The University of Jordan
Mathematics Department
Calculus III First Exam 6/7/2019

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Student's Name: حمزة ابراهيم سالم العمايرة Student's Nu.: 0182885 Instructor: 2

Q1: Let $A(3,0,1), B(0,4,2), C(5,1,-1)$ and $D(-1,2,5)$ be four points in space.

(a) (2 marks) Find the vector that has **magnitude 4** with the **opposite** direction of the vector $\overrightarrow{AB}$.

$-\overrightarrow{AB} = \vec{u} = \langle 3, -4, -1 \rangle$

$\rightarrow \hat{u} = \dfrac{\langle 3, -4, -1\rangle}{\sqrt{26}}$

$\rightarrow \hat{v} = 4\hat{u} = \dfrac{\langle 12, -16, -4\rangle}{\sqrt{26}}$

1.5

(b) (4 marks) Find the **volume** of **the parallelepiped** with adjacent edges $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$.

$$\text{Volume} = |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = \begin{vmatrix} -3 & 4 & 1 \\ 2 & 1 & -2 \\ -4 & 2 & 4 \end{vmatrix}$$

$$= |-3(8) - 4(\text{zero}) + 1(-8)|$$

$$= |-24 + 8| = \underline{16}$$

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Q2: (4 marks) Find an equation of the **sphere** that has center $C(1,0,-1)$ and touch the line L: $x = 1 + 3t, y = 2 - t, z = 4t$.

Solution

from the Line L $\rightarrow$ $P(1, 2, \text{zero})$.

$\text{Radius} = |CP| = \sqrt{\text{zero} + 4 + 1} = \sqrt{5}$

So

$(x-1)^2 + y^2 + (z^2+1) = 5$

1.5

Q3: (2 marks) Determine under what conditions the **two unit vectors** $\vec{a}$ and $\vec{b}$ satisfy $proj_{\vec{b}}\, \vec{a} = proj_{\vec{a}}\, \vec{b}$.

$\boxed{\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}}$

$proj_{\vec{b}}\vec{a} = proj_{\vec{a}}\vec{b}$

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{b}$

$\rightarrow \dfrac{\vec{a}}{|\vec{b}|^2} = \dfrac{\vec{b}}{|\vec{a}|^2} \rightarrow \vec{a}|\vec{a}|^2 = \vec{b}|\vec{b}|^2 \rightarrow \dfrac{\vec{a}}{\vec{b}} = \left|\dfrac{\vec{a}}{\vec{b}}\right|^2$

1

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Q4: (3 marks) Find **the intersection point** between the line $L: \underbrace{x-2}_{1} = \frac{y-1}{2}, z = 3$ and the plane $\boxed{2x - 3y + 4z = 5.}$

Solution: from the line $L \to P(2, 1, 3)$ & $\vec{u} = \langle 1, 2, \text{zero} \rangle$

So $\to$
$x = 2 + t$
$y = 1 + 2t$
$z = 3$

$\to 2(2+t) - 3(1+2t) + 4(3) = 5$

$\to 2t + 4 - 6t - 3 + 12 = 5$

$\to -4t = -8 \to \boxed{t = 2}$

from $t \to$ we find the intersection point

$\to$ $x = 4$, $y = 5$, $z = 3$ $(4, 5, 3)$

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Q5: (4 marks) Find the distance between **the two skew lines**

$$L_1: x = 2 + 2t, y = 4 - 2t, z = 4t,$$

$$L_2: \frac{4-x}{3} = \frac{y-4}{4} = \frac{z-1}{-2}.$$

Solution

$L_1 \to P_1(2, 4, \text{zero})$, $u_1 = \langle 2, -2, 4 \rangle$

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$L_2 \to P_2(4, 4, 1)$, $u_2 = \langle -3, 4, -2 \rangle$

So $d = \frac{|\vec{P_1P_2} \cdot (\vec{u_1} \times \vec{u_2})|}{|\vec{u_1} \times \vec{u_2}|}$

So $d = \frac{|-24 + \text{zero} + 2|}{\sqrt{212}}$

$= \frac{22}{\sqrt{212}}$

$\vec{P_1P_2} = \langle 2, \text{zero}, 1 \rangle$

$\vec{u_1} \times \vec{u_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 4 \\ -3 & 4 & -2 \end{vmatrix} = -12\hat{i} - 8\hat{j} + 2\hat{k}$

$\to |\vec{u_1} \times \vec{u_2}| = \sqrt{144 + 64 + 4}$

$-4 + 12$
-8

144
64
208

Q6: (3 marks) If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a} \cdot \vec{b} = -1$, $|\vec{a}| = 2$ and $|\vec{b}| = 3$, find $|\vec{a} + 2\vec{b}|$.

Solution $(|\vec{a} + 2\vec{b}|)^2 = |\vec{a}|^2 + 4|\vec{a}||\vec{b}| + 4|\vec{b}|^2$

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$= \cancel{4 + 4(-1)} + 36 = \boxed{36}$

$|a + 2b| = ?$

Q7: (a) (3 marks) **Describe** and **sketch** the region $4 \le x^2 + z^2 \le 16$.

Solution:

All set of points that lies between (on) the circular cylinders $x^2+z^2=4$ & $x^2+z^2=16$

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(b) (2 marks) **Identify** and **sketch** the graph of $x^2 + y^2 - 4x - 2y - z^2 + 4z = 0$.

Solutions:

$(x^2-4x+4)+(y^2-2y+1)+(z^2+4z+4) = \text{zero}+4+1+4$

$\to (x-2)^2+(y-1)^2+(z+2)^2 = 9$

So this equation is ~~circle~~ a sphere equation with center $(2,1,-2)$ and $r=3$

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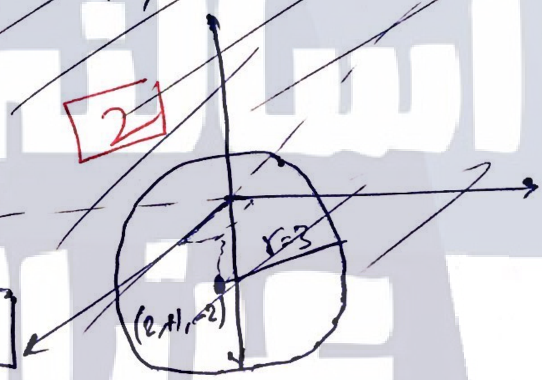

$\boxed{2}$

Q8: (4 marks) Find an equation of **the plane** containing the point $p(2,0,3)$ **and is perpendicular** to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

Solution:

$\pi_1: 2x - y + z - 1 = \text{zero} \to \vec{n}_1 = \langle 2, -1, 1 \rangle$

$\pi_2: x + y - 2z - 3 = \text{zero} \to \vec{n}_2 = \langle 1, 1, -2 \rangle$

So $\vec{n}_3 = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \langle 1, 5, 3 \rangle$

So π_3: ~~$x + 3y + 3z - (5) = \text{zero}$~~ $5y$

↳ : $x + 5y + 3z - (2 + 9) = \text{zero}$

$\to x + 5y + 3z - 11 = \text{zero}$

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Question 7 -b :

$$x^2 + y^2 - 4x - 2y - z^2 + 4z = \text{zero}$$

$$\rightarrow (x^2 + 4x + 4) + (y^2 - 2y + 1) - (z^2 - 4z + 4) = \text{zero} + \cancel{4} + -\cancel{4} + 1$$

$$\rightarrow (x+2)^2 + (y-1)^2 - (z-2)^2 = 1$$

So this equation is a (Hyperboloid with 1 sheet) equation with center $(-2, 1, 2)$

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The University of Jordan
Department of Mathematics
Calculus III, Second Exam

Student's Name:

Instructor's Nam

Q1) (4 marks) Find the arc length of $\vec{r}(t) =< \frac{1}{3}t^3, \frac{1}{\sqrt{2}}t^2, t >$, $1 \le t \le 4$.

$r'(t) = \langle t^2, \frac{2}{\sqrt{2}}t, 1 \rangle$

$\|r'(t)\| = \sqrt{t^4 + \frac{4}{2}t^2 + 1}$

$\sqrt{t^4 + 2t^2 + 1}$

$\sqrt{(t^2+1)^2}$

$S(t) = \int_1^4 t^2 + 1 \, dt$

$\frac{1}{3}t^3 + t \Big|_1^4 = \boxed{\frac{4^3}{3} + 4 - \left(\frac{1}{3} + 1\right)}$

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Q2)(3 marks) Let $h(t) = (2t+1)\vec{u}(t) \cdot \vec{v}(2t)$, $\vec{u}(t) =< t^2+1, e^t, cost >$, $\vec{v}(0) =< 1, 2, -3 >$, $\vec{v}'(0) =< -2, 5, 1 >$. Find $h'(0)$.

$u'(t) = 2t, e^t, -\sin t)$

$\langle 1, 1, 1 \rangle$ $\langle 0, 1, 0 \rangle$

$(\vec{v}(2t))\, 2\, \vec{u}(t) + (2t+1)\vec{u}'(t)\, (\vec{v}(2t)) + 2(2t+1)\, \vec{v}'(2t) \cdot \vec{u}(t)$

$2 \langle 1, 2, -3 \rangle \cdot \langle 1, 1, 1 \rangle + (1) \langle 0, 1, 0 \rangle \cdot \langle 1, 2, -3 \rangle + 2(1) \langle -2, 5, 1 \rangle \cdot \langle 1, 1, 1 \rangle$

$2\underbrace{(1+2-3)}_{0} + (0+2+0) + 2(-2+5+1)$

$2 + 8 = \boxed{10}$

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Q3)(3+2 marks) Let $\vec{r}(t) =< e^t cost, e^{2t}, e^t sint >$

a) Find the equation of the tangent line to the curve $\vec{r}(t)$ at the point $(1, 1, 0)$.

$t = 0$

$r'(t) = \langle e^t \cos t - e^t \sin t, 2e^{2t}, e^t \sin t + e^t \cos t \rangle$

$r'(1) = \langle e\cos 1 - e\sin 1, 2e^2, 1 \rangle$

$\boxed{(e\cos 1 - e\sin 1)(x-1) + (2e^2(y-1)) + 1(z-0)}$

0

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$x = e^t \cos t$ $\quad \sin^2\theta + \cos^2\theta = 1$ $\quad r(t) = \langle e^t \cos t, e^{2t}, e^t \sin t \rangle$

$z = e^t \sin t$

$x^2 + z^2 = y$ $\quad 1, 1, 0$

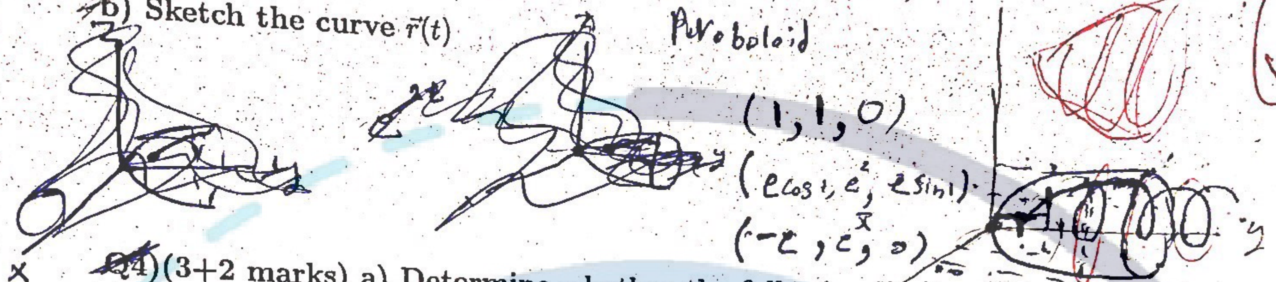
b) Sketch the curve $\vec{r}(t)$

Paraboloid

$(1,1,0)$

$(e\cos t, e^2, e\sin t)$

$(-e, e^{\pi}, 0)$



Q4)(3+2 marks) a) Determine whether the following limit exist or not.

$\lim_{(x,y)\to(0,0)} \frac{xy^2e^x}{x^2+5y^2}$

$y = mx$

$\lim_{x\to 0} \frac{m^2x^3e^x}{x^2+5m^2x^2} = \frac{x^3m^2e^x}{x^2(1+5m^2)} = 0$

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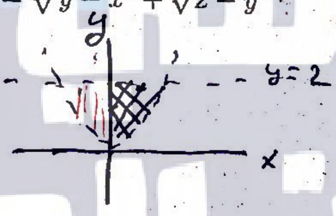
b) Sketch the domain of the function $f(x,y) = \sqrt{y-x^2} + \sqrt{2-y}$

$2 > y$ $\quad 2 - y > 0$

$y < 2$ $\quad y - x^2 > 0$ $\quad y > x^2$

$2 = y$ $\quad y = x^2$



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Q5)(3+2 marks) a) Find the equation of the tangent plane to the surface $xy^2 + z^3 = 12cos(xyz)$ at the point $(3,2,0)$.

$F(x,y,z) = xy^2 + z^3 - 12\cos(xyz)$

$F_x = y^2 + 0 + 12yz\sin(xyz) \to 4 + 0$

$F_y = 2xy + 12xz\sin(xyz) \to 12 + 0$

$F_z = 3z^2 + xy\sin(xyz) \to 0$

$0 = 4(x-3) + 12(y-2) + 0$

$4x - 12 + 12y - 24 = 0$

$\boxed{4x + 12y = 36}$

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b) If $D_uf(1,1,1) = 5$, $u = <4, -4, 2>$ and $|\nabla f(1,1,1)| = 5$. Find $\nabla f(1,1,1)$.

$\cos(\theta)$ $\quad \theta = 0$

$\hat{u} = \frac{<4,-4,2>}{\sqrt{36}} = <\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}>$ $\quad \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$

$<\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}> \cdot <x, y, z> = \cos(\theta)$.

$|\cos(\theta)| = 1$

$\theta = 0, \pi$

$5 = \hat{u} \cdot \nabla f(1,1,1)$ $\quad \boxed{\nabla f = 5\hat{u} = 5<\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}>}$

2

$f(2,5,4)$

$s=2$, $t=1$, $r=1$

$x=2$, $y=1$

Q6)(4 marks) Let f be a differentiable function of x, y and z, let $g(s,t,r) = f(st, s^2+r^3, 3t+r^4)$ and $f_x(2,1,1) = -4$, $f_x(2,5,4) = 2$, $f_y(2,1,1) = 3$, $f_y(2,5,4) = 6$, $f_z(2,5,4) = -3$, $f_z(2,1,1) = 7$, Find $g_r(2,1,1)$

$$f_y \cdot \frac{\partial y}{\partial r} + f_z \frac{\partial z}{\partial r}$$

$$6 \cdot 3r^2 + (-3)\, 4r^3$$

$r=1$

$$18 - 12 = \boxed{6}$$

4

Q7) (6 marks) Find and classify the critical points for the function $f(x,y) = x^2y - y^3 - 2x^2 + 3y$.

$F_x = 2yx - 4x = 0$

$2yx = 4x$, $y = 2$, $x^2 = 9$, $x = \pm 3$

$F_y = x^2 - 3y^2 + 3$

$x^2 - 12 + 3$

$F_{xx} = -4 + 2y$

$f_{yy} = -6y$

$F_{xy} = 2x$

$f_{yx} = 2x$

$(0,-1)$, $(0,+1)$, $(3,2)$, $(-3,2)$

	f_{xx}	f_{yy}	$(f_{yx})^2$	
$(0,-1)$	-6	6	0	S.P
$(0,+1)$	-2	-6	0	local max
$(3,2)$	0	-12	36	S.P
$(-3,2)$	0	-12	36	S.P

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