mathrm{x}^2+4x+3)}$$

$y=x^2+4x+3, y\to 0$

$$-2\cdot\lim_{y\to 0}\left(\frac{siny}{y}\right)$$

$$-2\cdot 1=-2$$

---

3) $\lim\limits_{x\to 0}x\cdot\cot\left(\frac{x}{3}\right)\Rightarrow\lim\limits_{x\to 0}\frac{x}{\tan\left(\frac{x}{3}\right)}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{\frac{x}{x}}{\frac{\tan\left(\frac{x}{3}\right)}{x}}=\frac{1}{\frac{1}{3}}=3$$

---

4) $\lim\limits_{x\to 0}\frac{\sin^2 x}{1-\cos(x)}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{1-cosx}\Rightarrow\lim_{x\to 0}\frac{\cancel{(1-cosx)}\cdot(1+cosx)}{\cancel{(1-cosx)}}=2$$

---

5) $\lim\limits_{x\to 0}\frac{2sinx}{x+tanx}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{\frac{2sinx}{x}}{\frac{x}{x}+\frac{tanx}{x}}=\frac{2}{1+1}=1$$

---

6) $\lim\limits_{x\to\frac{\pi}{2}}\frac{cosx-\frac{1}{2}sin2x}{1-sinx}=\frac{0}{0}$

$* \ sin2x = 2sinx\cdot cosx$

$$\lim_{x\to\frac{\pi}{2}}\frac{(cosx-sinx\cdot cosx)}{1-sinx}\Rightarrow\lim_{x\to\frac{\pi}{2}}\frac{cosx\cdot\cancel{(1-sinx)}}{\cancel{(1-sinx)}}=0$$

---

7) $\lim\limits_{x\to 0}\frac{sinx}{x^3}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{sinx}{x}\cdot\frac{1}{x^2}=1\cdot\frac{1}{0}=D.N.E$$

---

8) $\lim\limits_{x\to 0}\frac{\sec^2 x-\tan^2 x-1}{x^2}=\frac{0}{0}$

$* \ \sec^2 x-\tan^2 x=1$

$$\lim_{x\to 0}\frac{1-1}{x^2}\Rightarrow\lim_{x\to 0}0=0$$

Some examples for limits:

Ex 1: If a and b are real numbers such that $\lim\limits_{x\to 0}\frac{\sqrt{ax+b}-2}{x} = 1$, find a and b .

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0\ const. \qquad ,and\ \lim_{x\to 0} g(x) = 0$$

So, $\lim\limits_{x\to 0} f(x) = 0 \ \Rightarrow\ \lim\limits_{x\to 0}\sqrt{ax+b} - 2 = 0$

$$\sqrt{b} = 2 \Rightarrow \boldsymbol{b = 4}$$

$$\Rightarrow\ \lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x} = 1$$

$$\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x} = 1$$

$$\lim_{x\to 0}\sqrt{ax+4} - 2 = x$$

$$\sqrt{ax+4} = x+2 \ \Rightarrow \left(\sqrt{ax+4}\right)^2 = (x+2\,)^2$$

$$\underbrace{ax} + 4 = x^2 + \underbrace{4x} + 4$$

$$\boldsymbol{a = 4, b = 4}$$

Continuity

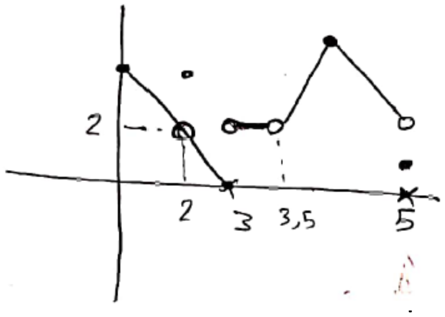
❖ Discontinuity types

1) Removable → when $f(a) \neq \lim\limits_{x\to a} f(x)$, but $\lim\limits_{x\to a} f(x) = constant$.
2) Non-removable → when $f(a) \neq \lim\limits_{x\to a} f(x)$, but $\lim\limits_{x\to a} f(x) = \pm\infty$ (infinity discontinuity).

   Non-removable → when $f(a) \neq \lim\limits_{x\to a} f(x)$, but $\lim\limits_{x\to a} f(x) = D.N.E$ (jump discontinuity).

Ex: As shown in the graph discuss the continuity of the following functions.

$f(x) \rightarrow$

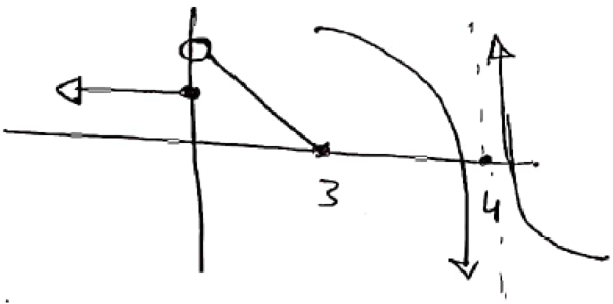


$f(x)$ is c $x = 5, 3.5, 3, 2$

$f(x)$ has a removable discontinuity at $x = 5, 2, 3.5$

$f(x)$ has a non-removable discontinuity at $x = 3$, (jump discontinuity)

$f(x) \rightarrow$



$f(x)$ is continuous in its domain except at $x = 0, 3, 4$

$f(x)$ hasn't any removable discontinuity

$f(x)$ has a non-removable discontinuity at $x = 4, 3, 0$

$x = 4 \rightarrow$ infinity and jump discontinuity, $x = 3, 0 \rightarrow$ jump discontinuity

*We can know the discontinuity from algebraic formula of the function such that

1) $f(x) = \frac{x^2-x}{x-1}$ → the discontinuity point is at $x = 1$

because $f(1)$ is an undefined value

*The limit of $f(x)$ when x approaches to zero equals 1, but $f(1)$ is undefined value

2) $f(x) = \frac{x^2+6x+9}{x^2+6x+9}$ → the discontinuity point is at $x = -3$

because $f(-3)$ is not defined value

* $\lim_{x \to -3} f(x) = 1$, but $f(1)$ is undefined value

3) $f(x) = \frac{x}{1-|x|}$ → discontinuity points are $x = 1\ and\ x = -1$

and so on

---

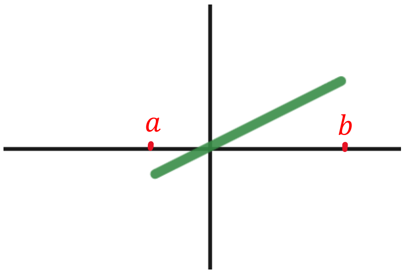
2. One sided continuity

*If $f(x) = \lim_{x \to c^-} f(x)$ and equal real number ⇒ left side continuity

*If $f(x) = \lim_{x \to c^+} f(x)$ and equal real number ⇒ right side continuity

as shown in the graph

$f(x) \rightarrow$



$f(a) = \lim_{x \to a^+} f(x)$ so $f(x)$ is continuous at $x = a$ from the right

$f(b) = \lim_{x \to b^-} f(x)$ so $f(x)$ is continuous at $x = b$ from the left

*إذا طلب السؤال البحث في اتصال الاقتران على مجاله:
الاقتراحات متصلة على مجالها.

*اذا طلب السؤال البحث في اتصاله على فترة:
نستخدم نظرية الاتصالات جهة واحدة للأطراف.

*إذا طلب السؤال البحث في اتصال الأطراف كنقاط:
الاقتران غير متصل عند الأطراف المختلفة.

*$f(x) = \sqrt{4 - x^2}$ →Domain→[-2,2]→continuous at [-2,2]

*$f(x) = \ In\ (3 + x)$ →Domain→[-3,∞]→continuous at [-3,∞]

*Show that $f(x) = \ \sqrt{9 - x^2}$ is continuous at [0,3].
It is continuous because the interval [0,3] is included in the domain.

f(0) = 3
$\lim_{x \to 0^+} f(x) = 3$

f(x) is continuous at x=0 from the right.

f(0) = 3
$\lim_{x \to 0^-} f(x) = 3$

f(x) is continuous at x=0 from the left.

So f(x) is continuous at [0,3].

*Show that $f(x) = \ \sqrt{9 - x^2}$ has a discontinuity at x= ±3.

$\lim_{x \to 3^+} f(x)$→Does Not Exist →Limit Does Not Exist → discontinuity at x =3.

$\lim_{x \to -3^-} f(x)$→Does Not Exist →Limit Does Not Exist → discontinuity at x =-3.

Extra Exercises

$$\mathrm{f(x)} = \begin{cases} \frac{\sin(ax)^2}{4x^2} & , \ x > 0 \\ 4 & , \ x \leq 0 \end{cases}$$

Find the value of a that makes f(x) continuous at x= 0.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^+} f(x) = \frac{a^2}{4} \ , f(0) = 4 \, , \frac{a^2}{4} = 4 \, , a = \pm 4$$

$$\mathrm{f(x)} = \begin{cases} \sin x + 5a & , \ x < \frac{\pi}{2} \\ 11 & , \ x = \frac{\pi}{2} \\ a + b & , \ x > \frac{\pi}{2} \end{cases}$$

Find the values of a and b that make f(x) continuous.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = 1 + 5a \ , f(x) = 11 \ \to \ 1 + 5a = 11 \to a = 2$$

$$\lim_{x \to 0^+} f(x) = 2 + b \ , f(x) = 11 \ \to \ 2 + b = 11 \ \to \ b = 9$$

Asymptotes → Horizontal Asymptotes H.A.

Asymptotes → Vertical Asymptotes V.A.

Asymptotes → Slant

خطوط التقارب

1.Verical Asymptotes (خطوط التقارب العاموديه):

X=a is a V.A. for f(x) if at least one of the following happens.

$$\lim_{x\to a} f(x) = \pm\infty \text{ OR } \lim_{x\to a^+} f(x) = \pm\infty \text{ OR } \lim_{x\to a^-} f(x) = \pm\infty$$

بالمختصر هو قيم x الي بتخلي الاقتران عنده مشاكل و بتعطيني جواب±∞ يعني مثلا اصفار نقام أو أماكن غير معرفة.

Steps:

1.We find the values that make the denominator equal zero. (اصفار مقام)

2.Check the values by limit. (بتاكد من جوابها بالنهايه).

Example: Find the V.A. for the following

a) $f(x) = \frac{1}{x}$

x=0 → Check

$$\lim_{x\to 0} f(x) = \frac{1}{0} = \infty$$

So, x=0 is a V.A.

b) $f(x) = \frac{\pi}{x-2}$

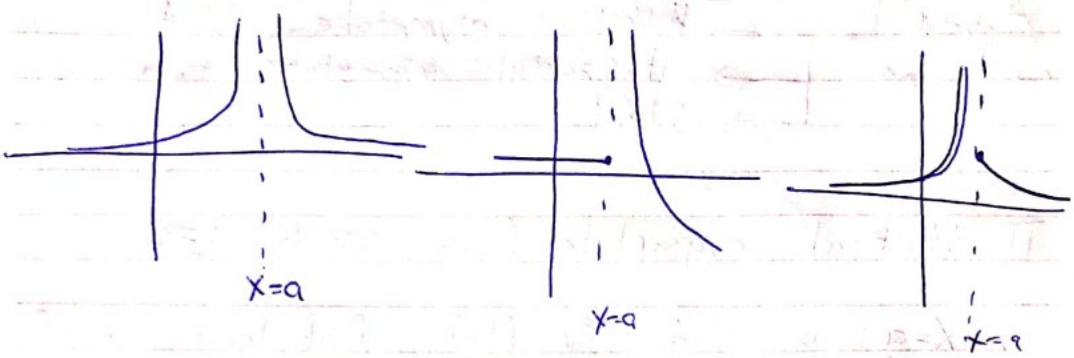
x=2 → Check

$$\lim_{x\to 2} f(x) = \frac{1}{0} = \infty$$

So, x=2 is a V.A.

٢

التوضيح:



لـ جميع ال ٣ حالات $X_1=a \leftarrow$ V.A

c) $f(x)=\frac{x-1}{x^2-1}$ → $x^2-1=0$ → x =1,-1

$$x=1, \lim_{x\to1} f(x)=\frac{0}{0}\to \lim_{x\to1}\frac{x-1}{(x-1)(x+1)}\to \lim_{x\to1}\frac{1}{(x+1)}\to\frac{1}{2}$$

So not V.A.

$$x=-1, \lim_{x\to-1} f(x)=\frac{-2}{0}=\infty=\pm\infty$$

So V.A.

V.A. only at x= -1

d) $f(x)=\frac{x^2-1}{x-1}$ → x= 1

$$x=1, \lim_{x\to1} f(x)=\frac{0}{0}\to \lim_{x\to1}\frac{(x-1)(x+1)}{x-1}\to \lim_{x\to1}\frac{(x+1)}{1}\to\frac{2}{1}$$

So, not V.A.

No V.A.

e) $f(x)=\tan x \ \to\ \tan x=\frac{\sin x}{\cos x}\ \to \cos x=0$

$$x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}=(2n-1)\frac{\pi}{2}$$

$$x=\frac{\pi}{2}, \lim_{x\to\frac{\pi}{2}}\tan x=\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos x}=\frac{1}{0}=\infty=\pm\infty, V.A.$$

$$x=(2n-1)\frac{\pi}{2}\ are\ V.A. for\ all\ integers\ n.$$

f) $f(x)=\ In\ (x-3)\to x=3$

$$x=3, \lim_{x\to3^+} In\ (x-3)=\lim_{x\to3^+} In\ (0^+)=\ \infty=\pm\infty, V.A.$$

بدي انتبه انه الاقتران هون معرف عند ال3 بس من اليمين.

g) $f(x)=\ In\ (2x+1)\to x=\frac{-1}{2}$

$$x=\frac{-1}{2}, \lim_{x\to\frac{-1}{2}^+} In\ (2x+1)=\lim_{x\to\frac{-1}{2}^+} In\ (0^+)=\ \infty=\pm\infty, V.A.$$

h) $f(x)=\frac{\sin x}{x}\to x=0$

$$x=0\ , \lim_{x\to0}\frac{\sin x}{x}=1\neq\pm\infty, no\ \ V.A$$

i) $f(x) = \frac{\cos x}{x} \to x = 0$

$$x = 0\,,\ \lim_{x \to 0} \frac{\cos x}{x} = \infty = \pm\infty, V.A\ at\ x = 0$$

Find V.A. for:

$$f(x) = \left\{ \begin{array}{ll} 2, & x \geq 1 \\ \frac{1}{1-x}, & x < 1 \end{array} \right\}$$

$$x = 1,\ \lim_{x \to 1^+} 2 = 2$$

$$x = 1,\ \lim_{x \to 1^-} \frac{1}{1-x} = \frac{1}{0^+} = \infty, so\ V.A.at\ x = 1$$

$$f(x) = \frac{\sin(x-1)}{x^2 - 1}\ , x = -1{,}1$$

$$x = 1,\ \lim_{x \to 1} f(x) = \frac{0}{0} \to \lim_{x \to 1} \frac{\sin(x-1)}{(x-1)(x+1)} \to \lim_{x \to 1} \frac{1}{(x+1)} \to \frac{1}{2}$$

$$x = -1,\ \lim_{x \to -1} f(x) = \frac{\sin -2}{0} = -\infty = \pm\infty$$

So, V.A. at x= -1

Limit at infinity.

2.Horizontal Asymptotes (خطوط التقارب الافقيه):

X=a is a H.A. for f(x) if at least one of the following happens.

$$\lim_{x \to \infty} f(x) = \text{ a OR } \lim_{x \to -\infty} f(x) = \text{ a, where a is a constant.}$$

*Note: If want to find H.A., you **must** find

$$\lim_{x \to \pm\infty} f(x)$$

Reminder:

V.A: x=a

H.A: y=a

Example: Find the H.A. for the following

a) $f(x) = \frac{2}{x+1}$

$$\lim_{x \to \pm\infty} f(x) = \frac{2}{\pm\infty} = 0$$

So, H.A. at y=0.

b) $f(x) = \tan^{-1} x$

$$x = \infty \quad \lim_{x \to \infty} \tan^{-1} x = \tan^{-1} \infty = \frac{\pi}{2}$$

$$x = -\infty \quad \lim_{x \to -\infty} \tan^{-1} x = \tan^{-1} -\infty = \frac{-\pi}{2}$$

So H.A at y = $\frac{\pi}{2}, \frac{-\pi}{2}$.

c) $f(x) = \frac{3x^{100}+100}{2x^{100}+x}$

$$\lim_{x \to \pm\infty} f(x) = \frac{\pm\infty}{\pm\infty} : \frac{\text{اعلى قوه}}{\text{اعلى قوه}}$$

$$\lim_{x \to \pm\infty} \frac{3x^{100}}{2x^{100}} = \frac{3}{2}$$

So, H.A at y = $\frac{3}{2}$

d) $f(x) = e^x$

$$x = \infty \quad \lim_{x \to \infty} e^x = e^\infty = \infty, no\ H.A$$

$$x = -\infty \quad \lim_{x \to -\infty} e^x = e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0, so\ H.A. at\ y = 0$$

e) $f(x) = \sqrt{x^2+1} - x$

$$\lim_{x \to \pm\infty} \sqrt{x^2+1} - x * \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \to \lim_{x \to \pm\infty} \frac{1}{\sqrt{x^2+1}+x}$$

$$\lim_{x \to \pm\infty} \frac{1}{|x|+x}$$

$$\lim_{x \to \infty} \frac{1}{x+x} = \frac{1}{\infty} = 0$$

$$\lim_{x \to -\infty} \frac{1}{-x+x} = \frac{1}{0} = \infty$$

H.A at y = 0

f) $f(x)=\frac{1-e^x}{1+2e^x}$

$$\lim_{x\to\infty}\frac{1-e^x}{1+2e^x}=\frac{-\infty}{\infty}\ undefined$$

$$\lim_{x\to\infty}\frac{e^x}{2e^x}\to\lim_{x\to\infty}\frac{-1}{2}=\frac{-1}{2},H.A.at\ y=\frac{-1}{2}$$

$$\lim_{x\to-\infty}\frac{1-e^x}{1+2e^x}=\frac{1}{1}=1,H.A.at\ y=1$$

g) $f(x)=\frac{\sqrt{2x^2+1}}{3x-5}$

$$\lim_{x\to\pm\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\pm\infty}{\pm\infty}\ undefined\to\lim_{x\to\pm\infty}\frac{\sqrt{2}\,|x|}{3x}$$

$$\lim_{x\to\infty}\frac{\sqrt{2}\,x}{3x}\to\lim_{x\to\infty}\frac{\sqrt{2}}{3}=\frac{\sqrt{2}}{3},H.A.at\ y=\frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty}\frac{\sqrt{2}-x}{3x}\to\lim_{x\to\infty}\frac{-\sqrt{2}}{3}=\frac{-\sqrt{2}}{3},H.A.at\ y=\frac{-\sqrt{2}}{3}$$

3.Slant Asymptotes (S.A.)

خط تقريب يفيد التقريب في الرسم في الاقترانات النسبيه

$$\left(\frac{Polynomial}{Polynomial}\right)$$

هذا الدرس ذاكره الكتاب في
الوحده الرابعه, لكن اضفناه مع
خطوط التقارب الاخرى.

باختصار بتعمل قسمه طويله و جواب القسمه بيكون هى الS.A, بيكون على صيغه

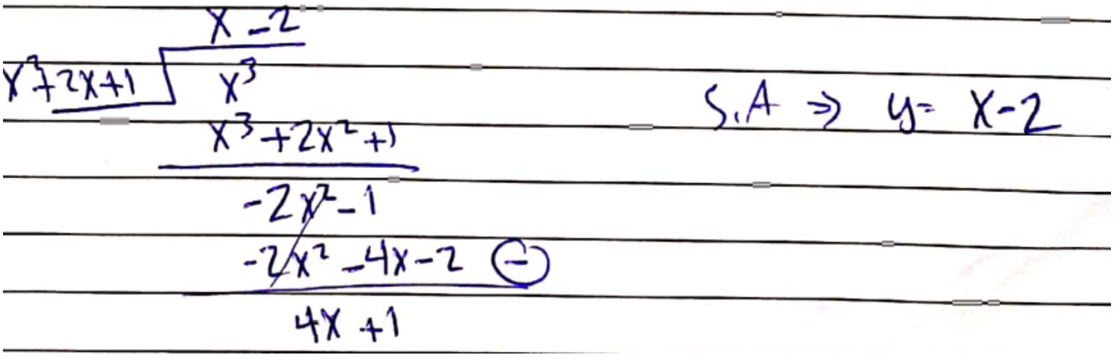
y=mx + b

1) $f(x)=\frac{x^3}{x^2+1}$

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & x^3+1 \quad ⊖ \\ \hline & -1 \to stop \end{array}$$

S.A ⇒ y=x

2) $f(x) = \frac{x^3}{(x+1)^2} = \frac{x^3}{x^2+2x+1}$



$x - 2$

$x^2+2x+1 \overline{) \; x^3}$

x^3+2x^2+1

$-2x^2-1$

$-2x^2-4x-2$ ⊖

$4x+1$

S.A ⇒ $y = x-2$

CHAPTER ‘3’

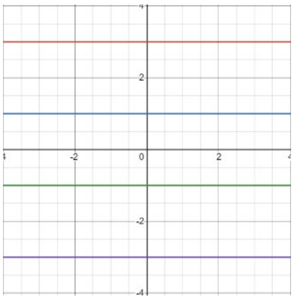
Differentiation Rules

Derivatives of Polynomials and Exponential Functions

1-Constant Function f(x) = c →Real Number

$f'(x) = 0$ or $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 \ = 0$$



Slope of tangent:

Limit = $\frac{\Delta y}{\Delta x}$ = $\frac{0}{\Delta x}$ = 0

2-Power Function $f(x) \ = \ x^n$, where n is a real number .

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n-(x)^n}{h}$$

$$= \lim_{h\to 0}\frac{(h+x-x)((h+x)^{n-1}) + (h+x)^{n-2}x) + \cdots + (x^{n-1}))}{h}$$

عدد الحدود :n, قيمة كل حد: x^{n-1}

$$= nx^{n-1}$$

Example $f(x) = x^3$, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-(x)^3}{h}$

$$= \lim_{h\to 0}\frac{x^3+3hx^2+3h^2x+h^3-x^3}{h} = \lim_{h\to 0} 3x^2+3hx+h^2 = 3x^2$$

3-Exponential Function $f(x) \ = \ a^{g(x)}$, where a is a positive real number .

$$f'(x) = g'(x)a^{g(x)}\ln(a)$$

Special Case (when a = e):

$$f'(x) = g'(x)e^{g(x)}\ln(e)$$

$$f'(x) = g'(x)e^{g(x)}$$

Find $f'(x)$ for:

1)f(x) $= e^{x^3}$, $f'(x) = 3x^2e^{x^3}$

2))f(x) $= e^{\sqrt{x}}$, $f'(x) = \frac{d}{dx}(\sqrt{x})(e^{\sqrt{x}}) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

4-The constant multiple rule. $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$, where f(x) is differentiable .

Example: $f(x) = 2x^2,\ f'(x) = \frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2*2x = 4x$

5-The sum rule. $\frac{d}{dx}(g(x) + f(x)) = \frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x))$, where f(x) and g(x) are differentiable .

Example: $f(x) = e^x + x^2,\ f'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\ e^x) = 2x + \ e^x$

5-The difference rule. $\frac{d}{dx}(g(x) - f(x)) = \frac{d}{dx}(g(x)) - \frac{d}{dx}(f(x))$, where f(x) and g(x) are differentiable .

Example: $f(x) = x^2 - \frac{1}{x},\ f'(x) = \frac{d}{dx}(x^2) - \frac{d}{dx}\left(\frac{1}{x}\right) = 2x + \frac{1}{x^2}$

6-The product rule. $\frac{d}{dx}\Big(\big(g(x)\big) * \big(f(x)\big)\Big) =$

$\frac{d}{dx}(f(x)) * \big(g(x)\big) + \frac{d}{dx}(g(x)) * \big(f(x)\big)$, where f(x) and g(x) are differentiable.

Examples:

1. $f(x) = xe^x\ , f'(x) = \frac{d}{dx}(x)e^x + \frac{d}{dx}(e^x)x = e^x + e^x x$

2. $f(x) = (2x)^2 * (x + e^x)$

$$f'(x) = \frac{d}{dx}(2x)^2(x + e^x) + \frac{d}{dx}(x + e^x)(2x)^2$$

$$= 8x(x + e^x) + (1 + e^x)4x^2$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Prove that $\frac{d}{dx}(\sin x) = \cos x$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos(\frac{2x+h}{2})}{h} = \cos x$$

Prove that $\frac{d}{dx}(\cos 3x) = -3\sin 3x$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(3(x+h))-\cos(3x)}{h}$$

$$= -3\sin(3x)$$

Find the derivative of each function:

1) $f(x) = 3 \to f'(x) = 0$

2) $f(x) = x \to f'(x) = 1$

3) $f(x) = x^6 \to f'(x) = 6x^5$

4) $f(x) = \pi x^5 \to f'(x) = \pi 5x^4$

5) $f(x) = 2^x \to f'(x) = 2^x \ln(2)$

$6) f(x) = 2^{2x} \rightarrow f'(x) = 2^{x}(\ln(2))(2)$

$7) f(x) = e^{x+3} \rightarrow f'(x) = e^{x+3}$

$8) f(x) = \ln(x) \rightarrow f'(x) = \frac{1}{x}, x > 0, because\ Domain(\ln(x)): x > 0$

$9) f(x) = \ln(ax) \rightarrow f(x) = \ln(a) + \ln(x), \ln(a)\ is\ a\ constant.$

$f'(x) = \frac{1}{x}, x > 0.$

$10) f(x) = \operatorname{Iog}_2(\mathrm{x}) \rightarrow f'(x) = \frac{1}{xln(2)}, x > 0$

$11) f(x) = \ln(2) \operatorname{Iog}_2(\mathrm{x}) \rightarrow f'(x) = \frac{1}{xln(2)} * \ln(2) = \frac{1}{x}, x > 0$

$12) f(x) = \sin x^2 \rightarrow f(x) = \sin x * \sin x$

$f'(x) = \sin x \cos x + \cos x \sin x$

$f'(x) = 2\sin x \cos x$

$f'(x) = \sin 2x$

$13) f(x) = \sin x^3 \rightarrow f(x) = \sin x * \sin x^2$

$f'(x) = \sin x\,(\sin x^2)' + \cos x \sin x^2$

$f'(x) = \sin x\,(\sin x^2)' + \cos x \sin x^2$

$f'(x) = \sin x(2\sin x \cos x) + \cos x \sin x^2$

$f'(x) = 3\cos x \sin x^2$

$$14) f(x) = \frac{2 - \sec x^2}{1 + \tan x}$$

$$f(x) = \frac{2 - (1 + \tan x^2)}{1 + \tan x}$$

$$f(x) = \frac{(1 + \tan x)(1 + \tan x)}{1 + \tan x}$$

$$f(x) = 1 + \tan x$$

$$f'(x) = -\sec x^2$$

$$15) f(x) = \frac{x + 1}{x}$$

$$f'(x) = \frac{(x)(1) - (x + 1)(1)}{(x)^2}$$

$$f'(x) = \frac{-1}{(x)^2}$$

$$16) f(x) = \frac{\sec x^2}{1 + \tan x^2}$$

$$f(x) = \frac{\sec x^2}{\sec x^2} = 1$$

$$f'(x) = 0, \sec x \ \neq 0$$

$$17) f(x) = \ln(\sec x + \tan x)$$

$$f'(x) = \frac{\sec x \tan x + \sec x^2}{\sec x + \tan x} = \sec x \, , \sec x \neq \ \tan x$$

$$18) f(x) = \mathrm{xln}(x)$$

$$f'(x) = (x) * \left(\frac{1}{x}\right) + (1)(\ln(x))$$

$$f'(x) = 1 + (\ln(x))$$

$$19)Find\ \lim_{x\to1}\frac{x^{1000}-1}{x-1}=\lim_{x\to1}\frac{x^{1000}-1^{1000}}{x-1}$$

$$f(x)=x^{1000}, f(1)=1^{1000}$$

$$=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=f'(1)$$

$$f(x)=x^{1000}\to f'(x)=1000x^{999}\to f'(1)=1000$$

$$20)Find\ \lim_{x\to0}\frac{e^x-1}{x}\to f(x)=e^x, f(0)=1$$

$$=\lim_{x\to0}\frac{f(x)-f(0)}{x}=f'(0)$$

$$f(x)=e^x, f'(x)=e^x, f'(0)=1$$

$$21)If\ \lim_{x\to1}\frac{(x^n-1)(x^{n-1}-1)(x^{n-2}-1)}{(x-1)^3}=24$$

Find the value of n where n is an integer.

$$\lim_{x\to1}\frac{(x^n-1)(x^{n-1}-1)(x^{n-2}-1)}{(x-1)^3}$$

$$=\boxed{\lim_{x\to1}\frac{(x^n-1)}{x-1}}*\boxed{\lim_{x\to1}\frac{(x^{n-1}-1)}{x-1}}*\boxed{\lim_{x\to1}\frac{(x^{n-2}-1)}{x-1}}$$

$$=\lim_{x\to1}\frac{x^n-1^n}{x-1}\to f(x)=x^n, f'(x)=nx^{n-1},$$

$$f'(1)=n$$

$$=\lim_{x\to1}\frac{x^{n-1}-1^{n-1}}{x-1}\to f(x)=x^{n-1}, f'(x)=(n-1)x^{n-2},$$

$$f'(1)=n-1$$

$$=\lim_{x\to1}\frac{x^{n-2}-1^{n-2}}{x-1}\to f(x)=x^{n-2}, f'(x)=(n-2)x^{n-3},$$

$$f'(1)=n-2$$

Therefore:

$(n)(n-1)(n-2)=24$

n=4

$22) If\ f(x+h)=f'(x+h), prove\ that\ f''(x)=f'(x)$

$$\lim_{h\to 0}\frac{f(x)-f(x+h)}{-h}=f'(x), thus: f'(x)=f(x)$$

$$\lim_{h\to 0}\frac{f'(x)-f''(x+h)}{-h}=f''(x), thus: f''(x)=f'(x)=f(x)$$

High Order Derivatives

Notations:

Where n is any integer greater than three.

$n \ni z, n > 3$

1)$f'(x), f''(x), f'''(x), f^n(x)$

2)y', y'', y''', y^n

3)$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^ny}{dx^n}$

$If\ f(x) = \sin x, show\ that\ f(x) = f^{(4)}(x)$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x = f(x)$

نلاحظ انه المشتقة الرابعة عادت إلى الشكل الأصلي للاقتران، و نلاحظ أنه سوف نعود للاقتران الأصلي كل ٤ مرات اشتقاقه.

$f^5(x) = \cos x$

$f^6(x) = -\sin x$

$f^7(x) = -\cos x$

$f^8(x) = \sin x = f^4(x)$

الآن في حالة وجود نمط اشتقاق كما في المثال السابق، و لكن في حال عدم عودته النمط سوف نحتاج إلى استخدام القسمة الطويلة و الباقي هو الذي يحدد.

Example:

$f^{31}(x) = ??$

7

4⟌31

28 -

$\boxed{3}$

اي هذا يعني $f^{31}(x)$ هي نفسها المشتقة الثالثة بعد $f^4(x)$.

$(((f^4(x))')')' = f^{31}(x)$

سيتم توضيح الاشتقاقات التي تحتوي على أنماط في الأمثلة القادمة.

$If\ f(x) = x^3 + x^2 + 1$ find $f'(x), f''(x), f'''(x), f^5(x).$

$f'(x) = 3x^2 + 2x, f''(x) = 6x + 2\ , f'''(x) = 6, f^5(x) = 0$

$If\ f(x) =\ x^n + c\ , f^4(x) = 120 *\ x^{(n-4)}$

Find f(x),n, c, if f(1) = 1.

$f^4(x) = n(n-1)(n-2)(n-3)x^{n-4} = 120x^{n-4}, n = 5$

$f(x) = x^5 + c,$ f(1) $=\ 1 =\ 1^5 + c \rightarrow c = 0$

$So, f(x) = x^5$

$If\ f(x) =\ x^n + x\ and\ f(5) = \frac{x^2}{16}\ l!\ Find\ n, f(x) and\ l.$

$f(5) = n(n-1)(n-2)(n-3)(n-4)x^{(n-5)} = \frac{x^2}{16}\ l!$

$(n-5) = 2 \rightarrow n = 7$

$\frac{l!}{16} =\ 7(7-1)(7-2)(7-3)(7-4)$

$l! = 8! \rightarrow l = 8$

الاشتقاقات التي تتضمن انماط و امثله عامه:

$If\ f(x) =\ x^3\ . Find\ f^3(x), f^4(x).$

$f'(x) = 3x^2, f''(x) = 6x, f'''(x) = 6, f^4(x) = 0.$

Therefore,

$If\ f(x) =\ x^n\ , f^h(x) = n!\ , f^{(h+1)}(x) = 0$

Example:

$If\ f(x) =\ x^{100} + x^{99} + x^{98}\ , find\ f^{99}(x)$

$f^{99}(x) = 100!\, x + 99! = 99!\,(100x + 1)$

If $f(x) =\ a \sin x - b \cos x\ \ and\ f'''(x) = -8 \sin x + 7 \cos x$

What is the value of: (a + b) and f(x)?

$f'''(x) = -b\sin x - a\cos x = -8\sin x + 7\cos x$

-b=-8→b=8, -a=7→ a=-7

(a + b) = (8-7) =1, $f(x) = -7\sin x + 8\cos x$

If $f(x) = b\sin x + b\cos x \quad and$

$f^4(x) = \sin x^{\,3} + \sin x^{\,2}\cos x + \sin x\cos x^2 + \cos x^3$

$f^4(x) = \sin x(\sin x^{\,2} + \cos x^2) + \cos x(\cos x^2 + \sin x^{\,2})$

$f^4(x) = \sin x + \cos x$

Therefore, b=1.

$If\ f(x) = xe^x, find\ f'''(x), f^{20}(x), f^{100}(x)$

$f'(x) = e^x + xe^x = e^x(1 + x)$

$f''(x) = e^x + e^x(1 + x) = e^x(2 + x)$

$f'''(x) = e^x + e^x(2 + x) = e^x(3 + x)$

نلاحظ انه النمط هو :

$f^{(u)}(x) = xe^x + ne^x$

$f^{20}(x) = xe^x + 20e^x$

$f^{100}(x) = xe^x + 100e^x$

If $f(x) = \sin x + \cos x \quad Find\ f^{140}(x), f^{139}(x), f^5(x), f^{18}(x)$

$f'(x) = \cos x - \sin x$

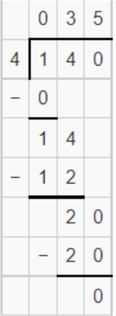
$f''(x) = -\cos x - \sin x$

$f'''(x) = \sin x - \cos x$

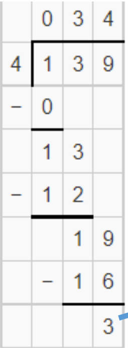
$f^4(x) = \sin x + \cos x$

نلاحظ انه النمط هو اننا نعود لل f(x), كل اربع مرات اشتقاق.

الباقي صفر, اي: $f^{140}(x) = f(x) = \sin x + \cos x$



	0	3	5
4	1	4	0
−	0		
	1	4	
−	1	2	
		2	0
	−	2	0
			0



	0	3	4
4	1	3	9
−	0		
	1	3	
−	1	2	
		1	9
	−	1	6
			3

الباقي ثلاث, اي: $f^{139}(x) = f'''(x) = \sin x - \cos x$

الباقي واحد, اي: $f^{5}(x) = f'(x) = \cos x - \sin x$

الباقي ثنين, اي: $f^{18}(x) = f''(x) = -\cos x - \sin x$

If $f(x)=e^{4x}$ → find: $f^{(5)}(x)$, $f^{(113)}(x)$

$\rightarrow f'(x) = 4e^{4x}$ $\quad \rightarrow f''(x) = 4^2e^{4x}$ $\quad \rightarrow f'''(x) = 4^3e^{4x}$

$f^{(4)}(x) = 4^{4} e^{4x}$

$f^{(5)}(x) = 4^5 e^{4x}$

$f^{(113)}(x) = 4^{113} e^{4x}$

نلاحظ أن النمط هو:

$$f^{(n)}(x) = 4^n e^{4x}$$

If f (x) = x sinx find: $f^{(91)}(x)$, $f^{(102)}(x)$

$\rightarrow f'(x) = xcosx - sinx$ $\quad \rightarrow f''(x) = 2cosx - xsinx$

$\rightarrow f'''(x) = -xcosx - 3sinx$ $\quad \rightarrow f^{(4)}(x) = -4cosx + xsinx$

$\rightarrow f^{(5)}(x) = xcosc + 5sinx$ $\quad \rightarrow f^{(6)}(x) = 6cosx - xsinx$

نلاحظ أن المشتقات الزوجية تكون على نمط:

$f^{(n)}(x) = ncosx - xsinx$ <u>or</u> $-(ncosx - xsinx)$

الان هذا يعني وجود $(-1)^n$ في النمط

$f^{(n)}(x) = (-1)^n(ncosx - xsinx)$ الان

الان يجب إيجاد علاقة $(-1)^n$ التي تحقق الشروط مثلا: $(-1)^{\frac{n+2}{2}}$

نعرفه بالتجربة

$\rightarrow \boldsymbol{f^{(n)}(x) = (-1)^{\left(\frac{n+2}{2}\right)}(ncosx - xsinx)}$

$f''(x) = (1)(2cosx - xsinx)$

$f^{(4)}(x) = (-1)(2cosx - xsinx)$ → $f^{(6)}(x) = (1)(2cosx - xsinx)$

ونلاحظ أن المشتقات الفردية تكون على نمط:

$f^{(n)}(x) = nsinx + xcosx$ **or** -($nsinx + xcosx$) → $(-1)^n$ وهذا يعني وجود نمط لِ

$f^{(n)}(x) = (-1)^{\square}(nsinx + xcosx)$ → مثلا

$$\boldsymbol{f^{(n)}(x) = (-1)^{\left(\frac{n-1}{2}\right)}(nsinx + xcosx)}$$

$$f^{(n)}(x) = \begin{cases} f^{(n)}(x) = (-1)^{\left(\frac{n+2}{2}\right)}(ncosx - xsinx) & \text{, when n → even integer} \\ f^{(n)}(x) = (-1)^{\left(\frac{n-1}{2}\right)}(nsinx + xcosx) & \text{, when n → odd integer} \end{cases}$$

$\rightarrow \boldsymbol{f^{(91)}(x) = (-1)(91sinx + xcosx)}$ $\quad \rightarrow \boldsymbol{f^{(102)}(x) = (1)(102sinx + xcosx)}$

If $\boldsymbol{f(x) = \frac{x^n}{a^2} - 4x^3 \rightarrow f'''(1) = \frac{60}{a^2} - 4}$ **,find the value of ‘a’ if** $\boldsymbol{f^{(5)}(1) = 30}$

Sol:

$$\rightarrow f'''(x) = \frac{1^{(n-3)}(n-1)(n-2)(n)}{a^2} - 4 = \frac{60}{a^2} - 4 \rightarrow n = 5$$

$$\rightarrow f'''(x) = \frac{x^2\ (60)}{a^2} - 4$$

$$\rightarrow f^{(4)}(x) = \frac{120x}{a^2}$$

$$\rightarrow f^{(5)}(x) = \frac{120}{a^2}$$

$$\rightarrow f^{(5)}(1) = 30 = \frac{120}{a^2}$$

$$\rightarrow a^2 = 4$$

$$\rightarrow a = \pm 2$$

The chain rule

تستخدم قاعدة السلسلة في حالات عدة منها:

1] وجود أكثر من علاقة تستطيع من خلالها الوصول الى علاقة بين متغيرين

$$\rightarrow \frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} \quad , \frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dn} * \frac{dn}{dx} , \ldots$$

2] وجود اقتران مركب←

$$f\big(g(x)\big) \rightarrow f'\big(g(x)\big)g'(x)$$

*) توضيح بسيط:

$$f(x) = x \ \rightarrow \ f'(x)dx = dx \ \rightarrow f'(x) = 1$$

في حالة وجود متغير واحد في المعادلة نشتق بدون مشاكل

لكن في حالة وجود أكثر من متغير مثلًا:

$$f(x) = u \quad \rightarrow \quad f'(x)\, dx = du$$

$$f'(x) = \frac{du}{dx}$$

(نحتاج الى ترتيب الحدود)

If $\boldsymbol{f(x) = sinx}$ $\rightarrow$ **find** $\boldsymbol{f'(x)}$

$f'(x)\ dx = \ cosx\ \ dx\ \ \rightarrow f'(x) = cosx$

$f(x) = sinu\ \rightarrow f'(x)dx = cosu\ du \rightarrow f'(x) = \cos(u)\ \frac{du}{dx}$

الآن 1] أمثلة متنوعة على العلاقة السابقة:

If $\boldsymbol{f(x) = sin\, x^2}$ $\rightarrow$ **find** $\boldsymbol{f^{'}(x)}$

$\mathrm{x}^2 = u\ \rightarrow\ 2x\ dx = du \rightarrow \frac{du}{dx} = 2x$

$f(x) = sinu\ \rightarrow\ f'(x)\ dx = cosu\ du\ \rightarrow f'(x) = cosu\ \frac{du}{dx} = \cos(x^2) * 2x$

If $\boldsymbol{f(x) = sin^3(x + sinx)}$ **find** $\boldsymbol{f'(x)}$

$\rightarrow$ u = (x+sinx) → du =dx(1+cosx) → $\frac{\mathrm{du}}{\mathrm{dx}} = 1 + cosx$

$f(x) = sin^3u$

$sinu = n \rightarrow du\ cosu = dn\ \rightarrow \frac{dn}{du} = cosu$

$f(x) = n^3 \rightarrow f'(x)dx = 3n^2\ dn \rightarrow f'(x) = 3n^2\ \boxed{\frac{dn}{dx}}$

$\rightarrow \frac{\mathrm{dn}}{\mathrm{dx}} = \frac{dn}{du} * \frac{du}{dx} = (cosu)(1 + cosx)$

$f'(x) = 3n^2(cosu)(1 + cosx)$

$f'(x) = 3sin^2u\ (cosu)(1 + cosx)$

$f'(x) = 3sin^2(x + sinx)\cos(x + sinx)\ \ (1 + cosx)$

Or

$f(x) = sin^3(u) = sinu * sin^2u$

$f'(x)\ dx = du(sinu)(2\ sinu\ \ cosu) + du\ \ cosu + sin^2u$

$f'(x)\ dx = 3du(sin^2u\ \ cosu) \rightarrow f'(x) = 3\frac{du}{dx}sin^2(x + sinx)\cos(x + sinx)$

$f'(x) = 3sin^2(x + sinx)\cos(x + sinx)((1 + cosx))$

Now if $f(x) = e^{g(x)}$ prove that $f'(x) = g'(x)\, e^{g(x)}$

$f(x) = e^{\,g(x)}$

$g(x) = u \rightarrow dx\, g'(x) = du \rightarrow \frac{du}{dx} = g'(x)$

$f(x) = e^{u} \rightarrow f'(x)dx = du\, e^{u} \rightarrow f'(x) = \frac{du}{dx} e^{u} = g'(x)\, e^{\,g(x)}$

Using chain rule show that if f(x)=$\ln g(x) \rightarrow f'(x) = \frac{g'(x)}{g(x)}$

$f(x) = \ln g(x) \rightarrow e^{f(x)} = g(x)$

$f'(x)\, \underbrace{e^{f(x)}}_{g(x)} = g'(x)$

$f'(x)\, g(x) = g'(x)$

$f'(x) = \frac{g'(x)}{g(x)}$

Using chain rule prove that if $f(x) = a^{g(x)}$, $a > 0$,

$$f'(x) = g'(x)\, a^{g(x)}\, ln(a)$$

$f(x) = a^{g(x)}$

$\log_a f(x) = g(x) \log_a a$

$\log_a f(x) = g(x)$

$\frac{\ln f(x)}{\underbrace{\ln a}_{\text{Constant}}} = g(x) \rightarrow \ln f(x) = g(x) \ln a$

$\frac{f'(x)}{f(x)} = g'(x)\ \ln a$

$f'(x) = g'(x)\, f(x)\ \ln a$

$f'(x) = g'(x)\ a^{g(x)}\ \ln a$

If $\boldsymbol{f(x) = sec^5\left(e^{3x^2-2}\right)}$ **find** $\boldsymbol{f'(x)}$

$$f'(x) = 5sec^4\left(e^{3x^2-2}\right) * \sec\left(e^{3x^2-2}\right) * tan\left(e^{3x^2-2}\right) * \left(e^{3x^2-2}\right) * (6x)$$

$$f'(x) = 30\,x\,e^{3x^2-2}\;sec^5\left(e^{3x^2-2}\right) tan\left(e^{3x^2-2}\right)$$

If $\boldsymbol{h(x) = \sqrt{4+3f(10x)}\ and\ f(10) = 7\,, f'(10) = 4}$

find $\boldsymbol{h'(1)}$

$$\acute{h}(x) = \frac{3 * f'(10x) * 10}{2\sqrt{4+3f(10x)}} \rightarrow h'(1) = \frac{30f'(10)}{2\sqrt{4+3f(10)}} = \frac{30*4}{2*\sqrt{25}} = \frac{120}{10} = 12$$

If $\boldsymbol{f(x) = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}}}$ **find** $\boldsymbol{f'(x)}$

$$f(x) = \sqrt{x+f(x)}$$

$$f^2(x) = x + f(x) \rightarrow f^2(x) - f(x) = x$$

$$2f'(x)f(x) - f'(x) = 1$$

$$f'(x) = \frac{1}{2f(x)-1}$$

If f(x)=$\boldsymbol{\sqrt{x^2+1}}$ **find**$\boldsymbol{f'(x)}$

$$f(x) = (x^2+1)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} * 2x * (x^2+1)^{\frac{-1}{2}}$$

$$f'(x) = \frac{x}{\sqrt{x^2+1}}$$

If $f(x) = e^{\sin(1+cosx)}$ find $f'(x)$

$$f'(x) = e^{\sin(1+cosx)} * \cos(1 + cosx) * -sinx$$

If $h(x) = \frac{x}{\sqrt{f(x)}}$, $f(x) > 0$, $h(x) \neq 0$

$and\ h(x) = 2h'(x)$ find $f'(x)$

$$\rightarrow f(x) = \frac{x^2}{h^2(x)} \rightarrow f'(x) = \frac{h^2(x)2x - x^2 * \overbrace{2\ h(x) * h'(x)}^{h(x)}}{h^4(x)}$$

$$\rightarrow f'(2) = \frac{h^2(2*4) - 4h^2(2)}{h^4(2)}$$

$$\rightarrow f'(2) = \frac{0}{h^2(2)} = 0$$

If $f(x) = \sin(e^{2x}\ sinx)$ find $f'(x)$

$$f'(x) = \cos(e^{2x}\ sinx)(e^{2x}cosx + 2e^{2x}sinx)$$

Now $\boldsymbol{f(x) = (x^2 + x + 1)^{\frac{-1}{3}}}$ find $\boldsymbol{f'(x)}$

u=$\mathrm{x}^2 + x + 1$ → $\frac{\mathrm{du}}{\mathrm{dx}}$ = 2x+1

$f(x) = u^{\frac{-1}{3}}$

$f'(x)\,dx = \frac{-1}{3} * u^{\frac{-4}{3}}\ du \ \rightarrow f'(x) = \frac{-1}{3} * u^{\frac{-4}{3}} * \frac{du}{dx}$

$f'(x) = \frac{-1}{3} * u^{\frac{-4}{3}} * (2x + 1)$

- **So that →** $\boldsymbol{\frac{\mathrm{d}}{\mathrm{dx}}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)g'(x)}$
- **So that →** $\boldsymbol{\frac{\mathrm{d}}{\mathrm{dx}}\left[\big(f(x)\big)^{n}\right] = n(f'(x))(f(x))^{n-1}}$
- **So that →** $\boldsymbol{\frac{\mathrm{d}}{\mathrm{dx}}\left[f(g(x))^{n}\right] = n(f(g(x)))^{n-1}\, g'(x)f'(g(x))}$

Or $\boldsymbol{\frac{\mathrm{d}}{\mathrm{dx}}\left[f(g(x))^{n}\right] = \frac{\mathrm{d}}{\mathrm{dx}}\left[u^{n}\right] = n\ u^{n-1}\ \frac{du}{dx}}$

u = $\boldsymbol{f\big(g(x)\big)}$ **→** $\boldsymbol{du = f'\big(g(x)\big)g'(x)\ dx}$

$\boldsymbol{\frac{du}{dx} = f'\big(g(x)\big)g'(x)}$

2] أمثلة على ربط الإقترانات

If $\boldsymbol{y = sin^2 m \ , x = cos m}$ **, find** $\boldsymbol{\frac{d^2y}{dx^2}}$

$y \to m \to x$

$dy = 2\sin m \ \cos m \ \ dm \ \to \frac{dy}{dm} = 2\sin m \cos m$

$x = \cos m \ \to \frac{dm}{dx} = \frac{-1}{\sin m} \ \ \to \frac{dy}{dx} = \frac{dy}{dm} * \frac{dm}{dx} = \frac{2\sin m \ \cos m}{-\sin m}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = -2\cos m \ \to \frac{d^2y}{dx^2} \ \ dx = 2\sin m \ \ dm \ \to \frac{d^2y}{dx^2} = 2\sin m \ \frac{dm}{dx}$

$= 2\sin m * \frac{-1}{\sin m} = \ -2$

Or $\ \ y \to x$

$y = sin^2 m = 1 - cos^2 m = 1 - x^2$

$y = 1 - x^2 \to \ \ y' = -2x \to \ \ y'' = -2$

If $\boldsymbol{y = secn \ ,n = lnf \ ,f > 0 \ ,\ \frac{df}{dx} = f \ tan \ n}$

show that $\boldsymbol{\frac{dy}{dx} = y^3 - y}$

$y \to n \to f \to x$

$\frac{\mathrm{dy}}{\mathrm{dn}} * \frac{dn}{df} * \frac{df}{dx} = \frac{dy}{dx} \ \to \ \frac{dy}{dn} = \sec n \ \tan n \ , \ \ \frac{dn}{df} = \frac{1}{f} \ , \ \ \frac{df}{dx} = f \tan n$

$\sec n * \tan n * \frac{1}{f} * f * \tan n = \frac{dy}{dx} = \sec n \ \tan^2 n$

$\boxed{\boldsymbol{tan^2 n = sec^2 - 1}}$

$\frac{dy}{dx} = \sec n \ \ (\sec^2 n - 1)$

$\frac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{y(y^2 - 1)}$

If $\boldsymbol{y = \frac{x+4}{x-2}\ ,\ n = \frac{y+2}{y-1}\ ,\ show\ that\ \frac{dn}{dx} = \frac{1}{2}}$

Sol:

$$\frac{dy}{dx} = \frac{-6}{(x-2)^2}\ ,\frac{dn}{dy} = \frac{-3}{\underbrace{(y-1)^2}_{y=\frac{x+4}{x-2}}} \rightarrow \frac{dy}{dn} = \frac{6^2}{(x-2)^2} * \frac{-1}{3} \rightarrow \frac{dn}{dy} = \frac{-3}{6^2(x-2)^2}$$

$$\frac{dy}{dx} * \frac{dn}{dy} = \frac{dx}{dn} = \frac{-6}{(x-2)^2} * \frac{-3(x-2)^2}{6^2} = \frac{1}{2}$$

If $\boldsymbol{f(x) = \tan x + \frac{tan^{2} n}{n}\ \ show\ that\ f'(x)|_{n=3} = sec^4(x)}$

$$f'(x) = sec^2x + sec^2x\ tan^{n-1}x$$

$$f'(x)|_{n=3} = sec^2x + sec^2x\ tan^2x = sec^2x\underbrace{(1+\ tan^2x)}_{sec^2x} = sec^4x$$

If $\boldsymbol{y = 1 + n^2\ ,n^3 = x + 3\ \ find\ \frac{dy}{dx}\Big|_{x=5}}$

$$y \rightarrow n \rightarrow x$$

$$\underbrace{\frac{dy}{dn}}_{2n} * \underbrace{\frac{dn}{dx}}_{\frac{1}{3n^2}} = \frac{dy}{dx}$$

$$2n * \frac{1}{3n^2} = \frac{2}{3n} = \frac{dy}{dx}$$

When $x = 5 \rightarrow\ n^3 = 8\ \rightarrow n = 2$

$$\frac{dy}{dx}\Big|_{n=2} = \frac{2}{3*2} = \frac{1}{3}$$

The implicit differentiation

$\frac{\mathrm{d}}{\mathrm{dx}}\,[x^2] = 2x\,\frac{dx}{dx} = 2x$ → Explicit

$\frac{\mathrm{d}}{\mathrm{dx}}\,[y^2] = 2y\,\left[\frac{dy}{dx}\right]$, $\frac{\mathrm{dy}}{\mathrm{dx}}$, $f'(x)$, y'

أي اشتقاق y بالنسبة الى x

Implicit

Find $\frac{\mathbf{dy}}{\mathbf{dx}}$:

1] **y=x** → $1\,.\,\frac{\mathrm{dy}}{\mathrm{dx}} = 1\,.\,\frac{dx}{dx} \rightarrow \frac{dy}{dx} = 1$

2] $\boldsymbol{x = 2y^{\frac{1}{2}}} \rightarrow \frac{dx}{dx} * 1 = 2 * \frac{1}{2} * y^{\frac{-1}{2}}\,\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \sqrt{y}\ or\ \frac{dy}{dx} = 2x$

3] $\boldsymbol{xy + y^2} = y \rightarrow \frac{dy}{dx} * 1 = y\frac{dx}{dx} + x\frac{dy}{dx} + 2y\frac{dy}{dx}$

$y' = y + xy' + 2yy'$

$y'(1 - x - 2y) = y \rightarrow y' = \frac{1}{1-x-2y}$

4] $\boldsymbol{x = \sqrt{y^2 - 1}} \rightarrow 1 = \frac{2yy'}{2\sqrt{y^2-1}} \rightarrow y' = \frac{\sqrt{y^2-1}}{y}\ or\ \frac{x}{y}$

5] find the slop of the function $\boldsymbol{y^3 = x^5 + 1}$ at the point (3,5)

$\rightarrow 3y^2\,y' = 5x^4 \rightarrow y' = \frac{5x^4}{3y^2} \rightarrow y'|_{(3,5)} = \frac{5*(3)^4}{3*(5)^2} = \frac{27}{5}$

6] find the third derivative of $\boldsymbol{sin\ x = cos\ y}$

$\cos x = y'(-\sin y) \ \rightarrow now \rightarrow \cos x + y' \sin y = 0 \ \rightarrow y' = \frac{\cos x}{\sin y}$

$\sin x = \cos y \rightarrow sin^2x = cos^2y \rightarrow 1 - cos^2x = 1 - sin^2y \rightarrow cos^2x = sin^2y$

$\rightarrow \frac{cos^2x}{sin^2y} = 1 \rightarrow\rightarrow (y')^2 = \frac{cos^2x}{sin^2y} \rightarrow (y')^2 = 1 \ , so\ y' = \pm 1$

$y' = \pm 1 \rightarrow y'' = 0 \rightarrow y''' = 0$

7] find $\left.\frac{\mathbf{d^2y}}{\mathbf{dx^2}}\right|_{x=0}$, $\left.\frac{\mathbf{d^3y}}{\mathbf{dx^3}}\right|_{x=\frac{\pi}{4}}$ for $\boldsymbol{y = \sin x + \cos x}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = \cos x - \sin x$

$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = -\sin x - \cos x \rightarrow \left.\frac{\mathrm{d^2y}}{\mathrm{dx^2}}\right|_{x=0} = -\sin 0 - \cos 0 = -1$

$\frac{\mathrm{d^3y}}{\mathrm{dx^3}} = \sin x - \cos x \rightarrow \left.\frac{\mathrm{d^3y}}{\mathrm{dx^3}}\right|_{x=\frac{\pi}{4}} = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

Extra exercises

If $x^2 - y^2 = 1$ prove that $y^3 y'' + 1 = 0$

$$\rightarrow y^2 = x^2 - 1 \rightarrow 2y\,y' = 2x \rightarrow y' = \frac{x}{y} \rightarrow y'' = \frac{y - xy'}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3}$$

Now $y^3 * y'' + 1 = y^3 * \frac{-1}{y^3} + 1 = 0$

If $x^2 + y^2 = 2y$ prove that $\frac{d^2y}{dx^2} = \frac{1}{(1-y)^3}$

$$y^2 - 2y = -x^2 \rightarrow (y-1)^2 = 1 - x^2$$

$$2(y-1)y' = -2x \rightarrow y' = \frac{-x}{y-1} \rightarrow y'' = \frac{-(y-1) + xy'}{(y-1)^2} = \frac{\frac{-x^2}{y-1} - (y-1)}{(y-1)^2}$$

$$y'' = \frac{-(x^2 + y^2 - 2y + 1)}{(y-1)^2} = \frac{-1}{(y-1)^3} = \frac{1}{(1-y)^3} \qquad \boxed{x^2 + y^2 - 2y = 0}$$

If $f^3(y^2) = 5x^3 + 3$, $y = 1$ *when* $x = 1$ *and* $f'(x)|_{x=1} = 5$

Find $\left.\frac{dy}{dx}\right|_{(1,1)}$

$$\rightarrow f'(y^2) * 3f^2(y^2) * 2y * y' = 15x^2$$

$$f'(1) * 3f^2(1) * 2 * 1 * y' = 15(1)^2 \rightarrow y' * 5 * 3\sqrt[3]{8^2} * 2 = 15 \rightarrow y' = \frac{1}{8}$$

If $\frac{dy}{dx} = L\sqrt{y}$, $(L \text{ is constant})$ *and* $\frac{d^2y}{dx^2} = 32$ find the value of L

$$\frac{dy}{dx} = L\sqrt{y}$$

$$\frac{d^2y}{dx^2} = L * \frac{y'}{2\sqrt{y}} = \frac{L * L\sqrt{y}}{2\sqrt{y}} = \frac{L^2}{2} \rightarrow \frac{d^2y}{dx^2} = 32 = \frac{L^2}{2}$$

$$L^2 = 64 \rightarrow L = \pm 8$$

5) If $2^y = xy - 4, find\ \frac{dy}{dx}$.

$$y' \cdot 2^y \cdot \ln 2 = xy' + y$$

$$y' \cdot 2^y \cdot \ln 2 - xy' = y$$

$$y' = \frac{y}{2^y \ln 2 - x}$$

OR

$$2^y = xy - 4$$

$$\ln 2^y = \ln(xy - 4)$$

$$y' \cdot \ln 2 = \frac{xy' + y}{xy - 4} \Rightarrow y' \cdot \ln 2 \cdot (xy - 4) - xy' = y, \qquad recall\ that\ 2^y = xy - 4$$

$$y' = \frac{y}{\ln 2 \cdot 2^y - x}$$

---

6) If $f(x^2 - 9) = x^2 + ax$, and $f'(16) = \frac{48}{32}$, find a.

$$2x \cdot f'(x^2 - 9) = 2x + a \rightarrow f'(x^2 - 9) = \frac{2x + a}{2x}$$

$$x^2 - 9 = 16 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

$$now\ f'(16) = 48 \begin{cases} \rightarrow x = 5 \rightarrow f'(16) = \frac{10+a}{a} = \frac{48}{32} \Rightarrow a = 5 \\ \\ \rightarrow x = -5 \rightarrow f'(16) = \frac{-10+a}{-10} = \frac{48}{32} \Rightarrow a = -5 \end{cases}$$

If $\sin(x + y) = y^2 \cdot cosx$, find $f'(x)$

$$\cos(x + y) \cdot (1 + y') = -y^2 \cdot sinx + cosx \cdot 2yy'$$

$$\cos(x + y) + \cos(x + y) \cdot y' = -y^2 \cdot sinx + 2yy' \cdot cosx$$

$$y'(2y \cdot cosx - \cos(x + y)) = \cos(x + y) + y^2 \cdot sinx$$

$$y' = \frac{\cos(x + y) + y^2 \cdot sinx}{2y \cdot cosx - \cos(x + y)}$$

Find y', y'' for each equation.

1) $\sqrt{y} + \sqrt{x} = y$

$$\frac{y'}{2\sqrt{y}} + \frac{1}{\sqrt{x}} = y' \Rightarrow \frac{y' - \sqrt[2]{y} - y'}{\sqrt[2]{y}} = \frac{1}{2\sqrt{x}}$$

$$y' = \frac{\sqrt{y}}{\sqrt{x}\left(2\sqrt{y} - 1\right)} = \frac{1}{\left(\sqrt{y} - 1\right)\left(2\sqrt{y} - 1\right)}$$

---

2) $x^2 + yx + \frac{y^2}{4} = 9$

$$\left(x + \frac{y}{2}\right)^2 = 9 \Rightarrow 2\left(x + \frac{y}{2}\right) \cdot \left(1 + \frac{y'}{2}\right) = 0$$

$$1 + \frac{y'}{2} = 0 \rightarrow y' = -2$$

OR $x + \frac{y}{2} = 0 \rightarrow y = -2x$

---

3) $\frac{y}{\sqrt{(x^4 - x^2)}} = \frac{1}{x^2} \rightarrow y = \frac{\sqrt{x^4(1 - x^{-2})}}{x^2} \rightarrow y = \sqrt{1 - x^{-2}}$

$$\frac{dy}{dx} = \frac{2}{x^3 \cdot \sqrt{1 - x^{-2}}} \Rightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{x^4 - x^2}}$$

$$\frac{dy}{dx} = \frac{1}{yx^3}$$

---

4) Given the curve $x^2 + 2y^2 = x$, find the point(s) on this curve where the range line has the slope 1

$$2x + 2yy' = 1 \rightarrow 2x + 2y = \frac{1 - 2x}{2} \Rightarrow \ @the\ point\ y = \frac{1 - 2x}{2}$$

$$x^2 + 2y^2 = x$$

$$x^2 + \frac{2(1 - 2x)}{4} = x$$

$$x^2 + \frac{1 - 4x - 4x^2}{4} = x$$

$2x^2 - 2x - 4x + 4x^2 + 1 = 0$

$$6x^2 - 6x + 1 = 0 \Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 24}}{2.6}$$

the point(s) are 1) $\left(\frac{1}{2} + \frac{\sqrt{2}}{\sqrt{6}}\right), f\left(\frac{1}{2} + \frac{\sqrt{2}}{\sqrt{6}}\right)$

2) $\left(\frac{1}{2} - \frac{\sqrt{2}}{\sqrt{6}}\right), f\left(\frac{1}{2} - \frac{\sqrt{2}}{\sqrt{6}}\right)$

L'hopital Rule

We use l'hopital rule in the limits when we see any indeterminate form and stop when the indeterminate from becomes any another value we can solve.

- L'hopital rule $\Rightarrow \lim\limits_{x \to \pm\infty, a} \frac{f(x)}{g(x)} = \lim\limits_{x \to \pm\infty, a} \frac{f\prime(x)}{g\prime(x)}$, where $\frac{f(x)}{g(x)}$ is indeterminate form.

Ex: Find each of the following limits.

1) $\lim\limits_{x \to \infty} \frac{x^2}{e^x} = \frac{0}{0}$

$\lim\limits_{x \to \infty} \frac{2x}{e^x} = \frac{2}{\infty} = 0$, stop l'hopital rule at here

2) $\lim\limits_{x \to 0} \frac{x - tanx}{x^3} = \frac{0}{0}$

$$\lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \to 0} \frac{-2\sec^2 x \cdot \tan x}{6x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \to 0} \frac{-2\sec^4 x + \tan^2 x \cdot \sec x}{6} = \frac{-2}{6} = \frac{1}{3}$$

3) $\lim\limits_{x\to\infty}\left(1+\frac{a}{bx}\right)^{x} = 1^{\infty}\ ??$

$$y = \lim_{x\to\infty}\left(1+\frac{a}{bx}\right)^{x}$$

$$\ln y = \lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{bx}\right) = 0.\infty$$

$\ln y = \lim\limits_{x\to\infty} \ln\frac{\ln\left(1+\frac{a}{bx}\right)}{\frac{1}{x}} = 0.\infty$→ l’hopital rule

$$\ln y = \lim_{x\to\infty}\frac{-\frac{a}{bx^2}}{1+\frac{a}{bx}.-\frac{1}{x^2}} \to \ln y = \lim_{x\to\infty}\frac{a}{b} \to e^{\frac{a}{b}} = y$$

so $\lim\limits_{x\to\infty}\left(1+\frac{a}{bx}\right)^{x} = e^{\frac{a}{b}}$

- $\lim\limits_{x\to\infty} 1^{x} = 1$, because 1 is constant and not became from a function like the previous exam

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The derivative of logarithmic functions

$f(x) = \log_a g(x), \;\; where\ a > 0$

$$f'(x) = \frac{g'(x)}{g(x) \cdot \log a}$$

when $a = e \Rightarrow f'(x) = \frac{g'(x)}{g(x)}$

EX: Find the derivative of the following functions

1) $f(x) = \ln(\ln x) \Rightarrow f'(x) = \frac{1}{x \cdot \ln x}$
2) $f(x) = \ln x^3 + x \;\Rightarrow f'(x) = \frac{3x^2+1}{x^3+x}$
3) $f(x) = \ln \sin e^x \;\Rightarrow\; f'(x) = \frac{e^x \cdot \cos e^x}{\sin e^x}$
4) $f(x) = \ln \sec x + \tan x \;\Rightarrow\; f'(x) = \sec x$
5) $f(x) = \ln \sec x \cdot \tan x$

$\rightarrow \ln \sec x \cdot \ln \tan x$

$$\rightarrow f'(x) = \tan x + \frac{\sec^2 x}{\tan x}$$

$\Rightarrow\; f'(x) = \frac{\tan^2 x + \sec^2 x}{\tan x}$

6) $f(x) = \ln(\ln x + e^x) \;\Rightarrow\; f'(x) = \frac{1+xe^x}{\ln x + e^x}$
7) $f(x) = \ln\left(x + \sqrt{x^2+1}\right) \;\Rightarrow\; f'(x) = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$
8) $f(x) = \ln(\tan(\ln x + x)) \Rightarrow f'(x) = \frac{\sec^2(\ln x + x) \cdot \left(\frac{1}{x}+1\right)}{\tan(\ln x + x)}$
9) $f(x) = \ln(x \cdot \ln x - x) \Rightarrow f'(x) = \frac{\ln x}{x(\ln x - 1)}$

Ex: If $f(x) = \log_a g(x)$, and $h(x) = \frac{\ln g(x)}{\ln a}$,show that $f'(x) = h'(x)$

$$f'(x) = \frac{g'(x)}{g(x) \cdot \ln a} = h'(x) = \frac{g'(x)}{g(x) \cdot \ln a}$$

Ex: If $f(x) = \ln g(x)$, and $f'(x) = \frac{2x+c}{x^2+6x}$, find the value of c.

$$g'(x) = 2x + 6 = 2x + c$$

$$c = 6$$

Ex: Find $\lim_{x\to1} \frac{\ln x}{x-1}$

$\lim_{x\to1} \frac{\ln x - 0}{x-1} = \lim_{x\to1} \frac{\ln x - \ln 1}{x-1}, f(x) = \ln x\,, f'(x) = \ln 1$

$$= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \Rightarrow f'(x) = \frac{1}{x}, f'(1) = 1$$

Ex: Show that if $f(x) = \ln ax\,, where\ a > 0, f'(x) = \frac{1}{x}\ (by\ logarithim\ laws)$

$$f(x) = \ln a + \ln x \to f'(x) = 1/x$$

$$*\ln a\ \ is\ a\ constant$$

Ex: Prove that if $f(x) = g(x) * l(x) * h(x) * m(x)$

$$f'(x) = g(x) * l(x) * h(x) * m'(x) + g(x) * l(x) * h'(x) * m(x) + g(x) * l'(x) * h(x) * m(x) \\ + g'(x) * l(x) * h(x) * m(x)$$

$\ln f(x) = \ln g(x) + \ln l(x) + \ln h(x) + \ln m(x)$

$$f'(x) = \left(\frac{g'(x)}{g(x)} + \frac{l'(x)}{l(x)} + \frac{h'(x)}{h(x)} + \frac{m'(x)}{m(x)}\right) * \big(g(x) * l(x) * h(x) * m(x)\big)$$

Ex: If $f(x) = x^x, find\ f'(x)$.

$$\ln f(x) = x \cdot \ln x$$

$\frac{f'(x)}{f(x)} = 1 + \ln x \rightarrow f'(x) = x^x + x^x \cdot \ln x$ OR $f'(x) = x^x + \ln x^{x^x}$.

Ex: If $f(x) = x^{x^x}, find\ f'(x)$

$$\ln f(x) = x^x \cdot \ln x$$

$$\frac{f'(x)}{f(x)} = \frac{x^x}{x} + (\ln x + x^x \cdot \ln x)$$

$$f'(x) = \frac{x^{x^x} \cdot x^x}{x} + \ln x \cdot x^{x^x} \cdot x^x + \ln x \cdot x^{x^x} \cdot x^x$$

$$f'(x) = x^{x^x} - x\left(\ln^2 x + \ln x + \frac{1}{x}\right)$$

Ex: If $x^y = y^x$, find $f'(x)$

$\ln x^y = \ln y^x$

$y \cdot \ln x = x \cdot \ln y \Rightarrow \left(\frac{y}{x} + y' \cdot \ln x = \frac{xy'}{y} + \ln y\right) * xy$

$y^2 + y \cdot \ln x \cdot xy = x^2 y' + \ln y \cdot xy$

$y'(\ln x \cdot yx - x^2) = \ln y \cdot xy - y^2$

$y' = \frac{\ln y \cdot xy - y^2}{\ln x \cdot xy - x^2}$

OR

$$x^y = y^x$$

$$\log_x x^y = \log_x y^x$$

$$y = x \cdot \log_x y$$

$$y = \frac{x \ln y}{\ln x} \to y' = x\left[\frac{\ln x \cdot \frac{y'}{y} - \frac{\ln y}{x}}{\ln^2 x}\right] + \frac{\ln y}{\ln x}$$

$$y' = \frac{x \cdot \ln x \cdot y' - \ln y \cdot y}{(\ln^2 x).xy} + \frac{\ln y * xy}{\ln x * (\ln^2 x\,.\,xy)}$$

Ex: If $y = x^{xxxxxxxxxx.....}$, find y'.

$$y = x^y$$

$$\ln y = y \cdot \ln x$$

$$\frac{y'}{y} = \frac{y}{x} + y' \cdot \ln x \Rightarrow xy' = y^2 + y' \cdot \ln x\,(xy)$$

$$y' = \frac{y^2}{x(\ln x \cdot y + 1)}$$

Ex: If $e^{\frac{x}{y}} = x - y$, find y'

$$\ln e^{\frac{x}{y}} = \ln(x - y)$$

$$\frac{x}{y} = \ln(x - y)$$

$$\to \frac{y - xy'}{y^2} = \frac{1 - y'}{x - y}$$

$$y^2 - y^2 \cdot y' = xy - y^2 \pm (x - y)(xy')$$

$$y'(x^2 - xy - y^2) = xy - 2y^2$$

$$y' = \frac{xy - 2y^2}{x^2 - xy - y^2} = \frac{y(x - y - y)}{x(x - y) - y^2} = \frac{y\left(e^{\frac{x}{y}} - y\right)}{xe^{\frac{x}{y}} - y^2}$$

Tangent line and normal line:

⇒ any line in the 2D system has an equation

⇒ line equation: $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point

⇒ m: slope الميل

slope= $\frac{\Delta y}{\Delta x}$

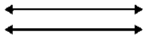
slpoe= $f'(x)$

slope= $\tan\theta\ , where\ \theta\ with\ the\ positive\ x - axis$

⇒ any two lines in space could be:

1. **Parallel**

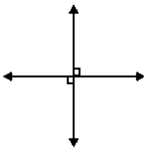
⇒ in this case $m_1 = m_2$

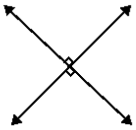


2. **Perpendicular**

⇒ in this case $m_1 * m_2 = -1$

or $m_1 = -\frac{1}{m_2}$



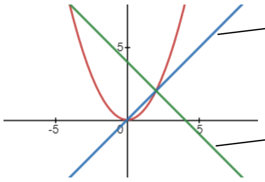


*إذا بدي أطلع خط عامودي على خط ثاني بقلب الميل وبضرب بسالب

- $m = \frac{dy}{dx} = f'(x)$

- Geometric meaning of $f'(a)$ (a) المشتقة عند

  $f'(a) = m$(ميل المماس)



- So $f'(a)$ is the slope (m) of the tangent line at point (a,f(a))

T.L: (y-f(a))=$f'(a)(x-a)$

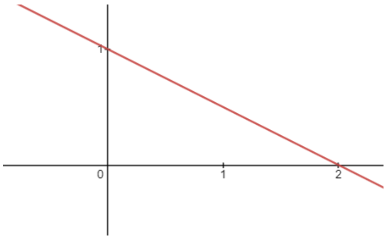
N.L: (y-f(a))=$\frac{-1}{f'(a)}(x-a)$

Remember $m_N = \frac{-1}{m_T}$

Conclusion:

$y - y_1 = f'(x)(x - x_1)$

Ex: find the equation of the line bellow:



$y - y_1 = m(x - x_1)$

The point we will take is (2,0)

The slope is $\frac{\Delta y}{\Delta x} = \frac{1-0}{0-2} = -0.5$

The equation becomes-> y=1-0.5x

Ex2: find the equation of the tangent line and normal line of $\boldsymbol{f(x) = x^2}$

At x=3

We need point and slope

Point -> (3,9)

Slope->6

T.L -> y=6x-9

N.L -> $\frac{19}{2} - \frac{x}{6}$

Ex3: find the equation of the t.l and n.l of:

1) $x^2 + 2xy - y^2 + x = 2$ at (1,2)
   Here he’s giving me a point so all I have to do is to find the slope
   اشتقاق ضمني-> $2x + 2xy' + 2y - 2yy' + 1 = 0$
   at (1,2) -> $2 + 2y' + 4 - 4y' + 1 = 0$

$$m_N = \frac{-2}{7}$$

T.L : $y = \frac{7}{2}x + \frac{11}{2}$
N.L: $y = \frac{16}{7} - \frac{2x}{7}$

2) $f(x) = \frac{e^x}{cosx}\ at\ (0,1)$

$$f'(x) = \frac{e^x cosx + e^x sinx}{cos^2x}$$

$$f'(0) = 1 = m_T, m_N = -1$$

T.L: y=x+1
N.L: y=1-x,

3) $y = 1.5\sqrt{x}\ at\ x = 4$

$m_T = \frac{3}{8}\ , m_N = \frac{-8}{3}$

T.L : $y - 3 = \frac{3}{8}(x - 4)$

N.L: $y - 3 = \frac{-8}{3}(x - 4)$

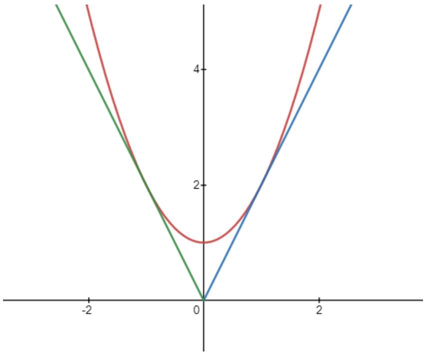
Ex 4: write the equation of tangent of $f(x) = x^2 + 1$ *that passes through the point* $(0, 0)$

$m = \frac{\Delta y}{\Delta x}, m = f'(x)$

$m = m$

$x^2 + 1 = 2x^2$

$x^2 = 1 \rightarrow x = \pm 1$



لويه الرسمة مثل هيك؟

the points are (1,2),(-1,2)

$T.L_1 \rightarrow y = 2x$

$T.L_2 \rightarrow y = -2x$

That’s why the drawing was like that

Ex5: find equation of T.L to the curve $y = x^{\frac{3}{2}}$ if the the tangent is parallel to $y = 3x + 1$ If L1//L2 → m1=m2

حتى نجد المعادلة نحتاج الى نقطة و ميل وبفضل كلمة موازي سنستطيع أن نجد الميل المنشود أما بالنسبة لنقطة التماس فيجب أن نولي الأهمية للميل لأنه هو من سيعطينا إياها بمعنى أننا نجدها بعد الميل

Y1=y2

$x^{\frac{3}{2}} = 1 + 3x$

X=4

P1 (4,1)

P2 (4,8)

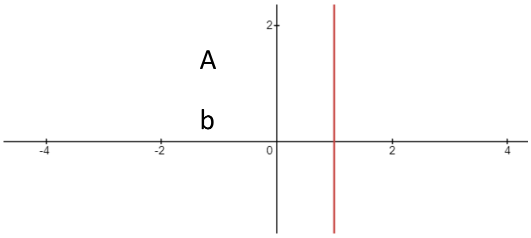
T.L → y=3x-4

Ex 6: if $f(x) = (x-2)^{\frac{2}{3}}$ find the equation of T.L at x=2

$$f'(x) = \frac{2}{3}(x-2)^{\frac{-1}{2}}$$

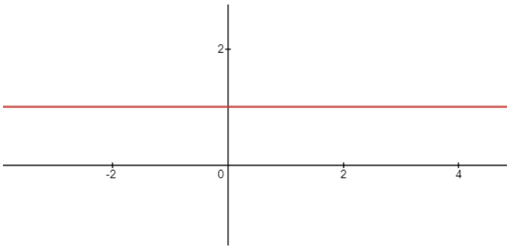
$f'(0) = \frac{2}{0} \longrightarrow$ this is a vertical tangent

Note: the tangent line is vertical when it's slope is $\frac{const.}{0} = \pm\infty$



slope is $\pm\infty$ v.t

Note: The T.L is horizontal when its slope is zero



Slope here is zero

h.t

ex: if $f(x) = 2x^2$ at what point is T.L horizontal?

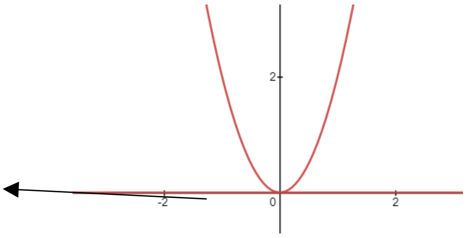
M=0

M=m

4x=0

X=0

(0,0) H.T



Derivative of inverse function

$f\left(f^{-1}(x)\right) = x$

$\frac{d}{dx}\, f'\left(f^{-1}(x)\right)(f^{-1})'(x) = 1$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Ex1: if f(4)=5, $f'(4) = \frac{2}{3}, find\ \frac{df^{-1}(5)}{dx}$

$(f'^{-1}(5))' = \frac{1}{f'(f^{-1}(5))}$

$\frac{1}{f'(4)} = 1.5$

Ex2: let $f(x) = x + e^{x}, find\ f^{-1'}(1)$

$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}\ , f^{-1}(1) = 0$

$\frac{1}{f'(0)} = 0.5$

Note: don't forget that all functions must be 1-1 to deal with inverse

Derivative of inverse trigonometric functions

$\sin^{-1} x$ / $\cos^{-1} x$ / $\tan^{-1} x$ / $\cot^{-1} x$ / $\csc^{-1} x$ / $\sec^{-1} x$

Ex: if $\boldsymbol{y = \sin^{-1} x}$, $\boldsymbol{find\ y'}$

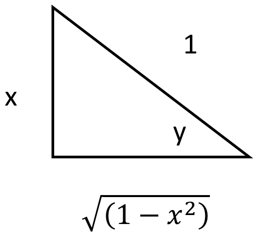
سوف نقوم بعمل خدعة الا وهي أن أضع معكوس اقتران على الجهتين

$y = \sin^{-1} x$

Siny=x

cosy= $* \ y' = 1$

$y' = \frac{1}{cosy} = \frac{1}{\sqrt{1-x^2}}$



F(x)	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$

but what if instead of x there was g(x)

➔ $f(x) = \sin^{-1}\big(g(x)\big), find\ f'(x)$

G(x) and x are the same so instead of x we multiply by the derivative of g(x)

$f'(x) = \frac{g'(x)}{\sqrt{1-g(x)^2}}$

Ex: if $\boldsymbol{f(x) = \csc^{-1}(g(x))}$,$\boldsymbol{find\ f'(x)}$

$f'(x) = \frac{-g'(x)}{g(x)*\sqrt{g^2(x)-1}}$

Ex: find $\boldsymbol{\frac{dy}{dx}}$ of:

1) $y = \sin^{-1} 2x$

   $y' = \frac{2}{\sqrt{1-4x^2}}$

2) $y = \tan^{-1} \sqrt{x}$

   $y' = \frac{1}{2(1+x^2)\sqrt{x}}$

3) $y = \cos^{-1}(e^x)$

   $y' = \frac{-e^x}{\sqrt{1-e^{2x}}}$

4) $y = \sqrt{\cos^{-1} x}$

   $y' = \frac{-1}{2\sqrt{1-x^2}\sqrt{\cos^{-1} x}}$

5) $y = \cos^{-1}(\sin^{-1} x)$

   $y' = \frac{-1}{\sqrt{(1-x^2)(1-(\sin^{-1} x)^2}}$

6) $y = 2(\sin^{-1} 3x)^6$

$$y' = \frac{36(\sin^{-1} 3x)^5}{\sqrt{1-9x^2}}$$

Ex: evaluate $\lim\limits_{h\to 0} \frac{\tan^{-1}(1+h)-\frac{\pi}{4}}{h}$

0.5 is the answer

Linear approximation and differentials

We know that equation of T.L is given by:

$y - y_1 = m(x - x_1)$

In other words -> $y = f(a) + f'(a)(x - a)$

➔ The equation above is the equation of linear approximation

حيث: 1) a عدد يتم اختياره من قبل حضرتكم
2) x هو العدد الأصلي في السؤال

Ex: $\sqrt{4.1}$ a=4 , x=4.1

الهدف من كل الدرس هو إيجاد قيم تقريبية بواسطة الاشتقاق واستعماله والدماغ دون الحاجة الى ما صنعه الانسان من عدة باهظة الثمن تكلفنا الكثير مثل الالة الحاسبة فكلنا يعرف ان المخ البشري اذكى بكثير من اي آلة صنعها هو التقريب الخطي هو احد تطبيقات الاشتقاق.

Ex: find linearization of

1) $f(x) = \sqrt{x+3}\ at\ a = 1$

   L(x) =$\frac{7}{4} + \frac{x}{4}$

2) $f(x) = sinx\ at\ a = 0$

   L(x) = x
3) Y= ln(x+1) at a=0

   L(x) = x

Ex2: using linear approximation find the value of the following

1) $\sqrt{17}$

   A=17, x=16

   L(x)= 4+ 1/8
   =4.125
2) $\sqrt{4.1}$

   L(x)= 2.025

EX: using linear approximation, find the value of the following:

1) $\sqrt{17}$

- ✓ We want to approximate $f(x) = \sqrt{x}$ at x=17 to get approximated value of $f(17) = \sqrt{17}$

- ✓ Following the linearization:

$f(x) \simeq l(x) = f(a) + f'(a)(x - a)$

Note: that we need values for a and x, the obvious thing that x=17 to get $\sqrt{17}$ And (a) must be a value near 17 and we know it's value and the closer it is to 17 the accurate the answer will be so a=16.

Using the calculator will give us answer near our answer which is a good approximation

Where, a=16 and $x\ at\ 17\ (x = 17)$

$$f(a) = \sqrt{16} = 4$$

$$f'(x) = \frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}, f'(a) = \frac{1}{8}$$

Then, $f(x) \simeq l(x) = 4 + \frac{x-16}{8}$

So, $f(17) \simeq \sqrt{17} \simeq 4 + \frac{17-16}{8} \simeq 4.125$

2) $\sqrt{4.1}$

- ✓ We want to approximate $f(x) = \sqrt{x}$ at x=4.1 to get approximated value of $f(4.1) = \sqrt{4.1}$.
- ✓ Following the linearization:

$$f(x) = \sqrt{x}, a = 4\ and\ x\ at\ 4.1$$

$$f'(x) = \frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}, f'(a) = \frac{1}{4}, f(a) = 2$$

$$l(x) = 2 + \frac{x-4}{4}, f(4.1) \simeq l(4.1) \simeq \sqrt{4.1} \simeq 2.025$$

3) $\sqrt[3]{63}$

- ✓ We want to approximate $f(x) = \sqrt[3]{x}$ at x=63 to get approximated value of $f(63) = \sqrt[3]{63}$.
- ✓ Following the linearization:

$$f(x) = \sqrt[3]{x}\,, a = 64\ and\ x\ at\ 63$$

$$f'(x) = \frac{d(\sqrt[3]{x})}{dx} = \frac{1}{3\sqrt[3]{x^3}}\,, f'(a) = \frac{1}{48}\,, f(a) = 4$$

$$l(x) = 4 + \frac{x-64}{48}\,, f(63) \simeq l(63) \simeq \sqrt[3]{63} \simeq 4 - \frac{1}{48}$$

4) $\ln(1.02)$

- ✓ We want to approximate $f(x) = \ln(x)$ at x=1.02 to get approximated value of $f(1.02) = \ln(1.02)$.
- ✓ Following the linearization:

$$f(x) = \ln(1.02)\,, a = 1and\ x\ at\ 1.02$$

$$f'(x) = \frac{d(\ln(x))}{dx} = \frac{1}{x}\,, f'(a) = 1\,, f(a) = 0$$

$$l(x) = x - 1\,, f(1.02) \simeq l(1.02) \simeq \ln(1.02) \simeq 0.02$$

5) $e^{-0.015}$

$f(x) = e^x, a = 0\ x\ at -0.015\ \ and\ \ f'(x) = e^x,$
$f(a) = 1\ and\ f'(a) = 1$
Then, $f(x) \simeq l(x) = 1 - x\ so, f(-0.015) \simeq 0.985$

6) $\sin(59^o)$

$f(x) = \sin(x), a = 60^o$ *and* x *at*59^o*and* $f'(x) = \cos(X)$, $f(a) = \frac{\sqrt{3}}{2}$ *and* $f'(a) = 0.5$

Then, $f(x) \simeq l(x) = \frac{\sqrt{3}}{2} + \frac{x-60^o}{2}$ *so*, $f(59^o) \simeq \frac{\sqrt{3}}{2} - \frac{\pi}{360}$

Note:

angels in degree are not real numbers so we can't use them in our calculations then a and x must be in radians not in degrees:

$$angel_{radians} = \frac{\pi}{180^o}(angel_{degree})$$

So, new angels in radian will be like this:

$$a = 60^o\frac{\pi}{180^o}$$

$$x = 59^o\frac{\pi}{180^o}$$

$$x - a = -\frac{\pi}{180^o}$$

Home work: approximate the following!

1) $(1.999)^4$ 2) $\cos(28^o)$

Ans,

1) 16-0.032 2) $\frac{\sqrt{3}}{2} + \frac{\pi}{180}$

Differentials (dx,dy):

- ✓ $\Delta x = x_2 - x_1 = dx$
- ✓ $f'(x) = \frac{dy}{dx}$ $, so\ dy = dx\ f'(x)$

EX1: $\boldsymbol{f(x) = x^2\ ,\ \Delta x = 5\ find\ dx\ and\ dy}$ **:**

$$dx = x_2 - x_1 = 5$$

$$dy = dx\ f'(x) = 5(2x) = 10x$$

EX2: $\boldsymbol{f(x) = e^{0.1x}\ , x_1 = 0\ , dx = 0.1\ find\ x_1\ and\ dy}$ **:**

$$dx = x_2 - x_1 = 0.1\ , x_2 = 0.1$$

$$dy = dx\ f'(x) = 0.1(0.1e^{0.1x}) = 0.01e^{0.1x}$$

Hyperbolic functions:

- ✓ Hyperbolic functions are functions that has same qualities as trigonometric functions and, in this section, we will have functions that are made from exponential functions.

Hyperbolic functions:	
sinhx	$\frac{e^x - e^{-x}}{2}$
coshx	$\frac{e^x + e^{-x}}{2}$
tanhx	$\frac{e^x - e^{-x}}{e^x + e^{-x}}$
cothx	$\frac{e^x + e^{-x}}{e^x - e^{-x}}$
sechx	$\frac{2}{e^x + e^{-x}}$
cschx	$\frac{2}{e^x - e^{-x}}$

Notes:

1) Sinhx=sin hyperbolic of x and coshx= cosin hyperbolic of x …etc.
2) Remember sinhx and coshx and you can derive the other functions just like trigonometry where:

$$tanhx = \frac{sinhx}{coshx}$$

EX: $\boldsymbol{evaluate}$ **:**

1) Sinh(0)=

We know from above table that $\sinh(x) = \frac{e^x - e^{-x}}{2}$ so

sinh(0)= $\frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$

2) Cosh(ln2)

$$\cosh(x) = \frac{e^{ln2} + e^{-ln2}}{2} = \frac{2 + e^{ln2^{-1}}}{2} = \frac{2 - 0.5}{2} = \frac{5}{4}$$

Notes:

- sinh(x)
  - domain is $\mathbb{R}$
  - range is $\mathbb{R}$
- cosh(x)
  - domain is $\mathbb{R}$
  - range is $[1,\infty)$
- tanh(x)
  - domain is $\mathbb{R}$
  - range is $\mathbb{R}$
  - it has H.Ass at y=1

Hyperbolic functions identities
$\boldsymbol{sinh(-x) = -sinh(X)\ \ , so\ it's an\ odd\ function}$
$\boldsymbol{cosh(-x) = cosh(X)\ \ , so\ it's an\ even\ function}$
$\boldsymbol{cosh(x)^2 - sinh(X)^2 = 1}$
$\boldsymbol{1 - tanh(x)^2 = sech(X)^2\ , try\ to\ prove\ it}$
$\boldsymbol{sinh(X \pm y) = \cosh(y)\sinh(X) \pm \cosh(X)\sinh(y)}$ $\boldsymbol{cosh(X \pm y) = \cosh(y)\cosh(X) \pm \sinh(X)\sinh(y)}$
$\boldsymbol{\cosh(X) + \sinh(X) = e^x}$ $\boldsymbol{\cosh(X) - \sinh(X) = e^{-x}}$

Proving the third identity:

- ✓ $\cosh(x) = \frac{e^x+e^{-x}}{2}$
- ✓ $\sinh(x) = \frac{e^x-e^{-x}}{2}$

$$cosh^2(x) - cosh^2(x) = (\frac{e^x + e^{-x}}{2})^2 - (\frac{e^x - e^{-x}}{2})^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

EX: $\boldsymbol{given\ that}\ \mathbf{cosh}(\boldsymbol{X}) = \frac{\mathbf{5}}{\mathbf{4}}\ \boldsymbol{and\ x > 0}\ ,\boldsymbol{find}\ \ \mathbf{sinh}(\boldsymbol{X})\,?$

Using the identity:

$$cosh(x)^2 - sinh(X)^2 = 1$$

By substitution:

$\frac{25}{16} - \sinh(X)^2 = 1$ then $\sinh(X) = \pm\frac{3}{4}$ But $\sinh(X) = -\frac{3}{4}$ is ignored as x>0

EX: $\boldsymbol{given\ that}\ \mathbf{tanh}(\boldsymbol{X}) = \frac{\mathbf{-3}}{\mathbf{5}}\ ,\boldsymbol{find}\ \ \mathbf{sinh}(\boldsymbol{X})\ \ \boldsymbol{and}\ \mathbf{cosh}(\boldsymbol{X})\,?$

Using the identity:

$$\mathbf{1} - \boldsymbol{tanh}(\boldsymbol{x})^{\mathbf{2}} = \boldsymbol{sech}(\boldsymbol{X})^{\mathbf{2}}$$

By substitution:

$1 - \frac{9}{25} = sech(X)^2$ then $sech(X)^2 = \frac{1}{\cosh(X)^2}$, so cosh(X)= $\pm\frac{5}{4}$ but cosh(X)=$-\frac{5}{4}$ is ignored as cosh(X) always positive

For sinhx we know that:

$$cosh(x)^2 - sinh(X)^2 = 1$$

$$\frac{25}{16} - sinh(X)^2 = 1$$

Then:

$sinh(X) = \ \pm\frac{3}{4}$ but $\frac{3}{4}$ is ignored as tanh(X) is negative and cosh(X)is positive so,

$$sinh(X) = cosh(X)tanh(X) = (+)(-) = -$$

Derivative of hyperbolic functions

$f(X)$	$f'(X)$
$sinh(x)$	$cosh(x)$
$cosh(x)$	$sinh(x)$
$tanh(x)$	$sech^2(x)$
$coth(x)$	$-csch^2(x)$
$sech(x)$	$-sech(x)tanh(x)$
$csch(x)$	$-\text{csch}\ (X)coth(x)$

- <u>Note</u>:
  Chain rule is applicable here for each hyperbolic function like any other function for example:

ويطبق على جميع
العلاقات بالأعلى

$$\frac{d(\sinh\ (g(X))}{dx} = \cosh\big(g(X)\big)\ .g'(X)$$

<u>EX:</u> find f’(X) of the following:

1) $f(X) = \cosh\ (\sqrt{x})$

$$f'(X) = \sinh\big(\sqrt{x}\big)\ .\frac{1}{2\sqrt{x}}$$

2) $f(X) = \sinh\ (\cosh(2x))$

$$f'(X) = 2\cosh(\cosh\ (2x))\ .\sinh\ (2x)$$

3) $f(X) = \ln(\tanh(x))$

$$f'(X) = \frac{\text{sech}\ (x)^2}{\tanh\ (x)}$$

4) $f(X) = \cosh(X)\ .e^{x}$

$$f'(X) = \cosh(X)\ .e^{x} + \sinh(x)\ .e^{x}$$

5) $f(X) = e^{\cosh(3x)}$

$$f'(x) = 3\sinh(3x).e^{\cosh\,(3x)}$$

EX: evaluate the following limits:

1) $\lim\limits_{x\to\infty} \sinh(X)$

$$\lim_{x\to\infty} \sinh(X) = \lim_{x\to\infty} \frac{e^{x} - e^{-x}}{2} = \frac{e^{\infty} - 0}{2} = \infty$$

2) $\lim\limits_{x\to\infty} \cosh\,(X)$

$$\lim_{x\to\infty} \cosh(X) = \lim_{x\to\infty} \frac{e^{x} + e^{-x}}{2} = \frac{e^{\infty} + 0}{2} = \infty$$

3) $\lim\limits_{x\to\infty} \frac{\sinh\,(X)}{e^{x}}$

Divide on $e^{x}$to get rid of $\frac{\infty}{\infty}$ situation

$$\lim_{x\to\infty} \frac{\sinh\,(X)}{e^{x}} = \frac{e^{x} + e^{-x}}{2e^{x}} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{e^{x} + e^{-x}}{2e^{x}} = \frac{1 + e^{-2x}}{2} = \frac{1 + 0}{2} = 0.5$$

4) $\lim\limits_{x\to -\infty} \tanh{(X)}$

$$\lim_{x\to -\infty} \tanh(X) = \lim_{x\to -\infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{\infty}{\infty}$$

$$\lim_{x\to \infty} \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = \lim_{x\to \infty} \frac{-e^{-x}}{e^{-x}} = -1$$

عوض السالب حتى تستطيع
تطبيق قاعدة اعلى قوة على
اعلى قوه.

Inverse of Hyperbolic Functions

$$\boldsymbol{sinh^{-1}(x) = ln(x + \sqrt{x^{2} + 1})}$$

$$\boldsymbol{cosh^{-1}(x) = ln(x + \sqrt{x^{2} - 1})}$$

$$\boldsymbol{tanh^{-1}(x) = \frac{1}{2} ln(\frac{1 + x}{1 - x})}$$

$$\boldsymbol{coth^{-1}(x) = \frac{1}{2} ln(\frac{x + 1}{x - 1})}$$

$$\boldsymbol{sech^{-1}(x) = ln(\frac{1 + \sqrt{1 - x^{2}}}{x})}$$

$$\boldsymbol{csch^{-1}(x) = ln(\frac{1}{x} + \frac{\sqrt{1 + x^{2}}}{x}}$$

proof:

$y = sinhx = \frac{e^{x} - e^{-x}}{2}$, so $x = \frac{e^{y} - e^{-y}}{2}$ for the inverse

and $2x = e^{y} - e^{-y}$ multiplying by e^{y} we get :

$2xe^{y} = e^{2y} - 1$, **notice that :** $\boldsymbol{(e^{y} \,.\, e^{-y} = e^{0} = 1)}$

Rearrange the equation to quadric form:

$0 = e^{2y} - 2xe^{y} - 1$ and let $r = e^{y}$ so we can end up with one y by general law or completing the square method if you want that method:

احفظ اول ثلاث بالجدول
اعلاه و اذا نسيتهم هي
الاثبات لاول وحدة وبتقدر
تطبق على الباقي:

Since $e^{y} > 0$ always and $x < 2\sqrt{x^{2} + 1}$, so we take the + term.

القانون العام

$0 = r^2 - 2xr - 1,\ r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ then:

$$r = \frac{2x \pm \sqrt{4(x^2+1)}}{2} = x \pm 2\sqrt{x^2+1}$$

$r = e^y = x + 2\sqrt{x^2+1}$ by ln for each side:

$$y = \boldsymbol{sinh^{-1}(x)} = \boldsymbol{ln}(x + 2\sqrt{x^2+1})$$

*عارف زنخين بس حاول احفظهم او أفهم طريقة الاثبات بحيث تستذكرهم بسرعة بالامتحان.

EX: find $\boldsymbol{sinh^{-1}(2)}$:

$\boldsymbol{sinh^{-1}(x) = ln\left(x + \sqrt{x^2+1}\right)}$ **at x=2 we get:** $\boldsymbol{sinh^{-1}(2) = ln\left(2 + \sqrt{5}\right)}$

ومن هذا المثال
رح نعمل لجدول
لمشتقات الانفيرس

EX: find $(\boldsymbol{sinh^{-1}(x)})'$:

$$\left(sinh^{-1}(x)\right)' = \frac{d\left(sinh^{-1}(x)\right)}{dx} = \frac{d\left(\ln\left(x+\sqrt{x^2+1}\right)\right)}{dx} = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} =$$

$$\frac{d(sinh^{-1}(x))}{dx} = \frac{(x+\sqrt{x^2+1})}{\sqrt{x^2+1}(x+\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$$

Derivative of inverse hyperbolic functions

$\boldsymbol{f(X)}$	$f'(X)$
$\boldsymbol{sinh^{-1}(x)}$	$\frac{1}{\sqrt{x^2+1}}$
$\boldsymbol{cosh^{-1}(x)}$	$\frac{1}{\sqrt{x^2-1}}$
$\boldsymbol{tanh^{-1}(x), coth^{-1}(x)}$	$\frac{1}{1-x^2}$
$\boldsymbol{sech^{-1}(x)}$	$\frac{-1}{x\sqrt{1-x^2}}$
$\boldsymbol{csch^{-1}(x)}$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

<u>EX:</u> find $\boldsymbol{\frac{dy}{dx}}$ of:

1) $y = sinh^{-1}(2x)$

$$\frac{dy}{dx} = \frac{2}{\sqrt{(2x)^2+1}}$$

2) $y = (tanh^{-1}(\sqrt{x}))^4$

$$\frac{dy}{dx} = 4(tanh^{-1}(\sqrt{x}))^3\ \frac{1}{2\sqrt{x}(1-x)}$$

3) $y = \ln(cosh^{-1}(x))$

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{x^2-1}}}{cosh^{-1}(x)}$$

4) $y = tanh^{-1}(sin(x))$

$$\frac{dy}{dx} = \frac{\cos(x)}{1-sin(x)^2} = \frac{\cos(x)}{\cos(x)^2} = \sec(x)$$

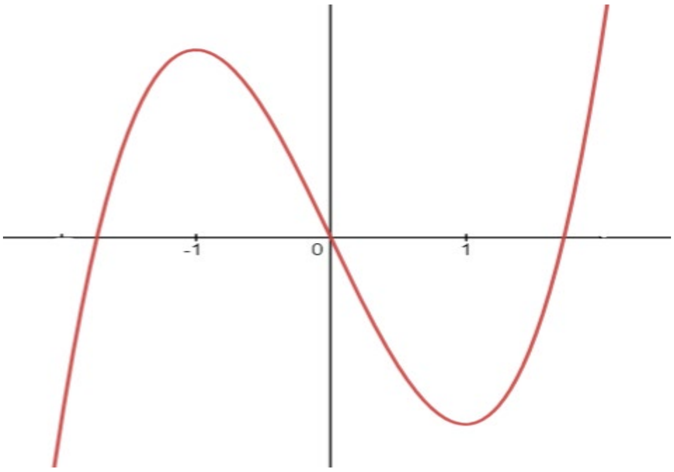
CHAPTER ‘4’

Applications of Differentiation

Maximum and minimum values

■introduction if we have the function

$f(x) = x^3 - 3x$, if we graph $f(x)$



We see that f (1) is less than the other values of f(x) when x approach to (1)

And $f(-1)$ is bigger than the other values of $f(x)$ when x approach to(-1) so what the name of $f(1)$, $f(-1)$

$f(1)$ → minimum

$f(-1)$ → maximum

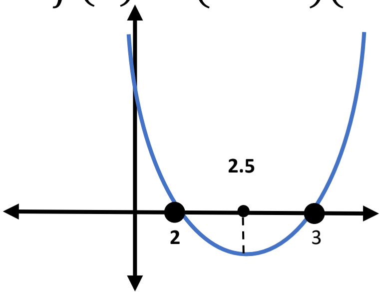
Now because {1,-1} are on the domain →R

So, we say that are local

$f(1)$ → Local Minimum

$f(-1)$ → Local Maximum

ow if $f(x) = (x-2)(x-3)$ → the graph is



Now $f(2.5)$ is the least value of $f(x)$on the domain

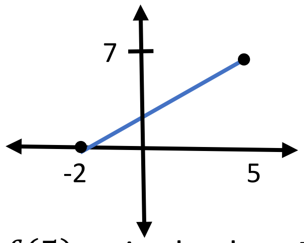
So $f(2.5)$ is absolute minimum and Local Value

لأنها أقل قيمة لـِ $f(x)$

لأنها أقل من القيم المجاورة لها

لأنها داخل المجال وليست طرف

Now if we have $f(x) = x + 2$ on$[-2,5]$



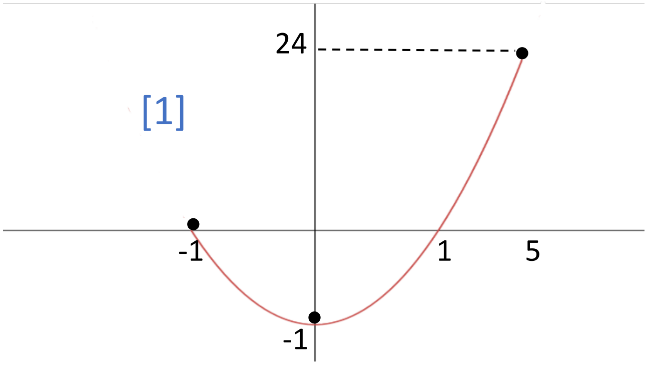
Now $f(-2)$ →is absolute minimum value

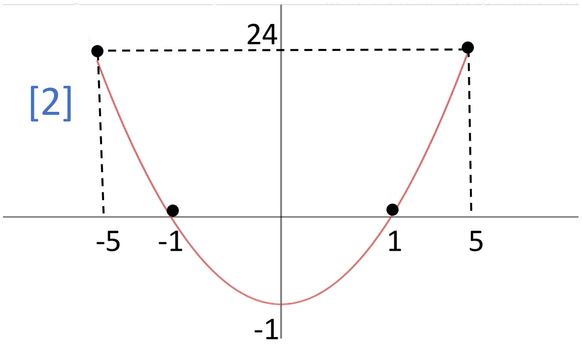
(Not Local because it’s end point pf the close interval)

طرف

$f(5) \to$ is absolute Maximum value (not Local like $f(-2)$)

Now if we have $f(x) = x^2 - 1$, on 1) [-1,5] , 2) [-5,5]


[1]


[2]

طرف مغلق ← مش محلي

$f(5) \to$ Local → absolute

X

1]Now Maximum

$f(-1) \to$ Local →الأطراف تهمل بما انها ليست مطلقة

X

طرف مغلق ← مش محلي

Minimum → $f(0) \to$ Local →absolute

2]Now Maximum

$f(5)$

$f(-5)$

→ absolute

يجوز أن توجد أكثر من قيمة مطلقة

Minimum → $f(0) \to$ Local →absolute

1] **Definition**: let **c** be a number in the Domail of $f(x)$

Then $f(c)$ is :

■ Absolute Maximum value of f on D if $f(c) \geq f(x)$

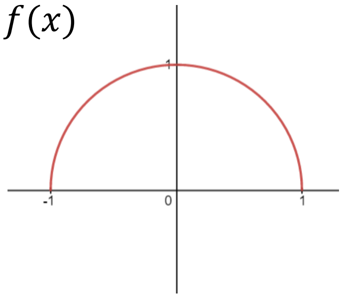
for all X in D (أكبر من أو تساوي جميع الصور في فترة المجال كاملة)

■ Absolute Minimum value of f on D if $f(x) \geq f(c)$

for all X in D (أقل من أو تساوي جميع الصور في فترة المجال كاملة)

Example: If $f(x) = \sqrt{-x^2 + 1}$,find the absolute Max and minimum values

$f(x)$



→$f(0)$ is the greatest values of $f(x)$ on D

(الأكبرمقارنة بمن حولها أيضًا)

So $f(0)$ → Absolute Maximum and Local

$\left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right]$ are the least values of $f(x)$ on D

so , $f(1), f(-1)$ →are absolute Minimum

2] **Definition**: The number $f(c)$ is a

∗ Local Maximum Value of f if $f(c) \geq f(x)$ when $f(x)$ is near c

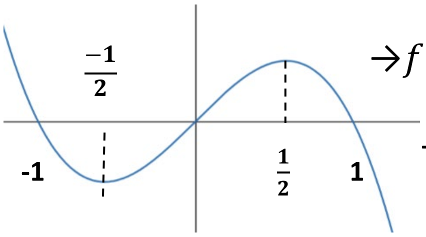
Or أي أكبرمقارنة بمن حوله

∗ Local minimum value of f if $f(x) \geq f(c)$ When $f(x)$ is near c

أي الأصغر مقارنة بما حوله

Example: if $f(x) = -x^3 + x$, find the Maximum Local Values

$f(x) \rightarrow$ graph



→$f\left(\frac{1}{2}\right) \rightarrow$ أكبر قيمة مقارنة بمن حولها لـِ $f(x)$

→$f\left(\frac{1}{2}\right) \rightarrow$ Max Local Value

$X \rightarrow \frac{1}{2}$

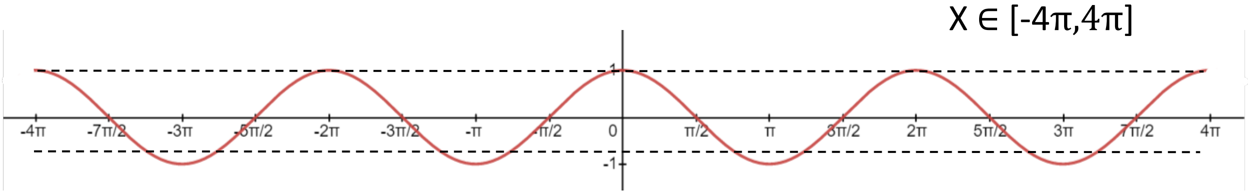
approach

→$f\left(\frac{-1}{2}\right) \rightarrow$ أقل قيمة مقارنة بمن حولها لـِ $f(x)$ →$f\left(\frac{-1}{2}\right) \rightarrow$ Min Local Value

Find the absolute Max &Minimum values and Local Max & Minimum values for each function:

1) $f(x) = \cos x$

X ∈ [-4π,4π]

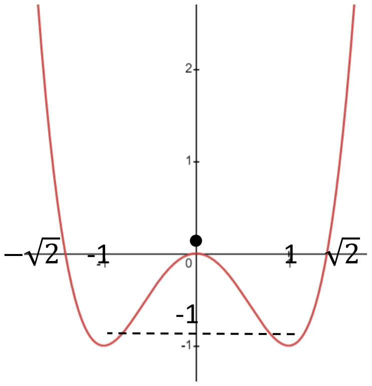


absolute max → $f(0), f(2\pi), f(-2\pi)$ →and Local

→ $f(4\pi), f(-4\pi)$

absolute min → $f(\pi), f(3\pi), f(-\pi), f(-3\pi)$ → and Local

2) $x^4 - 2x^2$

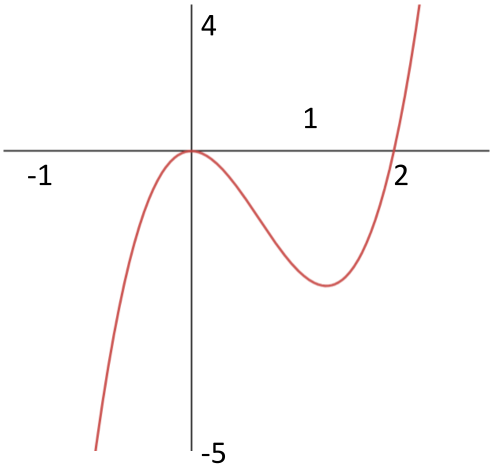


→absolute min → $f(-1), f(1)$ →and Local

Local Max → $f(0)$

absolute Max → { ﹏ }

3) $f(x) = 2x^3 - 3x^2$



absoulte max → $f(2)$

absolute min → $f(-1)$

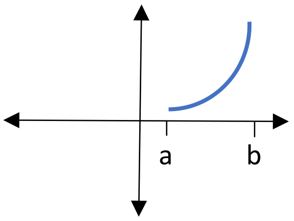
Local Max → $f(0)$

Local Min → $f(1)$

∗Note 1] Maximum and minimum values must be real numbers

2] If $f(x)$→D→[a,b] if$f(a), f(b)$ aren’t absolute values we ignores them.

3] The extrme values theorem: if f is continuous on a closed interval [a,b] then f attains an absolute Maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c , d on [a,b]

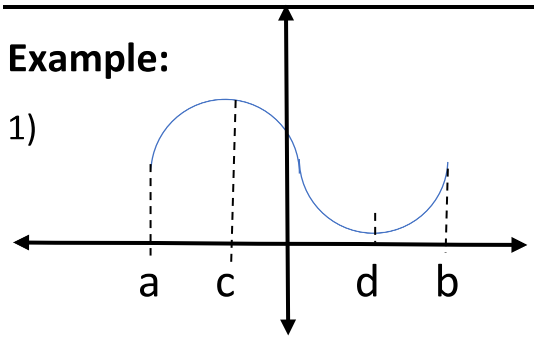


→$f(b)$ → absolute maximum

→ $f(c)$ → absolute minimum

Example:

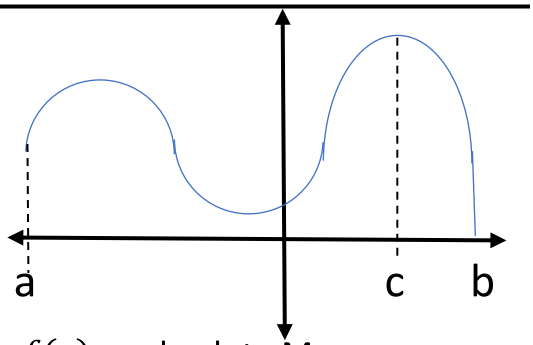
1)



$f(c)$ → absolute Max

$f(d)$ → absolute Min

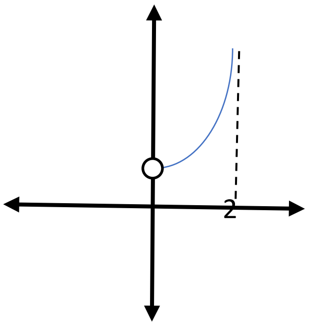
2)



$f(c)$ → absolute Max

$f(b)$ → absolute Min

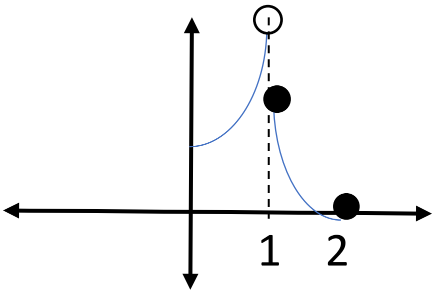
3)



$f(x)$ → continuos at (0,2)

has no absolute Max & Min

4)

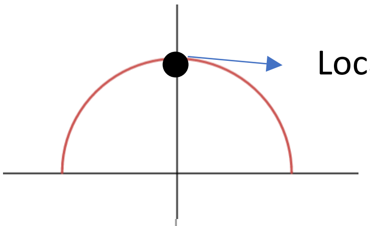


$f(x)$ → continuos at [1,2] – {1}

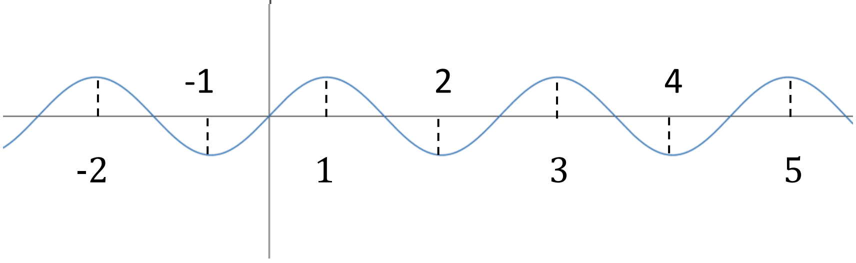
$f(x)$ → has minimum value →$f(2) = 0$

$f(x)$ → hasn’t maximum value

4] fermat's theorem : if f has a Local Maximum or minimum at c and $f'(c)$ exists , then $f'(c) = 0$



Local max → $f(0)$ →$f'(c)$ exists so $f'(0) = 0$



Local Max $f(-2), f(1), f(3), f(5)$

$f'(-2), f'(1), f'(3), f'(5)$ exists so

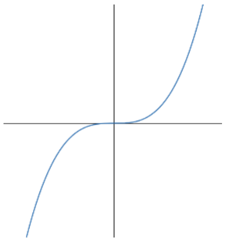
$f'(-2), f'(1), f'(3), f'(5) = 0$

Local min $f(-1), f(2), f(4)$

$f'(-1), f'(2), f'(4)$ exists so

$f'(-1), f'(2), f'(4) = 0$

If $f(x) = x^3$



→$f'(x) = 3x^2$

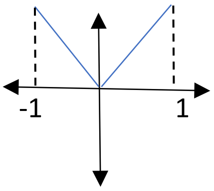
$f'(0) = 0$ → but $f'(0)$ isn't local max & min

So $f'(0)$ means that $f(x)$ has a horizantal tangent at x=0 instead of having a max or min Local values

5] Definition : a critical number of a function f is a number c in the Domin of f such the either $f'(c)$=0 or

$f'(c)$ does not exist , →$(c, f(c))$ critical point

If $f(x) = |x|$ on →[-1,1] →



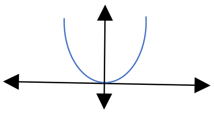
Critical number:

$f'(c)$ Dose not exist → $x = 1, -1, 0$

$f'(c) = 0$ →{ }

If $f(x) = x^2$, D ∈ R

Critical number

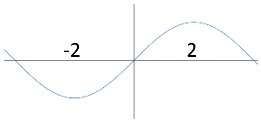


$f'(x) = 0 \rightarrow f'(x) = 2x = 0 \rightarrow x = 0$ $\underline{\underline{\text{or}}}$ $f'(x)$ → Does not exist → { }

6] if f has a Local Maximum or Minimum at c , then c is a critical number of f

Example:

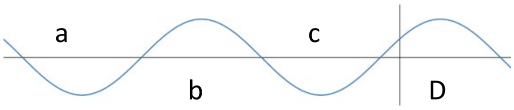
$f(x) \rightarrow$



$f(2) \rightarrow$ Local Maximum, 2 is a critical number

$f(-2) \rightarrow$ Local minimum, -2 is a critical number

$f(x) \rightarrow$



$f(a), f(c) \rightarrow$ Local Man →a, c are critical numbers

$f(b), f(D) \rightarrow$ Local Max → b, D are critical numbers

7] The close interval method : to find the absolute maximum and minimum values of a continuos function f on a closed

Interval [a , b]

1) find the values of f at the critical numbers of f in (a,b)

2) find the values of f at the endpoint of the interval($f(a)$ and $f(b)$)

3) The Largest of the values from step 1 and 2 is the absolute maximum value ,

the smallest of these value is the absolute minimum value.

Example : if $f(x) = \frac{x^3}{3} + \frac{5x^2}{2} + 6x$, on [-4 , 4] , find the absolute min & max values

Solution:

1] Critical numbers at $(-4 , 4)$, $f'(x) = 0 \rightarrow f'(x) = x^2 + 5x + 6$

$\rightarrow f'(x) = 0 \rightarrow x = -2 , -3$, $f'(x)$ does not exist → { }

$f(2) = -4.667$, $f(-3) = -4.5$

2] $f(-4) = -5.333$, $f(4) = 85.333$

→ Largest value → $f(4)$ → absolute max → 85.33

→smallist value→$f(-4)$→ absolute min → -5.333

Extra exercises:

1] If $f(x) = \sqrt{1 - x^2}$ on $\left[\frac{-1}{2}\ ,\ \frac{1}{2}\right]$, show that $f(x)$ attains absolute Max & min values

Solution:

using extreme value theorem → $f(x)$→ continuous on $\left[\frac{-1}{2}\ ,\ \frac{1}{2}\right]$ and differentable on $\left(\frac{-1}{2}\ ,\ \frac{1}{2}\right)$

then there are $\underbrace{f(c), f(d)}_{\text{on } \left[\frac{-1}{2}\ ,\ \frac{1}{2}\right]}$, where $f(c)$ → absolute max

$f(d)$ → absolute min

2] If $f(x) = x^3 - x$, [-3 , 3] , , show that $f(x)$ attains absolute Max & min values

Solution:

using extreme value theorem → $f(x)$→ continuous on $[-3\ ,\ 3\ ]$ and differentable on $(-3\ ,3)$

then there are $\underbrace{f(c), f(d)}_{\text{on } [-3\ ,\ 3]}$, where $f(c)$ → absolute max

$f(d)$ → absolute min

3] If $f(x) = \sqrt{x^2 + 1}$ and $f(x)$ has Local min at x=0

Where $f'(0)$ exist , show that $f'(0) = 0$

Solution:

Using fermat's theorem → f has local min at x=0 and $f'(0)$ exist

→then $f'(0) = 0$

4] let $f(x) = \frac{\tan x}{x}$, on which of the following intervals can we use the extreme value theorem

$*(0\,,\pi)\rightarrow$ ✗ becaues 1) The interval is not closed 2) $f(x)$ has discontinuity x=$\frac{\pi}{2}$

$*(0\,,\pi]\rightarrow$ ✗ becaues 1) The interval is not closed 2) $f(x)$ has discontinuity x=$\frac{\pi}{2}$

$*(0\,,\pi)\rightarrow$ ✗ becaues 1) $f(x)$ has discontinuity x=$\frac{\pi}{2}$

$*(1\,,2)\rightarrow$ ✗ becaues The interval is not closed

$*(1\,,2]\rightarrow$ ✗ becaues The interval is not closed

$*[1,2]\rightarrow$ ✓becaues 1) The interval is closed 2) $f(x)$ has continuous on the interval

5] If $f'(x)$ has 3 roots in (1 ,9) , show that $f(x)$ has at least max & min Local if $f'(2) = 4$, $f'(4) = -4$, $f'(5) = 1$

→ $f'(x)$ has 3 roots in (1,9) → $f'(x) = 0$, for 3 values on (1 , 9)

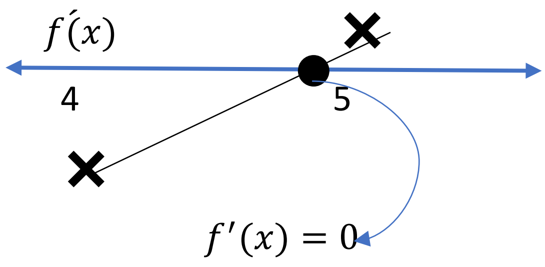
Now $f'(2) = 4\,, f'(4) = -4$

It means there are

$f'(c) = 0$ on (2 , 4)

So $f'(x) \rightarrow + \rightarrow \mathbf{0} \rightarrow -$

In [2 , 4] → max value



$f'(4) = -4$, and, $f'(5) = 1$

It there are $f'(d)$ on (4,5)

where

$f'(d) = 0$ so $f\acute{}(x) \rightarrow - \rightarrow$

$- \rightarrow +$

→ min value

So $f(x)$ has at least min and max Local values

If $\boldsymbol{f(x) = x^3 - 3x}$, on [-2 , 2] find the critical points

$f'(x) = 3x^2 - 3$, on(-2,2) how critical number

→ $f'(x) = 0$ **or** $f'(x)$ dose not exist

$f'(x) = 0 \rightarrow 3x^2 - 3 \rightarrow x = \pm 1$

$f'(x)$ does not exist → $x = \pm 2$

Critical point → $\{ (1, f(1)) , (-1, f(-1)) , (2, f(2)) , (-2, f(-2)) \}$

→ $\{ (1, -2) , (-1, -4) , (2, 2) , (-2 , -2) \}$

If $\boldsymbol{f(x) = \sqrt{1 - x^2}}$ on [-1 , 0) find the critical number

$f'(x) = \frac{-x}{\sqrt{1-x^2}}$, → $f'(x) = 0 \rightarrow -x = 0 \rightarrow x = 0$, but 0 is not include on the

interval so $f'(x) \neq 0$

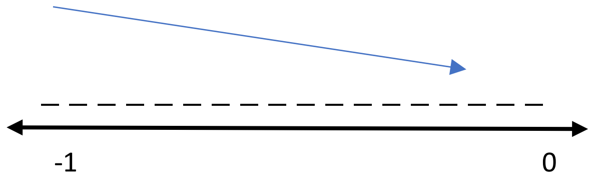
→ $f'(x)$ does not exist → $x = \pm 1 \rightarrow +1$ is not on the interval

So $x = -1$

Now critical number is $\underline{\underline{-1}}$

Critical point is $(-1 , f(-1)) \rightarrow (-1 ,\ 0)$

→ $f'(x)$



So $f(-1)$ is absolute max

Ex: If $f(x) = \sin x + \cos x$ on $[0.2\pi]$, find:

1) Critical numbers.
2) Critical points.
3) Local max & min.
4) Absolute max & min.

now → $f'(x) = \cos x + \sin x$ on $(0, 2\pi)$

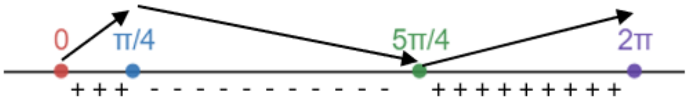
critical numbers → $f'(x) = 0 \rightarrow \sin x = \cos x \rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

→ $f'(x)$ doesn't exist → $x = 0, 2\pi$

1) Critical numbers are $\left\{0, \frac{\pi}{4}, \frac{5\pi}{4}, 2\pi\right\}$

2) Critical points $\left\{(0, f(0)), \left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right), \left(\frac{5\pi}{4}, f\left(\frac{5\pi}{4}\right)\right), (2\pi, f(2\pi))\right\}$

$$\left\{(0,1), \left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right), (2\pi, 1)\right\}$$



3) $f'(x) \rightarrow$

$f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ → local min

$f\left(\frac{\pi}{4}\right) = \sqrt{2}$ → local max

4) min → $f(0) = 1, f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ so $f\left(\frac{5\pi}{4}\right)$ is absolute min

max → $f\left(\frac{\pi}{4}\right) = \sqrt{2}, f(2\pi) = 1$ so $f\left(\frac{\pi}{4}\right)$ is absolute max

---

Ex: If $f(x) = x^2 + \frac{2}{x}$, Domain $= R - \{0\}$.

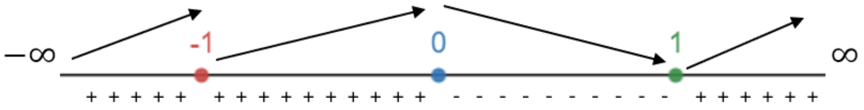
$f(x) = 2x + \frac{-2}{x} \to f'(x) = \frac{2x^2-2}{x}$

$f'(x)\ D.N.E \to x = 0$ but it's not included on the domain so $f'(x)$ exists.

$f'(x) = 0 \to 2x^2 - 2 = 0 \to x = \pm 1$

now

$f'(x) \to$



no absolute max & min

local min → $f(1) = 3$

If $f(x)$ is a polynomial of the nth degree, where $(n > 1)$ show that $f(x)$ has at most $n + 1$ critical numbers at $[a, b]$.

and at least 3 critical numbers when $n \to$ even

and at least 2 critical numbers when $n \to$ odd

***Explain $f(x) = x^3 - x^2 \to f'(x) = 3x^2 - 2x$**

$f(x)$ is a polynomial of the 3^{rd} degree

so $f'(x)$ is a polynomial of the 2^{nd} degree, so $f'(x)$ hat at most 2 roots, so $f'(x)$ on (a, b) has at most 4 critical numbers: 1) 2 roots of $f'(x)$

2) 2 → $f'(x)\ D.N.E$

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*مثال يتضمن إثبات سؤال كتاب

$f'(x)$ is a polynomial of the $(n - 1)$ degree?

so $f'(x)$ has at most $n - 1$ roots on (a, b) and $f'(x)$ has 2 critical numbers → $f'(x)$ does not exist at $x = a, b$.

so, the most number of critical numbers $= n - 1 + 2 = n + 1$

$f'(x)$ is a polynomial of $n - 1$ degree.

- When $n \to$ odd, $n-1 \to$ even

So $f'(x)$ has at least <u>no roots</u>
And 2 critical numbers → a, b

So, the least number of critical numbers $= 2 + 0 = 2$

- When $n \to$ even, $n-1 \to$ odd

So $f'(x)$ has at least <u>1 root</u>
And 2 critical numbers → a, b

So, the least number of critical numbers $= 2 + 1 = 3$

If $f(x) = \begin{cases} [x+2] & 0 > x \geq -3 \\ 2+x^2 & x > 0 \end{cases}$ find critical numbers of $f(x)$ when $f'(x)$ $D.N.E$

$$f(x) = \begin{cases} -1 & -2 > x \geq -3 \\ 0 & -1 > x \geq -2 \\ 1 & 0 > x \geq -1 \\ 2 & x = 0 \\ 2+x^2 & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} 0 & -2 > x \geq -3 \\ 0 & -1 > x \geq -2 \\ 0 & 0 > x \geq -1 \\ 2x & x > 0 \\ D.N.E & x = \{0, -1, -2, -3\} \end{cases}$$

so critical numbers of $f(x)$ when $f'(x)$ does not exist are $\{0, -1, -2, -3\}$

If $f(x) = \begin{cases} \frac{x^2}{2} & x \leq 1 \\ \frac{1}{2} & x > 1 \end{cases}$, find critical numbers

$$f'(x) = \begin{cases} x & x < 1 \\ 0 & x > 6 \\ D.N.E & x = 1 \end{cases}$$

Critical numbers: 1) $f'(x) = 0 \to (1, \infty)$ & $x = 0$

2) $f'(x)$ does not exist → $x = 1$

so that critical numbers $= [1, \infty) \cup \{0\}$.

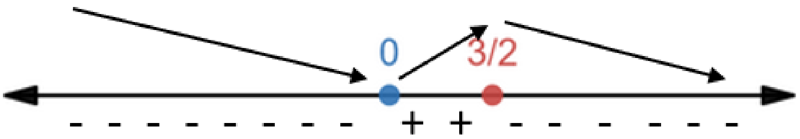
If $f(x) = x^{\frac{3}{5}} \cdot (4 - x)$, find critical numbers of $f(x)$.

$f'(x) = -x^{\frac{3}{5}} + \frac{3}{5} \cdot x^{\frac{-2}{5}} \cdot (x-4) = -x^{\frac{3}{5}} + \frac{3\cdot(x-4)}{5\cdot x^{\frac{3}{5}}}$

$= \frac{-5x+12-3x}{5\cdot x^{\frac{3}{5}}} = \frac{12-8x}{5\cdot x^{\frac{3}{5}}} \rightarrow f'(x) = 0 \rightarrow 12 - 8x = 0 \rightarrow x = \frac{3}{2}$

$\rightarrow f'(x)\ does\ not\ exist \rightarrow x = 0$

Critical numbers = $\left\{\frac{3}{2}, 0\right\} \rightarrow$



absolute values → $f\left(\frac{3}{2}\right) \rightarrow\ max$

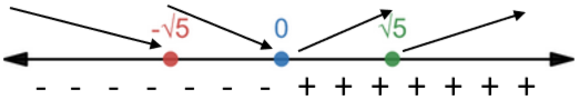
$f(0) \rightarrow local\ min$

$f\left(\frac{3}{2}\right) \rightarrow local\ max$

If $f(x) = (x^2 - 5)^3$ find critical numbers and absolute max & min values.

$f'(x) = 6x \cdot (x^2 - 5)^2 \rightarrow f'(x) = 0 \rightarrow 6x = 0 \rightarrow x = 0, x^2 - 5 = 0 \rightarrow x = \pm\sqrt{5}$

→ $f'(x)$ does not exist → { }



→ $f(0)$ → absolute min and local

If $f(x) = x^2 \cdot \ln x$, find critical umbers, $x > 0$.

$f'(x) = \frac{x^2}{x} + 2x \cdot \ln x = 0$

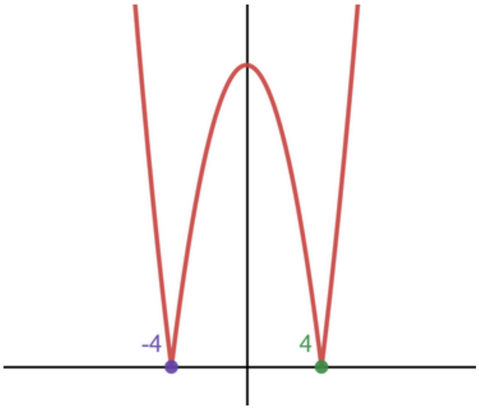
$x + 2x \cdot \ln x = 0 \rightarrow x(1 + 2\ln x) = 0$

$x \neq 0 \qquad 1 + 2\ln x = 0$

$\ln x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

If $f(x) = \left|16 - x^2\right|$, find critical numbers →



$$f(x) = \begin{cases} 16 - x^2 & 4 \geq x \geq -4 \rightarrow |x| < 4 \\ x^2 - 16 & |x| > 4 \end{cases}$$

$$f'(x) = \begin{cases} -2x & 4 > x > -4 \\ 2x & x > 4, x < -4 \\ D.N.E & x = -4, 4 \end{cases}$$

$f'(x) = 0 \rightarrow x = 0$

$f'(x)$ does not exist → $x = 4, -4$

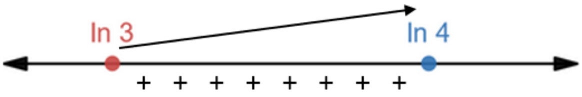
Critical numbers $= \{-4, 0, 4\}$.

Find the absolute max of $f(x) = 4 \cdot \sinh x$ on the interval $[\ln 3, \ln 4]$.

$f'(x) = 4 \cdot \cosh x = 4 \cdot \frac{1}{2}(e^x + e^{-x})$

$f'(x) = 0 \rightarrow$غير ممكن

$f'(x)$ does not exist → $x = \ln 3, \ln 4$



→ absolute max → $f(\ln 4)$

$$= 2 \cdot \left(e^{\ln 4} + e^{-\ln 4}\right)$$

$$= 2 \cdot \left(e^{\ln 4} + e^{\ln \frac{1}{4}}\right)$$

$$= 2 \cdot \left(4 + \frac{1}{4}\right) = \frac{17}{2}$$

The local maximum value of the function $f(x) = x^3 \cdot e^{-x}$

→$f'(x) = x^3 \cdot -e^{-x} + e^{-x} \cdot 3x^2 = 0 \rightarrow e^{-x} \cdot x^2(3 - x) = 0$

$x = 0 \qquad x = 3$

→$f'(x)$ does not exist → غير ممكن

→ local max → $f(3) = \frac{27}{e^3} = 27 \cdot e^{-3}$

If $g(x) = a\sin x - \cos ax$, $x \in \left[0, \frac{3\pi}{2}\right]$, $a = \frac{k}{8} \neq 0$.

if $g(x)$ has a critical number at $x = \frac{\pi}{3}$, the find k.

*$g(x) \rightarrow$ has critical number at $x = \frac{\pi}{3}$ and $g'\left(\frac{\pi}{3}\right)$ exists so $g'\left(\frac{\pi}{3}\right) = 0$

$$g'(x) = a\cosh x + a\sin ax \rightarrow g'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = -\sin\frac{a\pi}{3}$$

$$\cos\frac{\pi}{3} = -\sin\frac{7\pi}{6}$$

$$\frac{7\pi}{6} = a\frac{\pi}{3} \rightarrow a = \frac{7}{2}$$

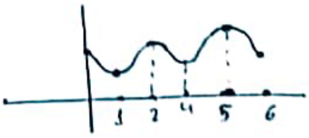
$$\frac{k}{8} = \frac{7}{2} \rightarrow k = 28$$

Sketch the graph of $f(x)$ such that $f(x)$ is continuous at $[0, 6]$, and $f(x)$ has absolute max at $x = 5$, absolute min at $x = 1$, local max at $x = 2$ and local min at $x = 4$.

يمكن رسم عدد لا نهائي من

الاقترانات وهذه واحدة فقط منها

بشكل أوضح



The Mean Value Theorem (M.V.T)

We will discuss many results of this chapter depend on one central fact which is M.V.T

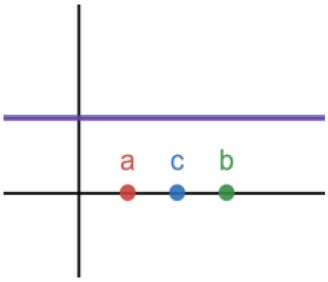
But to arrive at this M.V.T we first need to talk about:

Rolle’s Theorem

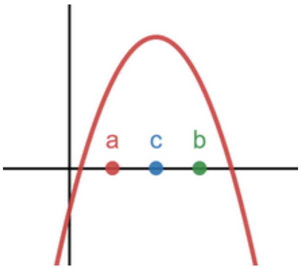
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), $f(a)=f(b)$ then there is a number that $c\in(a,b)$,

such that $f'(c)=0$

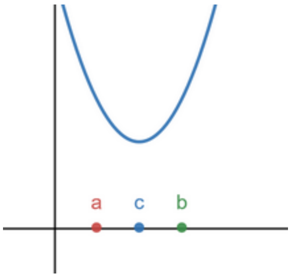
Case 1



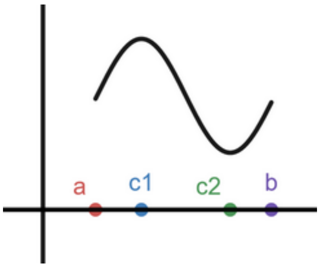
Case 2



Case 3



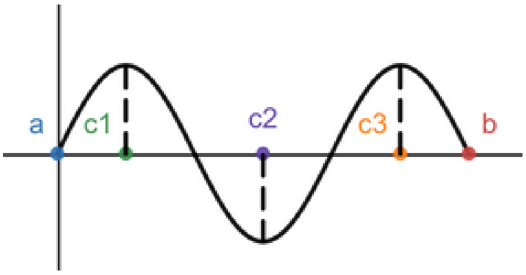
Case 4



We notice that in all cases $f(x)$ is:

1) Continuous
2) Differentiable
3) Has 1 number at least that $f'(c)=0$
4) $f(a)=f(b)$ شرط مهم جداً

Ex: Can we say that in the following graph of function that there are 3 numbers that satisfies Rolle's theorem ($f'(c) = 0$)??



$\rightarrow f(x)$

1) $f(x)$ cont. on $[a, b]$

2) $f(x)$ diff. on (a, b)

3) $f(a) = f(b)$

■ $f(x)$ satisfies Rolle's theorem

Ex: Find c that satisfies Rolle's theorem on $f(x) = 5 - 12x + 3x^2$ on $[1, 3]$.

1) $f(x)$ cont. 2) $f(x)$ is diff. 3) $f(1) = f(3) = -4$

$f'(c) = 0$

$f'(x) = -12 + 6x, \quad f'(x) = 0$

$6x = 12 \rightarrow x = 2$ So, at $c = 2 \rightarrow f'(c) = 0$

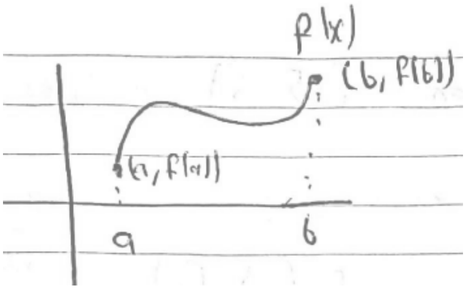
Mean Value Theorem (M.V.T)

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$, so that

ميل المماس $f'(c) = \frac{f(b)-f(a)}{b-a}$ ميل القاطع

وسيتم توضيحها بالرسم

*ملخص النظرية أنه يوجد مماس آخر يوازي ميل القاطع بين النقطة (a/b) وله نفس الميل لأنه يوازيه



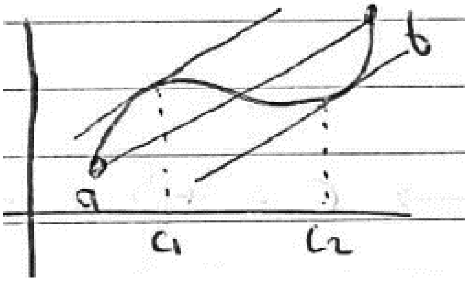
1) $f(x)$ is cont.

2) $f(x)$ is diff.

*لنفترض وصلنا خط بين a, b واجيت وطلعت ميله فبكون كالآتي:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

*بتحكيلك النظرية انه رح يكون في عدد المشتقة عنده (ميل المماس) يساوي ميل القاطع يعني:



*بلاحظ انه المشتقة عند c_1, c_2 (ميل المماس) الخط الي بعمله

بوازي خط ab يعني الهم نفس الميل فمن هون اجت القاعدة

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex: Find c that satisfies M.V.T for:

1) $f(x) = x^3 - x$ on $[0,2]$

- $f(x)$ is cont. on $[0, 2]$ → $p_1\ (0, 0)$, $p_2\ (2, 0)$
- $f(x)$ is diff. on $(0, 2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$(3x^2 - 1)|_{x=c} = \frac{f(2) - f(0)}{2 - 0}$$

$$3c^2 - 1 = \frac{6 - 0}{2}$$

$$3c^2 = 8 \rightarrow c = \frac{2}{\sqrt{3}}, \cancel{-\frac{2}{\sqrt{3}}}, \quad -\frac{2}{\sqrt{3}} \rightarrow \text{x outside the interval } [0, 2]$$

---

2) $f(x) = \sqrt{25 - x^2}$ on $[-5, 3]$

- $f(x)$ is cont. → $p_1\ (-5, 0)\ , p_2\ (3, \sqrt{4})$
- $f(x)$ is diff.

$$f'(c) = \frac{f(3) - f(-5)}{3 - -5}$$

$$\frac{-2c}{2\sqrt{25 - c^2}} = \frac{4 - 0}{8}$$

$$\sqrt{25 - c^2} = -2c$$

$$25 - c^2 = 4c^2$$

$$25 = 5c^2 \rightarrow c = \pm\sqrt{5}$$

$\sqrt{25 - c^2} = -2c$

$-\sqrt{5} \rightarrow\ \sqrt{20} = \sqrt{20}$

$\sqrt{5} \rightarrow \sqrt{20} \neq \sqrt{-20}$

so, the answer is at $c = -\sqrt{5}$

Ex: If $f(0) = -3, f'(x) \leq 5$, how large can $f(2)$ possible be??

a) $f(2) \geq 7$
b) $f(2) \leq 7$
c) $f(2) = 7$
d) $f(2) = 0$
e) can't be found

In this type of question, we must use M.V.T

ما دامت المشتقة مدركة امام الابصار فهذا معناه انها كانت متصلة وانها ليست المرة الاولى التي تم الاشتقاق بها

$f'\,(x) = \frac{f(b)-f(a)}{b-a}$ p1 (0,-3) / p2 (2,f(2))

$f'(x) \leq 5$

$\frac{f(b)-f(a)}{b-a} \leq 5$

$\frac{f(2)--3}{2-0} \leq 5$

$f(2) \leq 5$ so the max value of f(2) =7

Ex: find the value of c that satisfies the M.V.T

1) $f(x) = \frac{x^2}{x^2+1}$, [0,2]

F is cont -> f is differentiable

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$\frac{2x(x^2+1)-x^2*2x}{(x^2+1)^2} = \frac{\frac{4}{5}-1}{2}$ then we substitute c instead of x

$$c^2 + 20c + 1 = 0$$

$$c1, c2 = \frac{-20\pm\sqrt{396}}{2}$$

2) $f(x) = \frac{1}{x}$, $[1,3]$ H.W

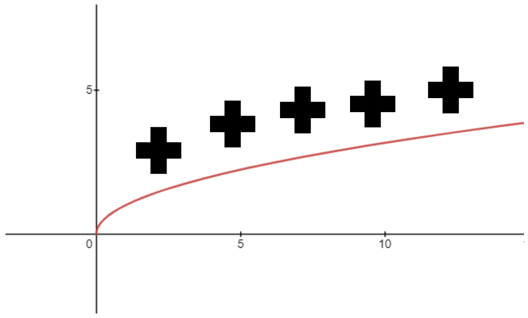
Ans $c = \pm\sqrt{3}$

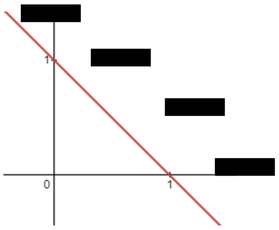
3) $f(x) = (x-3)^{-2}$, $[1,4]$

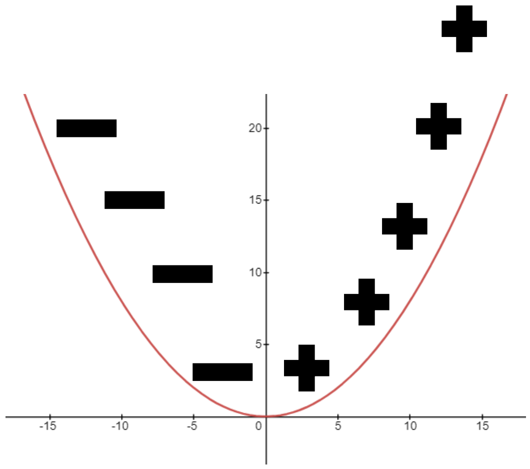
Not continuous on [1,4] because of 3

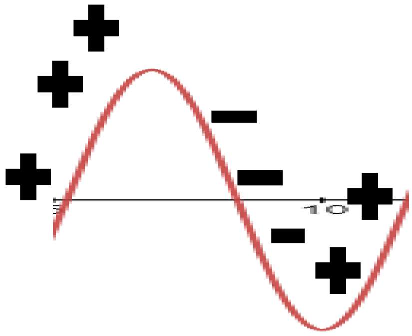
So we cant use M.V.T because it doesn't satisfy its terms

Increasing & decreasing







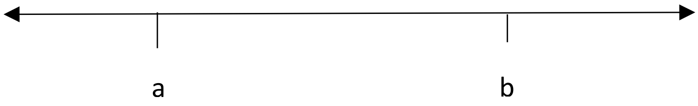


We can say that if $f'(x) > 0$, f $is\ increasing$

$f'(x) < 0, f$ $is\ decreasing$

To find the interval of increasing & decreasing we must :

$f(x) \rightarrow f'(x) \rightarrow f'(x) = 0 \rightarrow c.n \rightarrow$ دراسة الاشارة على خط الاعداد



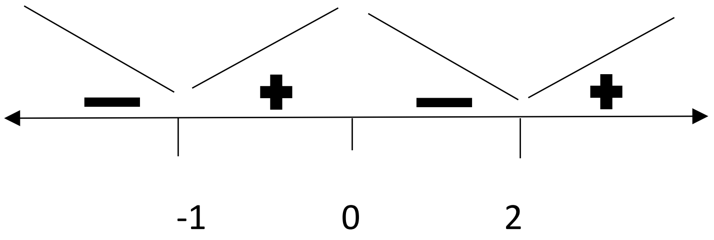
We take the signs of increasing and decreasing from $f'(x)$

Ex: find critical numbers and interval of increasing and decreasing and maximum and minimum values of:

$f(x) = 3x^4 - 4x^3 - 12x^2 + 4$

$f'(x) = 12x^3 - 12x^2 - 24x = 0$

c.n = 0,-1,-2



so f is increasing from (-1,0) U (2,∞) , and f is decreasing from

(-∞,-1) U (0,2)

ملاحظة: الامر ليس بيد احد منا ولكن الكتاب امرنا ان نبقي الفترات مفتوحة , فلا تطلق العنان لتفكيرك وتجعل دماغك يقنعك ان تجعلها مغلقة ففي النهاية لا عليم باحوال الدنيا كلها الا رب العالمين

Note:

-If the differential of f changes from + to – then f has a local max. at c

-If the differential of f changes from - to + then f has a local min. at c

So at the previous example we had:

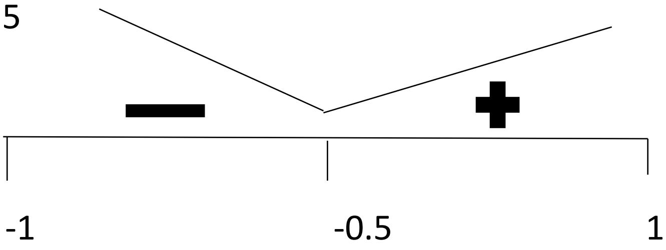
1) Local max at x=0 (not abs)
2) Local min at x=-1,2

2) $f(x) = ln(x^2 + x + 1)\ on\ [-1,1]$

$f'(x) = \frac{2x+1}{x^2+x+1}$

Zeros of numerator and denominator

c.n at x=-0.5



F is increasing from (-0.5,1) and decreasing from (-1,-0.5)

Also, we have local min. absolute at x=0.5

3) $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 0$

c.n -> x=0,2

f is increasing from (-∞,0)U(2,∞)

f is decreasing from (0,2)

f has local max at x=0, and local min x=2

4) f(x) = cosx ,[0,2π]

$f'(x) = -sinx = 0$, x=0,π,2π,3π,……

لكن ما يهمنا هو من لم يتجاوز حدود فترتنا

F is increasing from (π,2π)

F is decreasing from (0,π)

F has local min (abs) at x=π

5) g(x)= x+2sinx , [0,2π]

$g'(x) = 1 + cosx = 0$

Cosx=-0.5 , x=$\frac{2\pi}{3}$,$\frac{4\pi}{3}$

F increase from (0,$\frac{2\pi}{3}$)U($\frac{4\pi}{3}, 2\pi$)

Decrease from ($\frac{2\pi}{3}$,$\frac{4\pi}{3}$)

F has local min at x=4π/3

And local max at x=2π/3

Concavity (التقعر)

Definition: if the graph of ‘f’ lies above all of its tangents on an interval, then it’s called concave upward, if the graph of ‘f’ lies below all of its tangents it’s called concave downward.

In another meaning if $f''(x) > 0 \to concave\ upward$

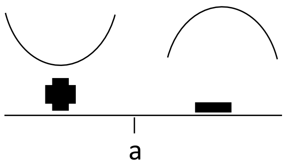
$$f''(x) < 0 \to concave\ downward$$

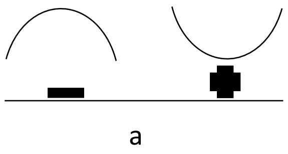
This is called concavity test

Steps $f(x) \to\ f'(x) \to\ f''(x) \to\ f''(x) = 0$ دراسة الاشارة على الخط

-inflection point: it’s a point that f(x) changes from concave upward

To concave downward, vice versa.





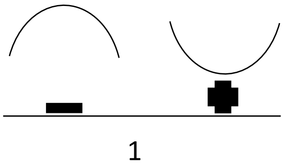
(a) is an inflection point

Ex: find interval of concavity and inflection points of following functions:

1)$f(x) = x^3 - 3x^2 - 9x + 4$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 6x - 6$



f(x) concaves upward (1,∞)

f(x) concaves downward (-∞,1)

f(x) has an infliction point at: (1,f(1)): (1,-7)

2)$f(x) = 3x^4 - 4x^3 - 12x^2 + 2$

$f'(x) = 12x^3 - 12x^2 - 24x$

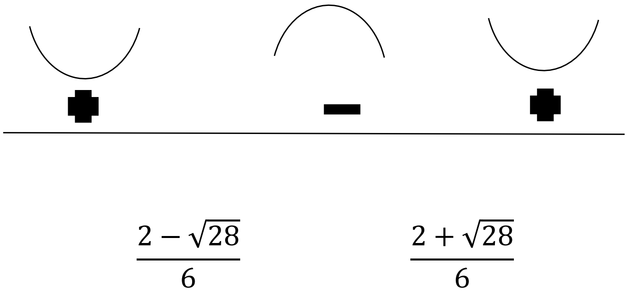
$f''^{(x)} = 36x^2 - 24x - 24 = 0$

$3x^2 - 2x - 2 = 0$, a=3, b=-2, c= -2.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 - -24}}{6}$$

$$x = \frac{2 \pm \sqrt{28}}{6}$$



f(x) concaves upward (-∞, $\frac{2-\sqrt{28}}{6}$)∪ ($\frac{2-\sqrt{28}}{6}$,∞)

f(x) concaves downward ($\frac{2-\sqrt{28}}{6}$, $\frac{2-\sqrt{28}}{6}$)

Infliction points at: x=$\frac{2\pm\sqrt{28}}{6}$

Examples:

Find:

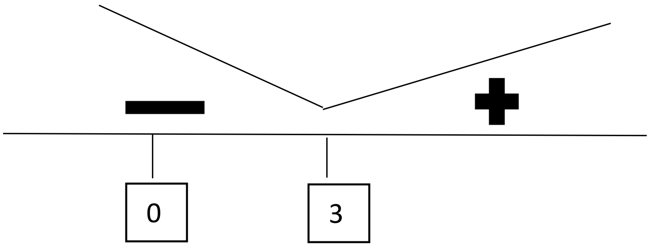
a) Interval of increasing and decreasing.
b) Local max and min values.
c) Interval of concavity.

$1 - f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 0 \rightarrow 4x^2(x-3) = 0$

$x = 0\ OR\ x = 3 \rightarrow Critical\ Numbers$

$(0, f(0))\ and\ (3, f(3)) \rightarrow Critical\ Points$



f is increasing in the interval (3, ∞)

f is decreasing in the interval (-∞,3)

Local min at (3, f (3))

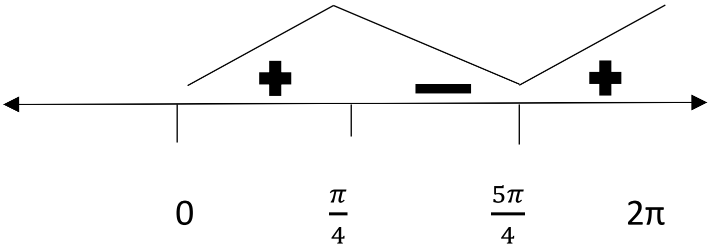
$f''(x) = 12x^2 - 24x^1 = 0 \rightarrow 12x$(x-2) =0

$x = 0\ OR\ x = 2$

$2 - f(x) = \sin x + \cos x\ ,\ \ 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x, \sin x = \cos x, x = \frac{\pi}{4}, \frac{5\pi}{4} \rightarrow Critical\ Numbers$

X

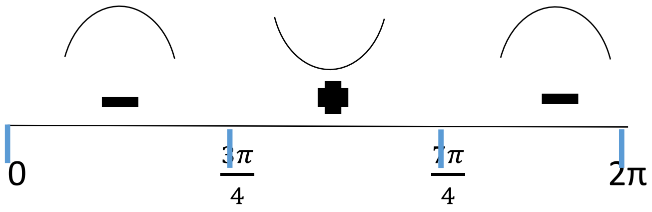


f is increasing in the interval $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi)$

f is decreasing in the interval $(\frac{\pi}{4}, \frac{5\pi}{4})$

Critical points at: $(\frac{\pi}{4}, f(\frac{\pi}{4}))$, $(\frac{5\pi}{4}, f(\frac{5\pi}{4}))$

$$f''(x) = -\cos x - \ \sin x, \sin x = -\cos x, x = \frac{3\pi}{4}, \frac{7\pi}{4}$$



f concaves downward in the interval $(0, \frac{3\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$

f concaves upward in the interval $(\frac{3\pi}{4}, \frac{7\pi}{4})$

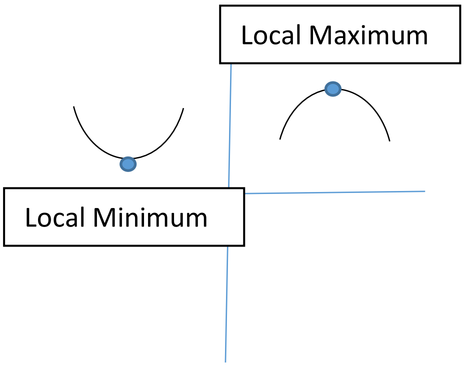
Infliction points: $\left(\frac{3\pi}{4}, \mathrm{f}\left(\frac{3\pi}{4}\right)\right), \left(\frac{7\pi}{4}, \mathrm{f}\left(\frac{7\pi}{4}\right)\right)$

The Second Derivative Test

اختبار المشتقه الثانيه

Suppose f’’(x) is continuous near c.

a) If f’(x)=0 and f’’(x)>0, then f has local maximum at c.
b) If f’(x)=0 and f’’(x)<0, then f has local minimum at c.



Example:

Using second derivative test, find all local extremes for:

$$f(x) = \sin x + \cos x \,, x \in [0,2\pi]$$

$$f'(x) = \cos x - \sin x, x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$f''(x) = -\sin x - \cos x$$

$$f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0 \;\;, local \max at\ x = \left(\frac{\pi}{4}\right)$$

$$f''\left(\frac{5\pi}{4}\right) = \sqrt{2} > 0, local \min at\ x = \left(\frac{5\pi}{4}\right)$$

The graphs of f(x), f’(x) and f’’(x).

If:

F(x) is increasing, f’(x) will be positive.

F(x) is decreasing, f’(x) will be negative.

Also, if:

F(x) concaves up, f’’(x) will be positive.

F(x) concaves down, f’’(x) will be negative.

Finally, if:

F’(x) is increasing, f’’(x) will be positive and f(x) will concave up.

F’(x) is decreasing, f’’(x) will be negative and f(x) will concave down.

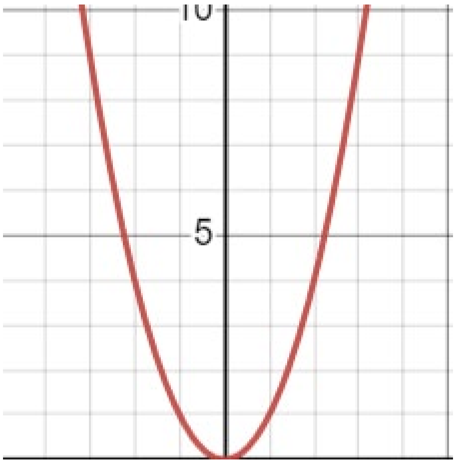
Example:

$f(x) = x^2$

f’(x) increases on (0, ∞)

f’(x) decreases on (-∞,0)

f’’(x) is positive everywhere.

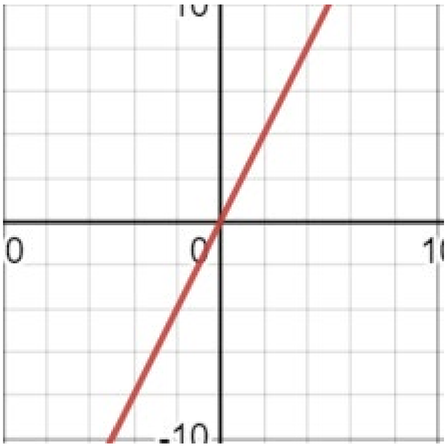


$f'(x) = 2x$

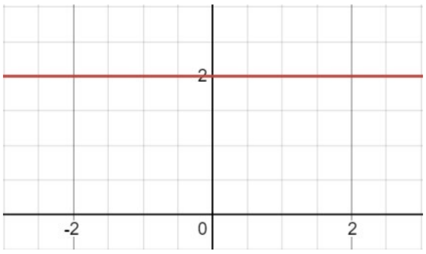
When f’(x) is above x-axis, f(x) is increasing.

When f’(x) is below x-axis, f(x) is decreasing.

f’(x) is increasing on R, thus f’’(x) is positive on R.



$f''(x) = 2$



When f’’(x) is above x-axis, f’(x) is increasing.

When f’’(x) is below x-axis, f’(x) is decreasing.

Since f’’(x) is positive on R, f(x) concaves up on R.

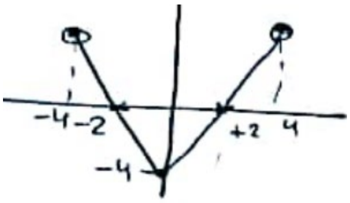
If the graph shown is the f’(x) graph,

Find:

a) Intervals of increasing and decreasing f(x).

2b) Intervals when f(x) concaves up and concave down.

c) Critical points.



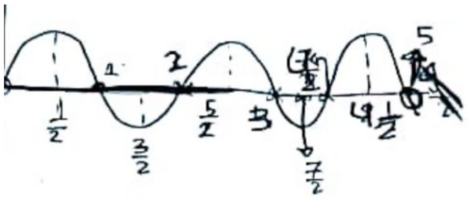
a) f(x) is increasing when f’(x) > 0: R\[-2,2]

b) f(x) concaves up when f’’(x) is positive AND when f’(x) is increasing. [0,4)

f(x) concaves down when f’’(x) is negative AND when f’(x) is decreasing. (-4,0]

c)f’(x) = 0 at x = ±2 AND when f’(x) D.N.E, at x =±4

If f(x):



f(x) is increasing on: [0,1] ∪ [2,3] ∪ [4,5]

f(x) is decreasing on: [1,2] ∪ [3,4]

f(x) concave down when f’’(x) is negative, when f’(x) is decreasing :

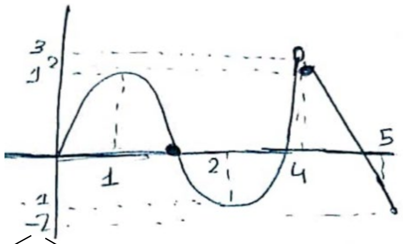
$[\frac{1}{2},\frac{3}{2}] \cup [\frac{5}{2},\frac{7}{2}] \cup [\frac{9}{2},5]$

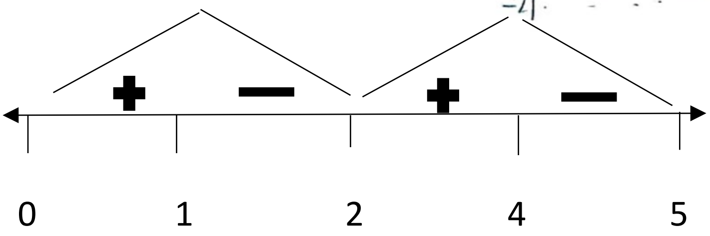
f(x) concave up when f’’(x) is positive, when f’(x) is increasing :

$[0,\frac{1}{2}] \cup [\frac{3}{2},\frac{5}{2}] \cup [\frac{7}{2},\frac{9}{2}]$

If f(x):

1)f’(x)





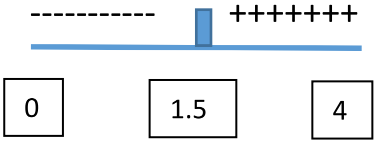
Local max at f(1) =1

Local min at f(2) =-1

No absolute max.

Absolute min f(5)=-2

2)f’(x)



Infliction point (1.5,f(1.5))→(1.5,0)

f(x) concave up when f’’(x) is positive, when f’(x) is increasing :

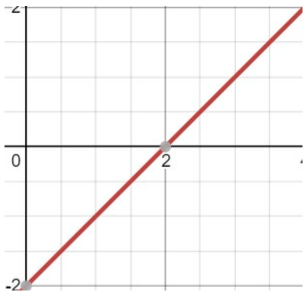
$[\frac{3}{2},4]$

f(x) concave down when f’’(x) is negative, when f’(x) is decreasing :

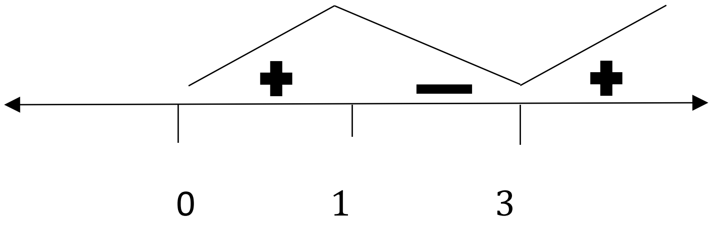
$[0,\frac{3}{2}]$

If f(x) has 2 critical points at x=1,3.

And the graph of f’’(x) is:



What is the value of f(1) and f(3)?



By second derivative test:

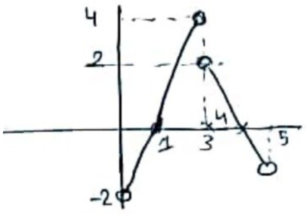
f’’ (1) is negative, thus f(1) is a max value.

f’’ (1) is positive, thus f(1) is a min value.

f(x)is increasing on R/ [1,3]

f(x)is decreasing on [1,3]

The graph shown is f’(x) where f(x) is continuous at [0,5]



a) Intervals of increasing and decreasing f(x).

b) Intervals when f(x) concave up and concave down.

c) Critical points.

a) f(x) is increasing when f’(x) > 0: [1,4]

f(x) is decreasing when f’(x) < 0: [0,1] ∪[4,5]

b) f(x) concaves up AND when f’(x) is increasing. [0,3)

 f(x) concaves down when f’(x) is decreasing. (3,5]

Infliction point: (3, f(3)).

c)f’(x) = 0 at x = 1,4 AND when f’(x) D.N.E, at x =0,3,5

CHAPTER ‘5’

The Integration

The Antiderivative

معكوس المشتقة

Definition: a function F called an antiderivative of f on an interval I if :

$F^1(x) = f(x)$ of all x in I

Let Take that $f(x) = x^2$, So F(x) maybe F(x) =$\frac{x^3}{3}$ or $\frac{x^3}{3} + 2$ or $\frac{x^3}{3} - \frac{1}{4}$ or $\frac{x^3}{3} + \sqrt{3}$ .....

So, we can say that the antiderivative of f(x) is F(x) +c

F(x) → f(x) → $f'(x)$ اشتقاق

F(x) ← f(x) ← $f'(x)$ تكامل

*بقدر احكي معكوس المشتقة هي الحالة التي كان عليها الاقتران قبل الاشتقاق او بالأحرى (التكامل)

* الدرس مقدمة عن التكامل

* الفرق بين معكوسات المشتقات هو الثابت

$$\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}$$

تذكر أن:

Ex: find f if $f^1(x) = e^x + 20(1+x^2)^{-1}$ and f(0)=-2

بدي ارجع خطوة لورى يعني بدي اصل f(x)

فبدي اتخيل كيف شكل الاشتقاق او بالأحرى (التكامل)

f(x) = $e^x + 20\,tan^{-1} x + c$

finding c by f(0)=-2

1+0+c=-2

C=-3

So $f(x) = e^x + 20\,tan^{-1} x - 3$

If $F(x) = c_1\, x \sin x + c_2 \cos x$, an antiderivative of f(x) = x cosx

Evaluate $3c_1+5c_2$

Sol F(x) → f(x)

فبدي اشتق F(x) وأساويها ب f(x)

$F^1(x)= f(x)$

$c_1 \, x \cos x + c_1 \sin x - c_2 \sin x = x \cos x$

بساوي المعاملات ببعض

$C_1 \, x \cos x = x \cos x$ so $c_1=1$

$C_1 \sin x - c_2 \sin x =0$

$C_1 \sin x= c_2 \sin x$

$C_1=c_2=1$

→ *$3c_1+5c_2=8$*

ومن هون هاد الدرس بنفوت على بوابة التكامل ورح نشرح قوانين التكامل والتكامل المحدد وغير المحدد

Definite integral:

التكامل المحدود

$\int_a^b f(x)\,dx$ → *area under the curve of f(x) on the interval [a,b]*

b → upper limit a→lower limit

properties (الخصائص)

1 $\int_a^a f(x)\,dx=0$

2 $\int_b^a f(x)\,dx= - \int_a^b f(x)\,dx$

3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4 $\int_a^b \quad ((f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5 $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$

Integration of circle:

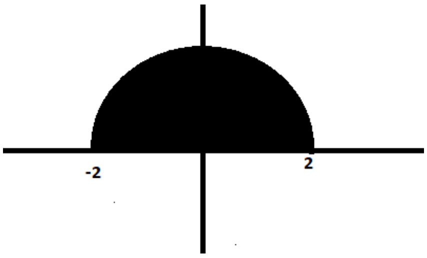
$x^2+y^2 = r^2 \rightarrow y = \pm\sqrt{r^2 - x^2}$

+ نصف علوي

- نصف سفلي

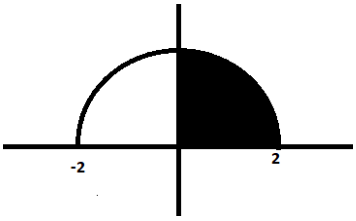
Ex : evaluate the following integrals :

1. $\int_{-2}^{2} \sqrt{4 - x^2}\, dx \longrightarrow$



$\int dx$=area $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4) = 2\pi$

2. $\int_{0}^{2} \sqrt{4 - x^2}\, dx \longrightarrow$



Area= $\frac{1}{4}\pi r^2$=$\frac{1}{4}\pi(4) = \pi$

3. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2 - x^2}\, dx$

Area = $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\sqrt{2}\right)^2 = \pi$

4. $\int_{0}^{3} -\sqrt{9 - x^2}\, dx$

$\int dx$=Area=$\frac{-1}{4}\pi r^2$=$\frac{-1}{4}\pi(9) = -\frac{9}{4}\pi$

لا يوجد مساحة سالبة ولكن تعني تحط خط الأعداد

Remark : $\int_{a}^{b} f(x)\, dx = \int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$

Ex : if $\int_{0}^{10} f(x)\, dx$ = 17 , $\int_{0}^{8} f(x)\, dx$ = 12 find $\int_{8}^{10} f(x) dx$

$\int_{8}^{10} dx \rightarrow 8 \rightarrow 0 \rightarrow 10$

$\int_{9}^{10} f(x)\, dx$ = $\int_{8}^{0} f(x)\, dx$ + $\int_{0}^{10} f(x)\, dx$

= -12+17=5

Ex : write the integration in one integration

$\int_{-2}^{2} f(x)\,dx$ + $\int_{2}^{5} f(x)\,dx$ - $\int_{-2}^{-1} f(x)\,dx$

Sol:-

$\int_{-2}^{5} f(x)\,dx$ - $\int_{-2}^{-1} f(x)\,dx \rightarrow \int_{-2}^{5} f(x)\,dx$ + $\int_{-1}^{-2} f(x)\,dx$

$\rightarrow \int_{-1}^{5} f(x)\,dx$

Indefinite integral : قواعد التكامل (التكامل الغير محدود)

Rules of integration

1.$\int k.dx$ = kx+c

2.$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

3.$\int \frac{1}{x+K} dx = ln|x+k| + c$

4.$\int (ax+b)^n\,dx = \frac{(aX+b)^{n+1}}{(n+1)a} + c$

5.$\int \frac{1}{ax+b} dx = \frac{ln|ax+b|}{a} + c$

6.$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

7.$\int a^x\,dx = \frac{a^x}{\ln a} + c$

8.$\int sinh\,x\,dx = cosh\,x + c$

9.$\int cosh\,x\,dx = sinh\,x + c$

10.$\int \frac{1}{1+x^2} dx = tan^{-1}x + c$

11..$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

12.$\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + c$

∫ f(x) dx	Ans
Sec^{2}x	tanx+c
secxtanx	Secx+c
csc^{2x}	-cotx+c
cscxcotx	-cscx+c

Ex: 1. $\int \left(\mathbf{10x^4 - \frac{1}{x^2} + \sqrt[3]{x} + 7}\right) \boldsymbol{dx}$

$\int 10x^4$ dx - $\int \frac{1}{x^2}$dx + $\int \sqrt[3]{x}$ dx + $\int$ 7 dx

= 2x^5+ $\frac{1}{x}$+$\frac{3}{4}$ $x^{\frac{4}{3}}$+7x+c

2.$\int [e^{2x+1} + (x+2)(2x+1)]dx$

$$\frac{e^{2x+1}}{2} + \frac{2x^3}{2} + \frac{5x^2}{2} + 2x + c$$

3.$\int \frac{\cos x}{\sin^2 x} dx$

$\int \cot x \csc x \, dx = -\csc x + c$

4.$\int 1 + \tan^2 x \, dx = \int \sec^2 x \, dx$= tanx+c

5.$\int \frac{1-\sqrt{x}+x^2e^x}{x^2} dx$

$$\int \frac{1}{x^2} - \frac{\sqrt{x}}{x^z} e^x \, dx$$

$$= \frac{-1}{x} + \frac{2}{\sqrt{x}} + e^x + c$$

6.$\int \frac{2e^x}{sinhx+coshx} dx$

$\sinh x + \cosh x = \frac{e^x-e^x}{2} + \frac{e^x+e^{-x}}{2}$ =e^x

$$\int \frac{2e^x}{e^x} dx = 2x + C$$

7) $\int \frac{1+cos^2x}{1-sin^2x}dx$

$\int \frac{1+cos^2x}{cos^2x}dx = \int sec^2x + 1\ dx = \tan x + x + c$

8) $\int_0^1 (2x+y)^3 + \sqrt{1-x^2}\,dx$ أي رمز غير ال x هو ثابت

$\frac{(2x+y)^4}{4(2)}\begin{cases}1\\0\end{cases} + \frac{1}{4}\pi$

$$\frac{(2+y)^4}{6} - \left(\frac{y}{6}\right)^4 + \frac{\pi}{4}$$

9) $\int \frac{x^2-1}{x^4-1}dx$

$$\frac{X^2-1}{(X^2-1)(x^2+1)}dx = \int \frac{1}{x^2+1}dx = \tan^{-1}x + c$$

10) $\int_{-1}^{2} |x|\,dx$

$$\int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx = -\frac{x^2}{2} + \frac{x^2}{2} = \frac{3}{2}$$

11) $\int_0^{\pi/3} \frac{sinx + sinx\ tan^2x}{sec^2x}dx$

$$\int_0^{\pi/3} sinx\ cos^2x + sinx\ \frac{sin^2x}{cos^2x}\ cos^2x\ dx$$

$$\int_0^{\pi/3} sinx\ cos^2x + sin^3x\ dx$$

$$\int_0^{\pi/3} sinx\ (cos^2x + sin^2x)dx$$

$$\int_0^{\pi/3} sinx\ dx = -cosx = \frac{1}{2}$$

12) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c \; or - \cos^{-1} x + c$

13) $\int \frac{dx}{a^2+x^2} = \tan^{-1}\frac{x}{a} + c$

14) $\int_{½}^{\frac{1}{\sqrt{2}}} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t = \frac{\pi}{12}$

15) f(x)= $\sin x$, $0 \leq x \leq \pi/2$

$\cos x$, π/2 ≤x≤ π

find $\int_0^{\pi} f(x)dx$

= $\int_0^{\pi/2} \sin x \, dx + \int_{\pi/2}^{\pi} cosx \, dx$

= $-\cos x + \sin x = 1 - 1 = 0$

16) $\int_0^{3\pi/2} |\sin x| \, dx$

$$\int_0^{\pi} sinx \, dx + \int_{\pi}^{3\pi/2} -sinx \, dx$$

$-\cos x + \cos x = 2 + 1 = 3$

17) $\int_0^1 (x^{10} + 10^x) \, dx$

= $\frac{x^{11}}{11} + \frac{10^x}{\ln 10}$ = $\frac{x}{11} + \left(\frac{10}{\ln 10} - \frac{1}{\ln 10}\right) = \frac{x}{11} + \frac{9}{\ln 10}$

18) $\int_0^1 \frac{sinh^3 x + \tanh^{-1} x - sec^2 x}{2e^x - \sqrt{2x} + \frac{1}{e}} dt$

$\frac{sinh^3 x + \tanh^{-1} x - sec^2 x}{2e^x - \sqrt{2x} + \frac{1}{e}}$= 1-0=1

Integral of symmetric functions:

$1) f(x) is\ odd\ (f(-x) = -f(x))$

$$\int_{-a}^{a} f(x)dx = 0\ (odd)$$

$$\int_{-a}^{a} f(x)dx\ \ 2)\ f(x) is\ even\ \big(f(-x)\big) = f(x)$$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

$* Evaluate\ the\ following:$

$$\mathbf{1)}\int_{-1}^{1} x^3 dx = \frac{x^4}{4} = 0$$

$$or\ \ x^3\ is\ odd\ so\ \int_{-a}^{a} f(x)dx = 0$$

$$\mathbf{2)}\ \int_{-1}^{1} \frac{\cos x}{sin^5 x}dx = \int_{-1}^{1} \frac{cosx}{sin^5 x}dx = 0$$

$$f(-x) = \frac{\cos(-x)}{sin^5(-x)} = \frac{-cosx}{sin^5 x} = -f(x) is\ odd$$

$$\mathbf{3)} \int_{-1}^{3} sin\pi x\ dx = \ \int_{-1}^{1} sin\pi x\ dx + \int_{1}^{3} sin\pi x\ dx$$

$$0 + \frac{-cos\pi x}{\pi} = \frac{2}{\pi}$$

$$\mathbf{4)}\int_{-\pi}^{\pi} x^{n+3}\ dx : if\ n\ is\ even\ \ then\ x^{n+3} = odd\ \ so\ \int_{-\pi}^{\pi} x^{n+3}\ dx = 0$$

$$if\ n\ is\ odd\ then\ n+3 = even\ then\ x^{n+3} = even$$

$$\int_{-\pi}^{\pi} x^{n+3}dx = 2\int_{0}^{\pi} x^{n+3}dx = \frac{2\pi^{n+4}}{n+4}$$

$$\mathbf{5)}\int_{-6}^{6}(x^3+1)\sqrt{36-x^3}\ dx$$

$$\int_{-6}^{6} x^3\sqrt{36-x^2}\,dx + \int_{-6}^{6}\sqrt{36-x^2}\ dx\ \Rightarrow\ \ 0 + \frac{1}{2}.6^2.\pi = 18\pi$$

$$\boldsymbol{Ex : if\ f(x) = \int_{-a}^{a} ax^3 + 2ax + 3x^2 dx\ , f(0) = 2}$$

$$\boldsymbol{find\ a^2}$$

$$f(x) = \int_{-a}^{a} ax^3 dx + \int_{-a}^{a} 2ax\,dx + \int_{-a}^{a} 3x^2 dx$$

$$odd + odd + even$$

$$f(x) = 0 + 0 + 2\int_{0}^{a} 3x^2 dx$$

$$f(x) = 6.\frac{x^3}{3} = 2a^3$$

$$f(0) = 2$$

$$2 = 2a^3$$

$$a = 1$$

Fundemental Theorem of Calculus

$part\ 1 : if\ f(x) is\ continuous\ and$

$$g(x) = \int_{u(t)}^{v(t)} f(t)dt\ \ then :$$

$$g'_{(x)} = f\big(v(x)\big)\,v'(x) - f\big(u(x)\big), u'(x)$$

اذا كان الاقتران على شكل تكامل وكانت حدود التكامل عبارة عن اقترانات,وطلب المشتقة ,بطبق القانون اللي فوق.

$\boldsymbol{ex: find\ g'(x)\ of\ :}$

1) $g(x) = \int_{1}^{x^4} \sec(t)\,dt\ \rightarrow f(x) = \sec(t)$
$g'(x) = \sec(x^4)\,.\,4x^3 - \sec(1)\,.\,0$
$g'(x) = 4x^3 \sec(x^4)$

اذا كانت حدود التكامل فيهم ثابت قيمة التكامل =صفر

2)$g(x) = \int_{sintx}^{2} \sqrt{1+t^2}dt$

$$g'(x) = \sqrt{1+4}.0 - \sqrt{1+sin^2x}\,.cosx$$

$$= -cosx\,.\sqrt{1+sin^2x}$$

3) $g(x) = \int_{\sqrt{x}}^{e^x} \tan^{-1} t\; dt$

$$= \tan^{-1} e^x.e^x - \tan^{-1}\sqrt{x}.1/2\sqrt{x}$$

$$= \;e^x \tan^{-1} e^x - \frac{\tan^{-1}\sqrt{x}}{2\sqrt{x}}$$

4) $g(x) = \int_{\pi}^{tanx} 3\sqrt{u^2+1}\,du$

$$g'(x) = \sqrt[3]{tan^2x+1}\,.sec^2x - 0$$

$$= \sqrt[3]{sec^2x}.sec^2x$$

$$= \sqrt[3]{sec^2x}.\sqrt[3]{sec^6x}$$

$$= \sqrt[3]{sec^8x}$$

Ex: if $\mathbf{f(x) = \int_0^x (1 - t^2)e^{t^2} dt}$ on what interval f is increasing?

$$f'(x) = 0$$
$$(1 - x^2)e^{x^2}.1 - 0 = 0$$
$$(1 - x^2)e^{x^2} = 0$$
$$x = 1, x = -1$$
$$ans: (-1,1) or [-1,1], (both\ correct)$$

$$\int_1^x (\frac{1}{4}t^2 - \frac{5}{3}t^3)dt$$

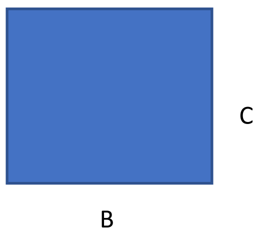
find the inflection points of $f(x)$

Ex if $\mathbf{f(x) = \int_1^x}$

$$f'(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3$$
$$f''(x) = x^3 - 5x^2 = 0$$
$$x = 0, x = 5$$
$$only\ inflection\ point\ at\ x = 5 = (5, f(5))$$

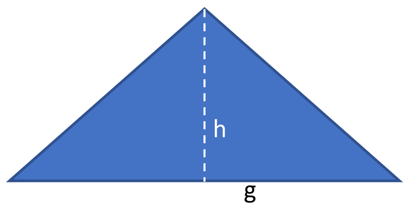
Area

We can calculate the area of regular shapes, such as:

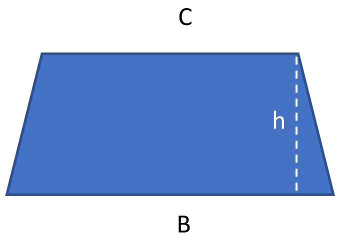


A=B*C

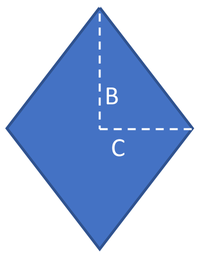
if B=C it's a square



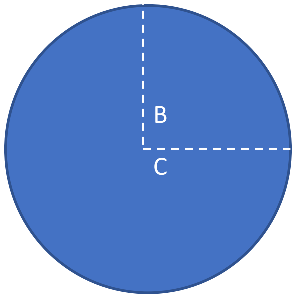
A=1/2 * g* h *sinΘ



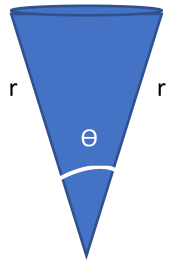
A=1/2 (B+C)h



A=B*C



A=πBC



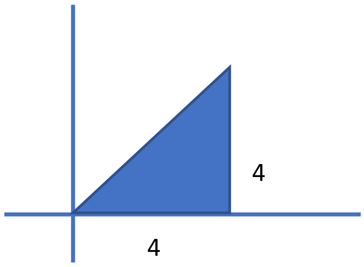
Θ: angle in radians

A=Θ r^2 /2, circle $\rightarrow$ Θ=2π

A=2π r^2/2 = π r^2

Now for irregular shapes how we calculate the area??

If we have f(x)=x on [0,4], [1,4]

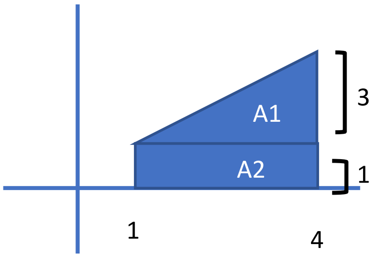


Then A=1/2*4*4 =8

If we integrate x on [0,4] what we get?

$\int_0^4 x\,dx = x^2/4 \Big]_0^4$ = 8 it's the same area

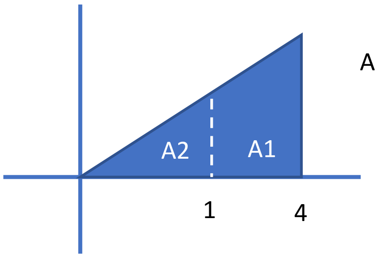
Let’s try on [1,4] →



A=A1+A2

=$\frac{1}{2}$*3*3+3*1=$\frac{15}{2}$

OR:



A=A1+A2 -A2

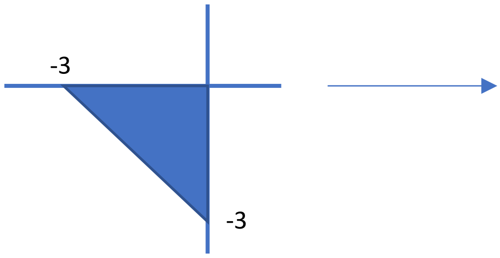
=$\frac{1}{2}$*4*4 - $\frac{1}{2}$ *1 *1 =15/2

How if we integrate x on [1,4] what we get??

$\int_1^4 x\,dx$ =$x^2/2$ $\Big]_1^4$ = 15/2 ,we get the area

So, Is integration = area all of time?

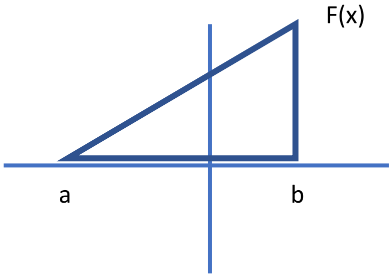
If f(x)=$|x|$ -3 on [-3,0], then what $\int_{-3}^{0} f(x)\,dx$ =?



Area =1/2*3*3=9/2

$\int_{-3}^{0} f(x)\,dx \;= \int_{-3}^{0}(-x-3)\,dx \;= \int_{-3}^{0} -(x+3)\,dx$ = $-(x+3)^2/2\Big]_{-3}^{0}$ = -9/2 in this case

$\int_{-3}^{0} f(s\,dx) \;=\; area$

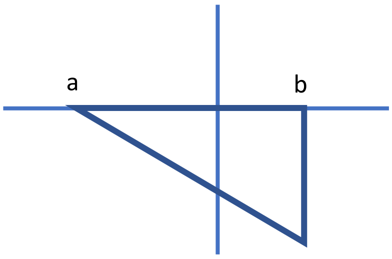


Then if f(x) ≥ 0 om [a, b]

Then $\int_{a}^{b} f(x)dx = the\ area$

And if f(x)≤ 0 on [a, b]

Then $\int_{a}^{b} f(x)\,dx \;=\; -area$

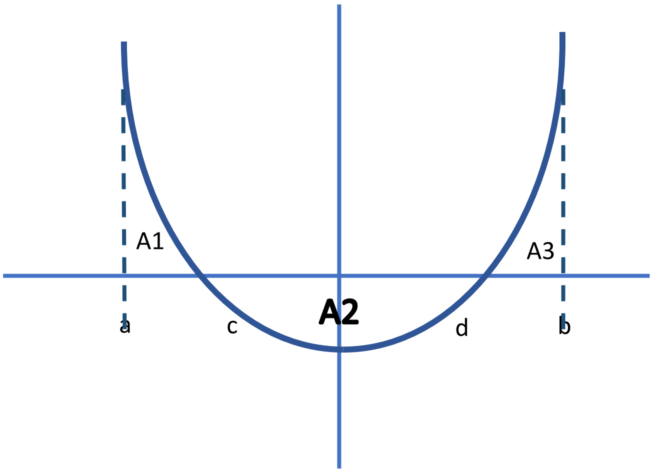


How if f(x) graph is:

Then $\int_{a}^{b} |f(x)|\,dx \;=??$

$\left|\int_{a}^{b} f(x)\,dx\right| =??$

Area =??

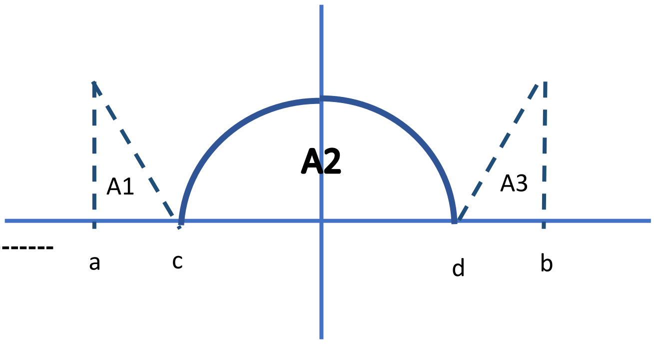


1) area = a1+a2+a3

2) $\left|\int_a^b f(x)\,dx\right| = \left|\int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx\right| =$ $|A1 - A2 + A3|$

3) $\int_a^b |f(x)\,dx| \; = ??$

then $\int_a^b |f(x)\,dx| \; = \int_a^c f(x)\,dx + \int_c^d -f(x)\,dx + \int_d^b f(x)\,dx \; = A1 + --A2 + A3 \; = A1 + A2 + A3$



So, we see that $\longrightarrow$ $\int_a^b |f(x)|\,dx \; = \; the\ area$

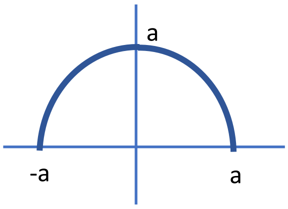
$\left|\int_a^b f(x)\,dx\right| = the\ absolute\ value\ of\ the\ integrate$ (لا يساوي المساحة)

If f(x)=x^2-1, the area of [-1,1] =??

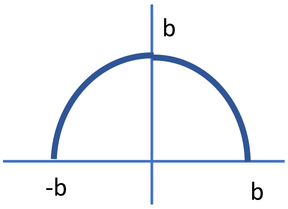
Area $|x^2 - 1|$ $\longrightarrow$ [-1 , 1]

$=\int_{-1}^{1} |x^2 - 1|\,dx \; = \int_{-1}^{1} -(x^2 - 1)\,dx = x - x^3/3\Big]_{-1}^{1} \; = 2/3 - (-2/3) = 4/3$

If f(x) =



g(x)=



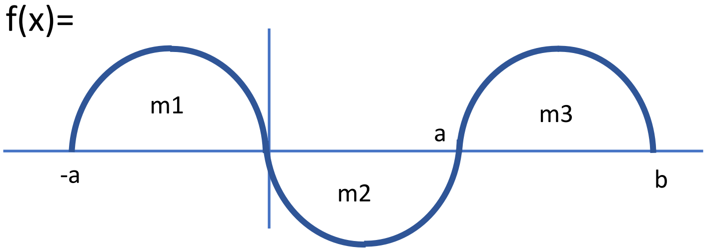
Find the value of b that make $\int_{-a}^{a} f(x)dx = \frac{1}{2}\int_{-b}^{b} g(x)dx$?

$\int_{-a}^{a} f(x)dx = \frac{1}{2}\int_{-b}^{b} g(x)dx$

π/2 a^2 = ½ (π/2 b^2)

b=$\sqrt{2}$ a

If f(x)=



Then $\int_{-a}^{b} f(x)dx = ??$, m1=m2=m3=m

$$\int_{-a}^{a} f(x)dx = ??$$

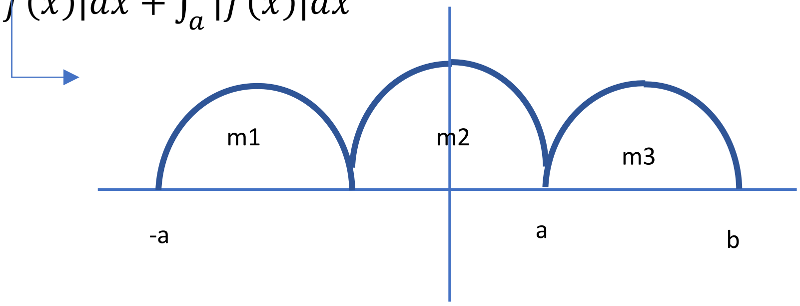
Is $\int_{-a}^{b} f(x)dx = \frac{1}{3}\int_{-a}^{b} |f(x)|$

1) $\int_{-a}^{b} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx + \int_{a}^{b} f(x)\,dx = m - m + m = m$

2) $\displaystyle\int_{-a}^{b} -f(x)dx = \int_{-a}^{0} -f(x)dx + \int_{0}^{a} -f(x)dx + \int_{a}^{b} -f(x)\,dx$
$= -m + m - m = -m$

3)$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = m - m = 0$

4) $\int_{-a}^{b} f(x)dx = m$, $\int_{-a}^{b} |f(x)|dx =$
$\int_{-a}^{0} |f(x)|dx + \int_{0}^{a} |f(x)|dx + \int_{a}^{b} |f(x)|dx$

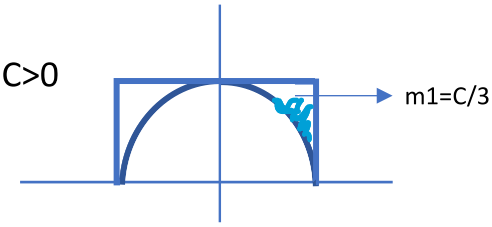


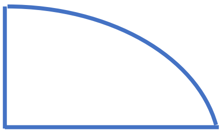
Now $\int_{-a}^{b} f(x)dx = 3m/3$, where 3m = $\frac{1}{3}\int_{-a}^{b}|f(x)|\,dx$

True

If f(x)=2 C, and g(x)=-x^2+C^2, C>0

Find the value of C?



m1=  - 

→ $\int_0^c 2C\,dx - \int_0^c C^2 - x^2dx = C/3$

=$2C^2$-C^3-C^3/3=C/3

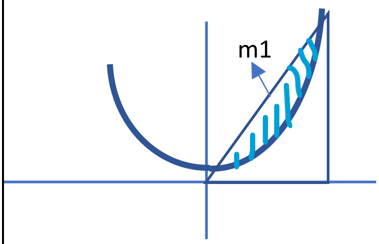
→ $4C^2$-$4C^3$=C → $4C^3$-$4C^2$+C=0

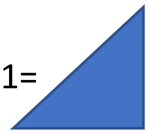
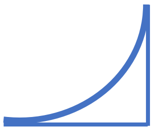
→ C($4C^2$-4C+1) =0

C=0 ✗ 2C-1=0

C=1/2

If f(x)=x^2 and g(x)=x, then the area between f(x) and g(x) on [0,1]=?



m1=  - 

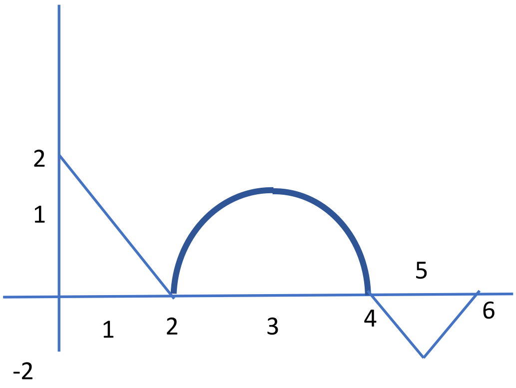
المساحة المحصورة بين الاقترانين*

* رح نكامل (الاقتران العلوي- الاقتران السفلي)

$\int_0^1 xdx - \int_0^1 x^2\, dx\Big]_0^1 = x^2/2\Big]_0^1 \quad - x^3/3 \quad = ½- 1/3 = 1/6$

قاعدة الاقتران في الأسفل*

If f(x) is piecewise function:



Then find:

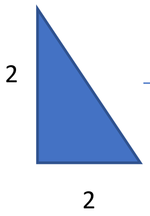
1) $\int_0^2 f(x)dx$

2) $\int_2^4 f(x)dx$

3) $\int_4^6 f(x)dx$

4) $\int_1^3 f(x)dx$

1) $\int_0^2 f(x)dx$ → the area of the triangle

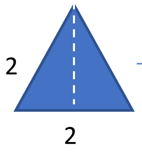


→ ½ *2*2=1

2) $\int_2^4 f(x)dx$ → the area of the half circle

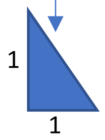
→ π/2*1² =π/2

3) $\int_4^6 f(x)dx$ → the area of triangle



→ -1/2*2*2=-2

4) $\int_1^3 f(x)dx$ → $\int_1^2 f(x)dx + \int_2^3 f(x)dx$ =π/4 +1/2



Semicircle

$\pi/4*1^2$

قاعدة الاقتران:*

F(x) on [0,2] → ax+b → a=$\frac{\Delta y}{\Delta x}$ = -tan α =-1 , f(x)=-x+b , f(2)=0 → f(x)=2-x

F(x) on [1,4] → $(y-h)^2 + (x-s)^2 = r^2$

(s,h) → the center

(3,0) → $y^2 + (x-3)^2 = 1$ → y= ±$\sqrt{1-(x-3)^2}$ = + $\sqrt{-8-x\wedge 2+6x}$

F(x) On [4,6] → f(x) =$2|x-5|-2$

$$F(x) = \begin{cases} 2-x, & 2>x\geq 0 \\ \sqrt{1-3-x}, & 4>x\geq 2 \\ 2|x-5|-2, & 6>x\geq 4 \end{cases}$$

Integration By substitution

If we have f(x)=C+$\sqrt{1+x^2}$ → df(x)/dx =x/$\sqrt{1+x^2}$ by using chain rule

So, f(x) =c+h(g(x))

$$\frac{df(x)}{dx} = \frac{dh(g(x))}{dx} \cdot \frac{dg(x)}{dx}$$

So

$$\int \frac{df(x)}{dx}\,\mathrm{dx} = \int \frac{d\left(h(g(x))\right)}{dx}\,dx$$

F(x)=h(g(x)) +c

$\int \frac{x}{\sqrt{1+x^2}}$ dx = $\sqrt{1+x^2}$ +c → by trying f(x)= $\sqrt{1+x^2}+c$

So how we can evaluate these integrations??

1) if u=g(x) is differentiable function whose range is an interval I and f is continuous on I, then

$$\int \frac{df(g(x))}{dx} \cdot \frac{dg(x)}{dx}\,dx = \mathrm{f(u)+c}$$

Example: if f(x)=3/2 $\sqrt{1+x^2}$ → $\frac{df(x)}{dx} = 3x/(2\sqrt{1+x^2})$

So $\int 3x/(2\sqrt{1+x^2})$ dx =3/2 $\sqrt{1+x^2}$ +c

Now $\int 3x/(2\sqrt{1+x^2})\ dx$

↓

$\frac{d(1+x^2)}{dx}$ =2x , so if u=1+x^2

Then du=2x dx

3/2 $\int\ 2x/(3\sqrt{1+x^2})dx$ = 3/4$\int du/(\sqrt{u})$ = ¾ $\int u^{-1/2}$du =$\frac{u^{1/2}}{1/2}$ *3/4 +c = 3/2 $u^{1/2}$+c =3/2$\sqrt{1+x^2}+c$

↓

$$\boxed{\int x^n \mathrm{dx} = \frac{x^{n+1}}{n+1} + \mathrm{c},\ \mathrm{n}\neq -1}$$

If f(x)=2x sin(x^2+4) , find $\int f(x)dx$?

$$\int f(x)dx = \int \sin(\underbrace{x^2+4}_{u})\,\underbrace{2xdx}_{du}$$

So u=x^2+4 then du=2x dx

=$\int sinu\ du$ =-cos u +c = -cos(x^2+4)+c

If $\frac{df(x)}{dx}$ = 1/x lnx then f(x)?

$$\int \frac{df(x)}{dx}\,dx = f(x)$$

= $\int \frac{1}{x} lnx\ dx$

So u=lnx then du=dx/x

F(x)=$\int \frac{1}{x}\ln x\ \ dx = \int u\,du = u^2 + c = \frac{(lnx)^2}{2} + c$

If $d^2f(x)/dx^2$ =6x^2 dx then f(x)??

$$\int \frac{d^2 f(x)}{dx^2} = \frac{df(x)}{dx}$$

=$\int 6x^2\ \ dx = \frac{6}{3}x^3 + c1 = 2x^3 + c1$

F(x)=$\int \frac{df(x)}{dx}\ dx = \int 2x^3 + c1\ dx = \frac{x^4}{2} + c1x + c2$

الحالة الأولى للتكامل بالتعويض*

وجود مشتقة أحد الاقترانات

نعوض الاقتران الذي مشتقته موجودة

Evaluate each integration

1.$\int \frac{ln(ln\,x)}{x\,ln\,x}dx$ → u=ln(lnx) → du=$\frac{1}{x\,ln\,x}dx$

$\int u\,du = \frac{u^2}{2} + c = \frac{((\ln\,(lnx))^2}{2} + c$

2.$e^{\sin x+3}\cos x\,dx$,$u = \sin x + 3$→$du = dx\cos x = \int e^u\,du = e^u +$
$C = e^{\sin x+3} + C$

3.$\int\ sin(x + cos\,x)(-1 + sin\,x)\,dx$ → u=x+cosx

du – (1-sinx)dx = $\int -siny\ du$ =cosu+c = cos(x+sinx)+c

4. $\int tanx\ \ sec^2x\ dx$ = $\int$ *tanx secx secx dx or* $\int tanx\ sec^2x\ dx$

u=tanx → $sec^2x\ dx$ = $\int u$ du =$\frac{u^2}{2} + c = \frac{tanx^2}{2} + c = \frac{secx^2}{2} + c$

5. $\int tan^{18}x + tan^{20}x\ dx = \int tan^{18}x\ (\ 1 + tan^2x)dx =$
$\int tan^{18}x\ \ sec^2x\ dx$

U= tanx ➔ du = $sec^2x\ dx$= $\int u^{18}du = \frac{u^{19}}{19} + c = \frac{tanx^{19}}{19} + c$

6. $\int \frac{x+1}{x}\cdot\frac{1}{x^2}dx$ → u=$\frac{x+1}{x}$ → $\frac{-1}{x^2}dx = -\int u\ du = -\frac{u^2}{2} + C =$
$\frac{-1}{2}\left(\frac{x+1}{x}\right)^2 + C$

7. $\int (x^2 + 2)(x^3 + 6x)^2$ → u=$x^3 + 6x$ du=$3x^2 + 6\ dx$

= $\int \frac{3(x^2+2)}{3}dx\ u^2 = \frac{1}{3}\int u^2\,du = \frac{u^3}{3} + c = \frac{(x^3+6x)^3}{3} + c$

$Note: \int f'(x)\,dx = f(x) + c$

$\int (f(x)g(x))'\,dx = f(x)g(x) + c$

1) $\displaystyle\int x\,e^x + e^x\,dx = \int \left(x\frac{d}{dx}[e^x] + e^x\frac{d}{d^x}[x]\right)dx = \int (xe^x)'\,dx$

$= xe^x + c$

2) $\int (lnx + 1)dx$ = $\int\ ln\,x\frac{d}{dx}[x] + x\frac{d}{dx}[ln\,x]\,dx = \int (x\,ln\,x)'\,dx =$
$x\ln x + c$

3) $\int (x(secx)^2 + tanx)dx = \int x\frac{d}{dx}[tanx] + tanx\frac{d}{dx}[x]\,dx =$
$\int (xtanx)'dx = xtanx + c$

الحالة التالية تشمل العودة الى الفرض او الحصول على تكامل ابسط من التكامل القديم

$\int x^{2^n-2}(ax^n + b)^R\,dx$, R,n real numbers

Such as $\int x^3(x^2+1)^4$→ u= $x^2 + 1$ → du = 2x dx

$$\int \frac{1}{2}x^2u^2\,du = \frac{1}{2}\int (u-1)(u^4)\,du = \frac{u^6}{12} + \frac{u^5}{10} + c$$
$$= \frac{x^2+1^{\ 6}}{12} + \frac{x^2+1^5}{10} + c$$

$\int x^{-5}(x^3+2x)^8\,dx = \int x^3(x^2+2)^8\,dx$ u=$x^2 + 2$ du =2x dx

$\int \frac{1}{2}x^2(u)^8 d_u = \int \frac{1}{2}(u-2)u^2\,du = \frac{(x^2+2)^{10}}{20}$-$\frac{(x^2+2)^9}{8} + c$

$\int x(x^3+3x^2+3x+1)^7\,dx = \int x((x+1)^3)^7\,dx = \int x^1(x^1+1)^{21}\,dx$

U=x+1 → du=dx= $\int u^{21}d_u = \int\ (u-1)4^{21}\,du = \frac{u^{23}}{23} - \frac{u^{22}}{22} + c =$
$\frac{(x+1)^{23}}{23} - \frac{(x+1)^{22}}{22} + c.$

If f(x)=ln$|g(x)| + c \longrightarrow \frac{df(x)}{dx} = \frac{dg(x)}{dx}$ .1/g(x)

Then $\int \frac{df(x)}{dx}\ dx = \int \frac{dg(x)}{dx}\ .\,1/g(x)dx$ =ln g(x) +c

$\int \frac{dg(x)}{dx}\ .\,1/g(x)dx$=ln$|g(x)| + c$

$\int \frac{x}{\frac{x^2}{2}+1}\ dx$=ln$\left|\frac{x^2}{2} + 1\right|$+c =ln $|x^2 + 2|$+c

$\int \frac{(\sec x+\tan x\ )\sec x}{\sec x+\tan x}\ dx$ =ln$|\sec x + \tan x|$+c

$\int \frac{(x^3+x^2+2\ )}{x^4+2x^2}$dx =$\int \frac{x^3}{x^4+2x^2}$ dx + $\int \frac{x^2+2}{x^4+2x^2}$ dx

= $\int \frac{x}{x^2+1}\ dx + \int \frac{1}{x^2}\ dx = \frac{1}{2}\int \frac{2x}{x^2+1}\ dx + \ \int \frac{1}{x^2}\ dx$

=1/2 ln$|x^2 + 1|$ -1/x +c

$\int \frac{2x^2\cos x^2+1}{x(\sin x^2+\ln x)}\ dx$= $\int \frac{2x\cos x^2+1/x}{\sin x^2+\ln x}\ dx$ = ln $|\sin x^2 + \ln x|$ +c

$\int \frac{1}{x+\sqrt{x}}\ dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\ dx = \int \frac{\frac{1}{\sqrt{x}}}{\sqrt{x}+1} dx = \int \frac{\frac{1*2}{2*\sqrt{x}}}{\sqrt{x}+1} dx$=2ln$\left|\sqrt{x} + 1\right| + c$

$\int \frac{2x}{sin^{-2}\,(x^2)\sqrt{1-x^4}} = \int \frac{\frac{2x}{\sqrt{1-x^4}}}{sin^{-2}\ x^2}\ dx$ =ln$|sin^{-1}\ x^2|$+c

$\int \frac{2x}{sinh^{-1}x\ \sqrt{x^2+1}}$ dx = 2$\int \frac{\frac{x}{\sqrt{x^2+1}}}{sinh^{-1}\ x}$ dx = 2ln$|sinh^{-1}\ x|$ +c

Ex: Evaluate the following integration:

1. $\int e^{sinx} sin2x + 2sinx\ cosx\ dx$

$$= \int \begin{matrix} 2e^{sinx} sinx\ cosx + 2sinx\ cosx\ dx \rightarrow \\ u = sinx\ , \\ du = dxcosx \end{matrix}$$

$=\int 2e^u u + 2e^u du = 2\int e^u u + u\ du \rightarrow ue^u \rightarrow ue^u + e^u$

$$2\int (ue^u) du = 2ue^u + c = 2sinx\ e^{sinx} + c$$

$$2. \int \frac{e^x - e^{-x}}{\cosh x} dx = \int \frac{e^x - e^{-x}}{\frac{e^{x-} + e^{-x}}{2}} dx = 2\ln|e^x + e^{-x}| + c$$

$$3. \int 5^t sin5^t\ dt = 1/\ln 5 \int 5^t \ln 5\ sin5^t\ dt = \frac{-1}{\ln 5}[cos5^t] + c$$

$$= \frac{cos5^t}{\ln \frac{1}{5}} + c$$

$$4. \int \frac{x}{1+x^4} dx = \int \frac{x}{1+x^{2^2}} dx \rightarrow u = x^2 \rightarrow du = 2xdx$$

$$\frac{1}{2}\int \frac{1}{1+u^2} du = \frac{1}{2}\tan^{-1} u + c = \frac{1}{2}\tan^{-1} x^2 + c$$

$$5. \int \frac{1 - sinx}{cosx(1 - sinx)} dx \rightarrow \int \frac{1}{\frac{cosx}{1 - sinx}} * \frac{1}{1 - sinx} dx = \ln(\frac{cosx}{1 - sinx}) + c$$

$$6. \int \frac{1 + tanx}{1 - tanx} dx = \int \tan\left(\frac{\pi}{4} + x\right) dx = \int \frac{\sin\left(\frac{\pi}{4} + x\right)}{\cos\left(\frac{\pi}{4} + x\right)} dx = -\ln\cos\left(\frac{\pi}{4} + x\right)$$

$$+ c$$

$$7. \int \frac{sinx}{cosx + cos^2x} dx$$

$$= \int \frac{\frac{sinx}{cos^2x}}{\frac{cosx}{cos^2x} + \frac{cos^2x}{cos^2x}} dx = \int \frac{tanx\ secx}{secx + 1} dx = \ln secx + 1 + c$$

$$8. \int \frac{sin2x}{cosx + cos^2x} dx$$

$$= \int \frac{2sinx\ cosx}{cosx + cos^2x} dx = \int \frac{2sinx}{1 + cosx} dx = -2\ln|1 + cosx| + c$$

subtitution Rule for Definite

$if\ g' is\ continuous\ in\ [a,b] and\ f\ is\ continuous\ on\ the\ range\ of\ g(x)$

$$= u, then \int_a^b f'\big(g(x)\big)\, g'(x)dx = f(u) from\ g(a) to\ g(b)$$

$$if\ f(x) = \int_0^3 6x(x^2+1^{)2}dx\ then\ f(x) =?$$

$$f(x) = \int_0^3 6x\,(x^2+1^{)2}dx \rightarrow u = x^2+1, du = 2xdx$$

$$x \rightarrow 3, u \rightarrow 3^2+1 \rightarrow 10$$

$$x \rightarrow 0, u \rightarrow 0^2+1 \rightarrow 1 = \int_1^{10} 3u^2du = u^3 = 999$$

$$if\ f(x) = \int_{-1}^{1} x^2 sin\pi x^3\, dx\ then\ f(x) =?$$

$$u = \ \pi x^3\,, du = 3\pi x^2\, dx,$$

$$\int_{-\pi}^{\pi} \frac{1}{3\pi} sinu\ du = \frac{1}{3\pi}(-cosu) = 0$$

Evaluate each integral

1. $\int_{0}^{\pi} e^{cosx}dx \;\rightarrow u = cosx \rightarrow du = -sinxdx$

$$\int_{-1}^{1} -e^{u}du = e^{1} - e^{-1}$$

2. $\int_{e^1}^{e^2} \frac{\ln x}{x}dx \rightarrow u = \ln x \rightarrow du = \frac{1}{x}dx$

$$\int_{1}^{2} udu = \frac{3}{2}$$

3. $\int_{-1}^{1} \frac{1}{x+x^9}dx = \int_{-1}^{1} \frac{1}{x^9(1+x^{-8})}dx = \int_{-1}^{1} \frac{x^{-9}}{1+x^{-8}}dx$

$$u = 1 + x^{-8} \rightarrow du = -8x^{-9}dx$$

$$\int_{2}^{2} \frac{-1}{8}\frac{1}{u}du = 0$$

4. $\int_{-1}^{1} \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}1 - \cos^{-1}-1 = 0 - \pi = -\pi$

5. $\int_{\ln 5}^{\ln 10} cschx\, dx$

$$= \int_{\ln 5}^{\ln 10} \frac{2}{e^{x} - e^{-x}}dx = \int_{\ln 5}^{\ln 10} \frac{2e^{x}}{e^{2x} - 1}dx\; ,u = e^{x} \rightarrow du = dxe^{x}$$

6.$\int_5^{10}\frac{2}{u^2-1}du=\int_5^{10}\frac{2-2u}{u^2-1}+\frac{2u}{u^2-1}du=\int_5^{10}\frac{2(1-u)}{(u-1)(u+1)}du+\int_5^{10}\frac{2u}{u^2-1}du=\ln\frac{9}{11}-\ln\frac{4}{6}$

$$\int f'(x)f(x)^{n-1}\sqrt{a-f(x)^{2n}}dx\ \rightarrow Area$$

$$\int f'(x)f(x)^{n-1}\sqrt{a-(f(x)^n)^2}dx\ \rightarrow u=f(x)^n\rightarrow du=nf(x)^{n-1}f'(x)$$

$1/n \int\sqrt{1-u^2du\rightarrow}circles$

$$\int_0^1 x^3\sqrt{1-x^8}dx\rightarrow\int_0^1 x^3\sqrt{1-x^{4^2}dx\ \rightarrow}u=x^4\rightarrow du=4x^3$$

$$\int_0^1\sqrt{1-u^2}\,du=\frac{\pi}{16}$$

$$\int_{-2}^2 2x\sqrt{1-x^4}dx\rightarrow\int_{-2}^2 2x\sqrt{1-x^{2^2}dx\ }\ \rightarrow u=x^2\rightarrow du=2xdx$$

$$\int_4^4\sqrt{1-u^2}du=0$$

$$\int_0^{\frac{\pi}{16}}\frac{sin\sqrt{x}}{\sqrt{x}}dx\rightarrow u^2=x\rightarrow 2udu=dx\rightarrow dx=2udu$$

$$\int_0^{\frac{\pi}{4}}\frac{2u\ sinu\ du}{u}=\int_0^{\frac{\pi}{4}}2sinu\ du=-2cosu=2-\sqrt{2}$$