Jordan University
Mathematics Department

Final Exam
Math 102

Summer Semester
2009/2010

Section: ....

....

: Number: ....

n the blank: (2 points each)

1- The polar equation for the curve $x^2+y^2=4x$ is $r^2=4r\cos\theta$ $(r=4\cos\theta)$

2- If $f(0)=f(3)=3$ and $f'(0)=f'(3)=1$, then

$\int_0^3 xf''(x)\,dx$ $x f'(x)\Big|_0^3-\int_0^3 f'(x)\,dx = xf'(x)\Big|_0^3-f(x)\Big|_0^3 = 3\times1-[f(3)-f(0)] = 3-[3-3]=3$

3- The set of values of C such that $\int_2^6 \frac{1}{2x+c}\,dx$ is an improper integral is

$-12\le c\le -2$

4- The sum of the series $\sum_{k=1}^{\infty}\frac{2^{k+3}}{4^{k+1}}$ is $\sum_{k=0}^{\infty}\frac{2^{k+4}}{4^{k+2}}=\frac{2^4}{4^2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=\frac{1}{1-\frac{1}{2}}=2$

5- The partial fraction decomposition (do not evaluate the constants) of

$\frac{x+4}{(x^2+11)^2}$ is $\frac{AX+B}{(X^2+11)}+\frac{CX+D}{(X^2+11)^2}$

6- The sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{3^{2n}(2n)!}$ is $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

7- The limit of the sequence $\left\{\left(\frac{n-2}{n}\right)^n\right\}$ is e^{-2}

8- The polar coordinates of the point P with rectangular coordinates $(-\sqrt{3},-1)$ are $\left(2, \frac{7\pi}{6}\right)$

9- Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^{2n+1}}{n^2+4}$.

(5 points)

$$\lim_{n\to\infty} \frac{\dfrac{\left|(-1)^{n+1}(x+1)^{2(n+1)+1}\right|}{(n+1)^2+4}}{\dfrac{\left|(-1)^n (x+1)^{2n+1}\right|}{n^2+4}} < 1 \iff \lim_{n\to\infty} \frac{n^2+4}{(n+1)^2+4} \left|(x+1)^2\right| < 1$$

$$\iff \left|(x+1)^2\right| < 1 \iff |x+1| < 1$$

$$\iff -1 < x+1 < 1 \iff -2 < x < 0$$

when $x=-2$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+4}$ which is absolutely convergent

when $x=0$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ which is convergent

So, the interval of convergence is $[-2, 0]$

and the radius of convergence is 1

10- Test the series for convergence or divergence $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$. (5 points)

using the root test

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty} \frac{n!}{n^4} = \lim_{n\to\infty} \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \frac{n-3}{n} \times (n-4) \times \dots 1$$

$$= 1 \times 1 \times 1 \times 1 \times \infty = \infty > 1$$

So, the series diverges

11- Test the series for convergence $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$.

Suggested answer

$$\frac{|\sin 2n|}{1+2^n} < \frac{|\sin 2n|}{2^n} \le \frac{1}{2^n} \qquad n \ge 1$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent [Geometric Series with $|r| = \frac{1}{2} < 1$]

So, by the comparison test, $\sum_{n=1}^{\infty} \frac{|\sin 2n|}{1+2^n}$ is convergent

Thus, $\sum \frac{\sin 2n}{1+2^n}$ is convergent (it's absolutely conver

12- Evaluate the integral $\int \frac{x\,dx}{\sqrt{x^2+x+1}}$. (5 points)

$x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$ (completing square)

$$\int \frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,dx$$

using the method of trigonometric substitution

$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \frac{\sqrt{3}}{2}\sec^2\theta$

$$= \int \frac{(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2})\frac{\sqrt{3}}{2}\sec^2\theta}{\sqrt{(\frac{\sqrt{3}}{2}\tan\theta)^2 + \frac{3}{4}}}\,d\theta = \int \frac{(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2})\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\sqrt{\frac{3}{4}(\tan^2\theta+1)}}$$

$$= \int \frac{(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2})\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{\sqrt{3}}{2}\sec\theta} = \int \frac{\sqrt{3}}{2}\sec\theta\tan\theta - \frac{1}{2}\sec\theta\,d\theta$$

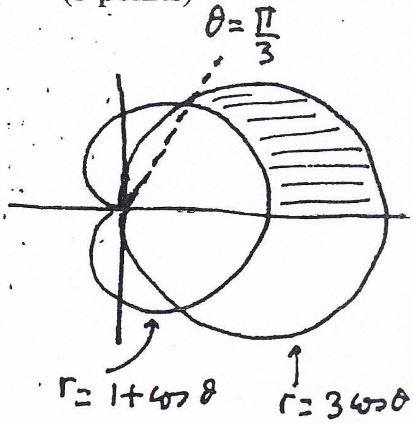
$$= \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + c$$

$\tan\theta = \frac{x+\frac{1}{2}}{\sqrt{3}/2}$

$$= \frac{\cancel{\sqrt{3}}}{\cancel{2}}\frac{\sqrt{x^2+x+1}}{\sqrt{3}/2} - \frac{1}{2}\ln\left|\frac{\sqrt{x^2+x+1}}{\sqrt{3}/2} + \frac{x+\frac{1}{2}}{\sqrt{3}/2}\right|$$

13- Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$. **(5 points)**

Area = 2 × area of the shaded region



$$= 2\left[\int_0^{\pi/3} \tfrac{1}{2}(3\cos\theta)^2\,d\theta - \int_0^{\pi/3} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta\right]$$

$$= \int_0^{\pi/3} 9\cos^2\theta - \cos^2\theta - 2\cos\theta - 1\,d\theta$$

$$= \int_0^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1\,d\theta$$

$$= \int_0^{\pi/3} 8 \times \tfrac{1}{2}[1+\cos 2\theta] - 2\cos\theta - 1\,d\theta$$

$$= \int_0^{\pi/3} 4\cos 2\theta - 2\cos\theta + 3\,d\theta$$

$$= 2\sin 2\theta - 2\sin\theta + 3\theta \Big|_0^{\pi/3}$$

$$= 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} + 3\left(\frac{\pi}{3}\right) - 0 = \pi$$

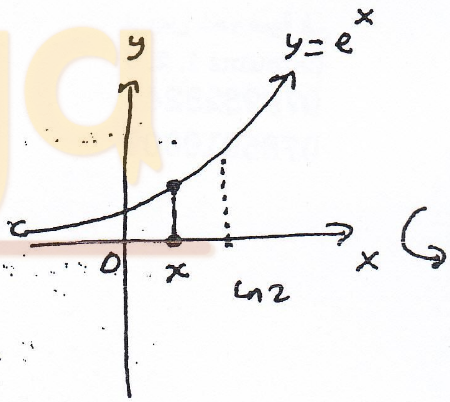
$1 + \cos\theta = 3\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

14- Find the area of the surface of the solid that results by revolving the region enclosed by the curves $y = e^x$, $x = 0$, $x = \ln 2$ about the $x-axis$. **(5 points)**

Using the method of disks



$$\text{volume} = \pi \int_0^{\ln 2} (e^x)^2 - 0^2\,dx$$

$$= \pi \int_0^{\ln 2} e^{2x}\,dx = \frac{\pi}{2} e^{2x}\Big|_0^{\ln 2}$$

$$= \frac{\pi}{2}\left[e^{2\ln 2} - 1\right] = \frac{\pi}{2}[4-1]$$

$$= \frac{3\pi}{2}\ \text{unit}^3$$

15- Evaluate the integral $\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx$. (5 points)

$$\int \frac{(e^x)^2}{(e^x)^2+3e^x+2}\,dx$$

$$u=e^x \Rightarrow \frac{du}{dx}=e^x \Rightarrow dx=\frac{du}{u}$$

$$\int \frac{u^2}{u^2+3u+2}\left(\frac{du}{u}\right)=\int \frac{u}{u^2+3u+2}\,du=\int \frac{u}{(u+1)(u+2)}\,du$$

Using the method of partial fractions

$$\frac{u}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2} \Rightarrow A(u+2)+B(u+1)=u$$

$$u=-1 \Rightarrow A=-1$$
$$u=-2 \Rightarrow B=2$$

$$\int \frac{u}{(u+1)(u+2)}\,du=\int \frac{-1}{(u+1)}+\frac{2}{(u+2)}\,du$$
$$=-\ln|u+1|+2\ln|u+2|+c$$
$$=-\ln(e^x+1)+2\ln(e^x+2)+c$$
$$=\ln\left(\frac{(e^x+2)^2}{e^x+1}\right)+c$$

Jordan University
Mathematics Department

Final Exam
Math 102

2^{nd} Semester
2009/2010

Name: ....

Student Number: ....

I- Fill in the blank: (2 points each)

1- The polar coordinate (r,θ) for the point $P(-1,-\sqrt{3})$ is ...$\left(2, \frac{4\pi}{3}\right)$........

2- The equation $r = 2\sin(\theta) + 6\cos(\theta)$ represents a circle with center ....(3, 1)........... and radius ...$\sqrt{10}$.......

3- $\sum_{n=1}^{\infty} 6(0.9)^{n+1} =$ ...$\sum_{n=0}^{\infty} 6(0.9)^{n+2} = 6(0.9)^2 \frac{1}{1-0.9} = \frac{6(0.9)^2}{0.1} = 48.6$

4- If $\sum_{n=1}^{\infty} 2a_n = 3$, then $\lim_{n\to\infty}(a_n + 6) =$ .. 6............................

5- The sum of the series $\sum_{n=1}^{\infty} \frac{(\ln 3)^n}{n!} =$ $\sum_{n=0}^{\infty} \frac{(\ln 3)^n}{n!} - 1 = e^{\ln 3} - 1 = 3 - 1 = 2$

6- $\int \frac{1}{x^2 + 2x + 2} dx =$ ...$\int \frac{1}{(x+1)^2 + 1} dx = \tan^{-1}(x+1) + c$

7- The integral that gives the arc length of the curve $x = e^y$, $0 \le y \le 1$ is

.... $\int_0^1 \sqrt{1+e^{2y}}\,dy$ ....

8- The maclaurin series for the function $f(x) = x\sin(4x)$ is ...... $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{2n+2}}{(2n+1)!}$

9- The substitution $x = \sin(\theta)$ transforms the integral $\int \frac{dx}{(1-x^2)^{3/2}}$

into $\int$L L L L L L L $d\theta$. $\int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta.$

10- $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n^2}\right)-\cos\left(\frac{\pi}{(n+1)^2}\right)\right) =$ ...... $\lim_{k\to\infty} \cos\pi - \cos\frac{\pi}{(k+1)^2} = -1-1=-2$ ....

11- The Taylor series for the function $f(x) = 3e^x$ about $x = 2$ is ..... $3e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ . . .

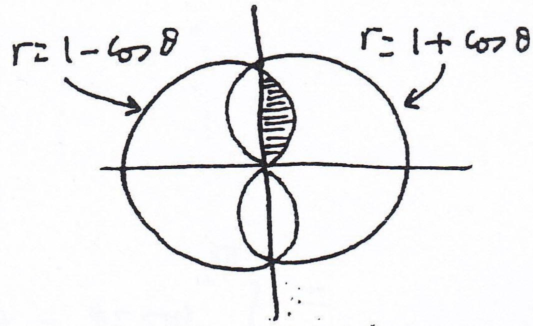
12- The partial fraction decomposition for $\frac{1}{(x^3-1)(x-1)}$ is $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1}$

$\downarrow$

$\frac{1}{(x-1)^2(x^2+x+1)}$

II- **Set up the integrals that represent: - (do not evaluate the integrals).**

a- the area of the region, lies inside the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$. **(3 points)**



Area = 4 × area of the shaded region.

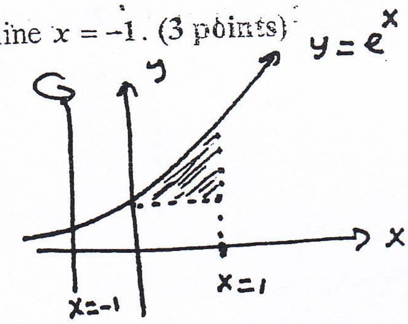
$$= 4 \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2 \, d\theta$$

$$= 2 \int_0^{\pi/2} \cos^2\theta - 2\cos\theta + 1 \, d\theta$$

$$= 2 \int_0^{\pi/2} \frac{1}{2}[1+\cos(2\theta)] - 2\cos\theta + 1 \, d\theta$$

continue →

b- The volume of the solid results when the region bounded by the curves $y = e^x$, $y = 1$, and $x = 1$ is revolved about the line $x = -1$. (3 points)



using cylindrical shells

$$\text{Volume} = 2\pi \int_0^1 (e^x - 1)(x+1)\, dx$$

$$= 2\pi \int_0^1 (x+1)e^x - x - 1 \, dx$$

$$= 2\pi \left[(x+1)e^x - e^x - \frac{x^2}{2} - x\right]_0^1 = 2\pi\left[xe^x - \frac{x^2}{2} - x\right] = 2\pi(e - 3/2)$$

III- **Find the interval of convergence for the power series** $\sum_{n=1}^{\infty} \frac{(x+4)^n}{\sqrt{n}\ 3^n}$. **(5 points)**

$$\lim_{n\to\infty} \frac{\left|\frac{(x+4)^{n+1}}{\sqrt{n+1}\ 3^{n+1}}\right|}{\left|\frac{(x+4)^n}{\sqrt{n}\ 3^n}\right|} < 1 \iff \lim_{n\to\infty} \frac{1}{3}\frac{\sqrt{n}}{\sqrt{n+1}}|x+4| < 1$$

$$\iff \frac{1}{3}|x+4| < 1$$

$$\iff |x+4| < 3 \iff -3 < x+4 < 3$$

$$\iff -7 < x < -1$$

when $x=-1$, the series becomes: $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\ 3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent

when $x=-7$, the Series becomes: $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\ 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is conditionall converge.

So, the interval of convergence is $[-7, -1)$

$$= \int_0^{\pi/2} \cos 2\theta - 4\cos\theta + 3\, d\theta$$

$$= \frac{\sin 2\theta}{2} - 4\sin\theta + 3\theta \Big|_0^{\pi/2} = -4 + \frac{3\pi}{2}$$

IV- a- Find the Manclaurin series for the function (3 points)

$$f(x) = \frac{x}{(1+2x)^2}, \qquad -\frac{1}{2} < x < \frac{1}{2}$$

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n \qquad -\frac{1}{2} < x < \frac{1}{2}$$

$$\frac{d}{dx}\frac{1}{1+2x} = \frac{d}{dx}\sum_{n=0}^{\infty} (-2)^n x^n \implies \frac{-2}{(1+2x)^2} = \sum_{n=1}^{\infty} (-2)^n n x^{n-1}$$

$$\implies \frac{x}{(1+2x)^2} = -\frac{1}{2}\sum_{n=1}^{\infty} (-1)^n n (2x)^n$$

b- Use part (a) to evaluate the sum of the series $\sum_{n=1}^{\infty} (-1)^n n \left(\frac{2}{3}\right)^n$. (2 points)

From (a) we have $\sum_{n=1}^{\infty} (-1)^n n (2x)^n = \frac{-2x}{(1+2x)^2}$

$$x = \frac{1}{3} \implies \sum_{n=1}^{\infty} (-1)^n n \left(\frac{2}{3}\right)^n = \frac{-2/3}{(5/3)^2} = \frac{-6}{25}.$$

V- Test the following series for convergence

a- $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ (3 points)

Using the root test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

So, the series converges

b- $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. (3 points)

Using the integral test.

$$\int_2^{\infty} \frac{1}{x\,Ln(x)}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\,Ln(x)}\,dx$$

$$= \lim_{b\to\infty} Ln|Ln(b)| - Ln|Ln(2)|$$

$$= \infty$$

So the series diverges.

$$\int \frac{1}{x\,Ln(x)}\,dx$$

$$u = Ln(x) \Rightarrow \frac{du}{dx} =$$

$$\int \frac{1}{x\,u}(x\,du)$$

$$= \int \frac{1}{u}\,du = Ln|\ldots$$

$$= Ln|Ln\ldots$$

VI- Evalute the integral $\int x^3 \tan^{-1}(x)\,dx$ (5 points)

Using the integration by parts

$u = \tan^{-1}(x)$ $\quad dv = x^2\,dx$

$du = \frac{1}{1+x^2}\,dx$ $\quad v = \frac{x^3}{3}$

$-\int$

$$= \frac{x^3}{3}\tan^{-1}(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$$\begin{array}{r} x \\ x^2+1 \,\overline{)\, x^3 \quad} \\ \underline{x^3 + x} \\ -x \end{array}$$

$$= \frac{x^3}{3}\tan^{-1}(x) - \frac{1}{3}\left[\int x - \frac{x}{x^2+1}\,dx\right]$$

$$= \frac{x^3}{3}\tan^{-1}(x) - \frac{1}{3}\left[\int x - \frac{1}{2}\,\frac{2x}{x^2+1}\,dx\right]$$

$$= \frac{x^3}{3}\tan^{-1}(x) - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}Ln(x^2+1)\right] + c$$