CHAPTER 7 :

TEQNIUQUES OF INTEGRATION

7.1 *Integration by parts*

Note: $(u.v)^1 = udv + vdu$

مشتقة ضرب

Integ . $\int udv = \int (u.v)^1 - \int v\, du$

$\int udv - u.v - \int vdu \rightarrow$

$(int.\ by\ parts$ قانون$)$

لما أفرض الu بختار اقتران سهل اشتقاقه

لما أفرض الdv بختار اقتران بقدر أكامله

$Ex : find :$

1) $\int xe^x\, dx$

$u = x$ $\quad$ $dv = e^x dx$

اشتقاق $\quad$ تكامل

$du = dx$ $\quad$ $v = e^x$

$= u.v - \int vdu$

$= xe^x - \int e^x\, dx$

$= xe^x - e^x + c$

2) $\int xsinx\, dx$

$u = x$ $\qquad\qquad dv = sinx\, dx$

$du = dx$ $\qquad\qquad v = -cosx$

$= -xcosx - \int -cosx\, dx$

$= -xcosx + sinx + c$

3) $\int x \ln x \, dx$

$u = \ln x$ $\qquad\qquad dv = xdx$

أخذت $u = \ln x$ عشان ما بقدر أكامل ال $\ln x$ فاشتقاقه سهل

$du = \frac{dx}{x}$ $\qquad\qquad v = \frac{x^2}{2}$

$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2x} dx$

$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

4) $\int x \tan^{-1}(x)\ dx$

$\underline{u = x \ \rightarrow wrong}$

$u = \tan^{-1} x \qquad dv = xdx$

$du = \frac{dx}{1+x^2} \qquad v = \frac{x^2}{2}$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\ dx$

$= \frac{x^2}{2} tan^{-1} x - \frac{1}{2}\ [\int \left(\frac{x^2+1}{x^2+1}\ \frac{-1}{x^2+1}\right) dx]$

$= \frac{x^2}{2} tan^{-1} x - \frac{1}{2}[x - \tan^{-1} x]$

5) $x^2 e^{2x}\ dx \rightarrow by\ parts\ 2\ times\ or\ Tables\ Method$

$Method\ 1:$

$u = x^2 \qquad dv = e^{2x}\ dx$

$du = 2xdx \qquad v = \frac{e^{2x}}{2}$

$\frac{x^2}{2} e^{2x} - \int xe^{2x} dx \rightarrow (one\ more\ by\ parts)$

$u = x \qquad dv = e^{2x} dx$

$Method\ 2:$

بشتق	Sign	بكامل
		e^{2x}
x^2	+	$\frac{e^{2x}}{2}$
$2x$	−	$\frac{e^{2x}}{4}$
2	+	$\frac{e^{2x}}{8}$
0		

$= \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{e^{2x}}{4} + c$

6) $\int x^3 \sin 3x \, dx \rightarrow tables$

u	sign	dv
x^3	+	$\sin 3x$
$3x^2$	−	$-\frac{cos3x}{3}$
$6x$	+	$-\frac{sin3x}{9}$
6	−	$\frac{cos3x}{27}$
0		$\frac{sin3x}{81}$

$$-\frac{x^3}{3}\cos 3x + \frac{x^2}{3}\sin 3x + \frac{6x}{27}\cos 3x - \frac{6}{81}\sin 3x + c$$

7) $\int x^5 e^x \, dx \rightarrow try\ it$

8) $x^5 \ln x \, dx$

$u = \ln x \qquad\qquad dv = x^5 dx$

$du = \frac{dx}{x} \qquad\qquad v = \frac{x^6}{6}$

$= \frac{x^6}{6}\ln x - \int \frac{x^5}{6} dx$

$= \frac{x^6}{6}\ln x - \frac{x^6}{36} + c$

9) $\int e^{\sqrt{x}}dx \longrightarrow$ *If we subs.* $\sqrt{x}$ *by z it turns into int. by parts*

$z = \sqrt{x}$

$dz = \frac{dx}{2\sqrt{x}}$

$\Longrightarrow \int e^{z}.2\sqrt{x}\,dz$

$\int e^{z}.2z\,dz$

$u = 2z \qquad dv = e^{z}dz$

$du = 2dz \qquad v = e^{z}$

$= 2ze^{z} - \int 2e^{z}dz \rightarrow 2ze^{z} - 2e^{z} + c$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

10) $\int \frac{\sin^{-1}(\ln x)}{x}\,dx$ ⟶ $important$!

$z = \ln x$

$dz = \frac{dx}{x}$ ⟶ $dx = x dz$

$\int \frac{\sin^{-1} z\, x\, .dz}{x}$

$\int \sin^{-1} z\, dz$?? ⟶ هون بقدر أحله $parts$ ولكن ماذا أفرض

$\sin^{-1} z$ ⟶ اشتقاقه سهل ولكن تكامله ما بعرفه

$u = \sin^{-1} z$ $\quad\quad dv = dz$

$du = \frac{dz}{\sqrt{1-z^2}}$ ⟵ $v = z$

$= z\sin^{-1} z - \int \frac{z}{\sqrt{1-z^2}}\,dz$

بالتعويض

$t = 1 - z^2$

$dt = -2z\, dz$

$= z\sin^{-1} z - \int \frac{z}{\sqrt{t}} . \frac{-2dt}{2z}$

$= z\sin^{-1} z + \int \frac{t^{-\frac{1}{2}}}{2}\, dt$

$= z\sin^{-1} z + 2\frac{t^{\frac{1}{2}}}{2} + c$

$= \ln x\ \sin^{-1}(\ln x) + \sqrt{1-z^2} + c$

* هون رح نيجي لفكرة اقترانين دوريات مضروبين اشتقاقه\تكامله بخليه يعيد نفسه

$Ex : 1) \int e^x sinx\, dx$ → لما اشوف اقترانين دوريات مضروبين بحل بالطريقة الآتية

$\downarrow$

I

$u = e^x \qquad dv = sinx\, dx$

$du = e^x dx \qquad v = -cosx$

$I = -e^x cosx + \int e^x\, cosx\, dx$

$\downarrow$

one more parts

$u = e^x \qquad v = cosx\, dx$

$du = e^x dx \qquad v = sinx$

$I = -e^x cosx + e^x sinx - \int \underline{e^x sinx\, dx}$

$\downarrow$

I

$I = e^x(sinx - cosx) - I$

$2I = e^x(sinx - cosx)$

$I = \frac{1}{2} e^x(sinx - cosx) + c$

$2) \int e^{2x} cos3x\, dx \rightarrow try\ it$

3) $\int \cos(\ln x)\, dx \longrightarrow \int \cos z \,.\, x \,.\, dz$

$z = \ln x$

$dz = \frac{dx}{x}$

$x = e^{z}$

$\int e^{z} \cos z \, dz \longrightarrow$ نفس تكملة مثال رقم واحد

4) $\int \frac{(\ln x)^2}{x^3}$

$z = \ln x \rightarrow x = e^{z} \,, dz = \frac{dx}{x}$

$\int \frac{z^2}{\cancel{x}^3} . \cancel{x} . dz$

$\int \frac{z^2}{e^{2z}} dz \rightarrow \int e^{-2z} . z^2 . dz$

u	sign	dv
		e^{-2z}
z^2	+	$\frac{-e^{-2z}}{2}$
$2z$	−	$\frac{e^{-2z}}{4}$
2	+	$-\frac{e^{-2z}}{8}$
0		

5) $\int \tan^{-1}\left(\frac{1}{x}\right) dx$

$Method\ 1 : z = \frac{1}{x} \rightarrow parts$

$Method\ 2 : parts$:

$M.2$:

$$\frac{d}{dx}\tan^{-1}(u) = \frac{u}{u^2+1}$$

$$u = \tan^{-1}\left(\frac{1}{x}\right) \quad , \quad du = \frac{\frac{-dx}{x^2}}{1+x^2} \quad \rightarrow du = \frac{-dx}{x^2+1}$$

$$dv = dx \rightarrow dv = x$$

$$= x\tan^{-1}\frac{1}{x} + \int \frac{x\,dx}{x^2+1}$$

$$= x\tan^{-1}\frac{1}{x} + \frac{1}{2}\ln\,|x^2+1| + c$$

6) $\int \frac{x}{10^x} dx$

$\int x10^{-x} dx \qquad u = x\,, dv = 10^{-x}$

7) $\int x(\tan x)^2 dx$

$u = x \qquad dv = (tanx)^2 \rightarrow (\sec x)^2 - 1$

8) $\int x^{\frac{2}{3}} \ln x \, dx$

$u = \ln x \qquad dv = x^{\frac{2}{3}}$

$*$ *More problems* :

evaluate :

a) $\int \cos(\ln x)\, dx \longrightarrow z = \ln x$ أجزاء مرتين

b) $\int_0^2 y \sinh(y) dy \longrightarrow u = y\,, v = \sinh(y)\, dy$ لا تنس حدود التكامل

c **) $\int \frac{xe^{2x}}{(1+2x)^2} dx \longrightarrow u = xe^{2x}, dv = \frac{1}{(1+2x)^2} \rightarrow trick$

* أي استفسار أو سؤال لا تتردد ....

في هذا الجزء سنتحدث عن التكاملات المثلثية المرفوعة الى قوة مثال :

$$(\sec x)^3 , (tanx)^4, (sinx)^5$$

$$\int (sinx)^n dx \qquad or \qquad \int (cosx)^n dx$$

$\underline{n = odd}$

$$ex: \int (sinx)^3 dx$$

$$\int sinx(1 - (cosx)^2)dx$$

$$z = cosx$$

$\underline{n = even}$

$ex: \int (sinx)^2 dx \rightarrow$ أنت وشطارتك بالمتطابقات

$$\int \frac{1}{2}(1 - \cos(2x))dx$$

$$= \cos(2x)$$

$$= 1 - 2(sinx)^2$$

$*$ *In General*:

$$\int (sinx)^n x\, dx = I$$

غير مطالب بحفظها ابدا ولكن فهم كيفية اشتقاقها

الهدف منها تسهيل التكامل

$$u = (sinx)^{n-1} \qquad dv = sinx\, dx$$

$$du = (n-1)(sinx)^{n-2} cosxdx \qquad v = -cosx$$

$$I = -cosx\, (sinx)^{n-1} + \int (n-1)(cosx)^2 (sinx)^{n-2} dx$$

$$I = -cosx\, (sinx)^{n-1} + (n-1) \int (1-(sinx)^2)(sinx)^{n-2} dx$$

$$I = -cosx\, (sinx)^{n-1} + (n-1) \int (sinx)^{n-2} dx - (n-1) \int (sinx)^n d$$

$$I = -cosx\, (sinx)^{n-1} + (n-1) \int (sinx)^{n-2}\, dx - In + I$$

$$I = -cosx\, (sinx)^{n-1} + \frac{n-1}{n} \int (sinx)^{n-2} dx \ \rightarrow General\ formula$$

$for\ (cosx)^n same\ steps$

$* \int (secx)^2 dx$

$n = even$

$ex: \int (secx)^6 dx \rightarrow \int (1 + (tanx)^2)^2 (secx)^2 dx$

$z = tanx$

$n = odd$

$ex: \int (secx)^2 dx \rightarrow sol. is\ formula$

$In\ General$:

$\int (secx)^n dx = I$

$u = (secx)^{n-2}$ $\qquad dv = (secx)^2 dx$

$du = (n-2)(secx)^{n-3}. secx\ tanx\ dx$

$du = (n-2)(secx)^{n-2} tanx\ dx$ $\qquad v = tanx$

$I = tanx\ (secx)^{n-2} - \int (n-2)(secx)^{n-2}. (tanx)^2 dx$

$I = tanx\ (secx)^{n-2} - (n-2)(I - \int (secx)^{n-2} dx)$

$I = tanx\ (secx)^{n-2} - In + 2I + (n-2) \int (secx)^{n-2}\ dx$

$(n-1)I = tanx\ (secx)^{n-2} + (n-2) \int (secx)^{n-2}\ dx$

$$I = \frac{1}{n-1} tanx\ (secx)^{n-2} + \frac{n-2}{n-1} \int (secx)^{n-2} dx$$ → ليست للحفظ

$* \int (secx)^5 dx \longrightarrow formula$ بهيك أسئلة فقط افهم طريقة الاشتقاق لل

$$formula \longrightarrow \frac{1}{4} tanx\ (secx)^3 + \frac{3}{4}\int (secx)^3 dx$$

$$\downarrow$$

$$formula$$

$$\frac{1}{4} tanx\ (secx)^3 + \frac{3}{4}\left[\frac{1}{2} tanx\ secx + \frac{1}{2}\int secx\ dx\right] + c$$

$$\downarrow$$

$$\ln|secx + tanx|$$

$* \int (tanx)^n\ dx$

<u>*In General*</u> سهلة

$$\int (tanx)^n\ dx$$

$$= \int (\tan x)^{n-2}\ (tanx)^2 dx$$

$$= \int (tanx)^{n-2}((secx)^2 - 1)\ dx$$

$$= \int (tanx)^{n-2}(secx)^2\ dx - \int (tanx)^{n-2} dx$$

$$= \frac{(tanx)^{n-1}}{n-1} - \int (tanx)^{n-2} dx$$

$$* \int (tanx)^6 dx$$

$$\int (tanx)^4 ((\sec x)^2 - 1)\ dx$$

$$\int (tanx)^4 (secx)^2 - \int (tanx)^4 dx$$

$\downarrow$ $z = tanx$ $\qquad$ $\downarrow$ $formula$

في هذا الجزء سنراجع المتطابقات التي نحتاجها لبعض أشكال التكامل

$Ex: \int sin3x\ cos4x\ dx$

1) $\sin(a + b) = \sin a \cos b + \cos a \sin b$

2) $\sin(a - b) = \sin a \cos b - \cos a \sin b$

3) $\cos(a + b) = \cos a \cos b - \sin a \sin b$

4) $\cos(a - b) = \cos a \cos b + \sin a \sin b$

$$1 + 2 \quad \sin a \cos b = \frac{1}{2}(\sin(a + b) + \sin(a - b))$$

$$3 + 4 \quad \cos a \cos b = \frac{1}{2}(\cos(a + b) + \cos(a - b))$$

$$4 - 3 \quad \sin a \sin b = \frac{1}{2}(\cos(a - b) - \cos(a + b))$$

* جاهزات للحل المباشر

$$* \int \sin 3x \cos 4x \, dx$$

$$= \frac{1}{2}\int (\sin 7x - \sin x)\, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 7x}{7} + \cos x\,\right] + c$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + c$$

$$* \int \cos 5x \cos 4x \, dx \quad \longrightarrow \textit{try it}$$

<u>*7.2 Trigonometric integrals*</u>

$$\int (\sin x)^n\,(\cos x)^m\, dx$$

<u>*Case 1 :*</u>

if m is odd or n is odd

Ex: $\int (sinx)^4 (cosx)^5\, dx$

$$\int (sinx)^4 (cosx)^4 cosx\, dx$$

$$\int (sinx)^4 (1-(sinx)^2)^2 cosx\, dx$$

$$z = \cos x$$

Case 2 :

$if\ n\ and\ m\ are\ both\ even$ بتحول الأصغر قوة

$$Ex\ \int (sinx)^4 (cosx)^6\, dx$$

$$= \int (1-(cosx)^2)^2 (cosx)^6 dx$$

$$= \int (cosx)^6 dx - 2\int (cosx)^8 dx + \int (cosx)^{10}\, dx \longrightarrow formula$$

غير مطالب مثل هيك بالامتحان ولكن ممكن يطلب منك شكل التكامل فقط

$$\int (secx)^n (tanx)^m\, dx$$

Case 1 :

$n\ and\ m\ both\ even$

$Ex{:}\ \int (tanx)^4 (secx)^6\, dx \longrightarrow z$ بسحب $(secx)^2$ عامل مشترك ومتطابقة وبفرض

$$\int (tanx)^4 (secx)^4 (secx)^2\, dx$$

$$\int (tanx)^4 (1+(tanx)^2)^2 (secx)^2 dx$$

$$z = tanx$$

Case 2 :

n and m both odd

$Ex: \int (tanx)^3 (secx)^5 dx \rightarrow$ zوبفرض عامل مشترك و متطابقة$secx\ tanx$ بسحب

$$\int (tanx)^2 (secx)^4 secx\ tanx\ dx$$

$$\int ((secx)^2 - 1)(secx)^4 secx\ tanx\ dx$$

$$z = secx\ , \quad dz = secx\ tanx\ dx$$

Case 3 :

n odd and m even

بوحد الاقترانات الى الاقتران ذو القوة الفردية

ثم $formula$

$Ex: \int (secx)^3 (tanx)^4 dx$

$$\int (secx)^3 ((secx)^2 - 1)^2 dx$$

فك التربيع

$$\int (secx)^7 dx - \int 2(secx)^5 dx + \int (secx)^3 dx$$

برجع بكرر غير مطالب بالحل للنهاية ولكن يجب معرفة الوصول لهذا الشكل

$Examples : evaluate :$

$$1)\int (\csc x)^4\,(\cot x)^5\ dx$$

$$\int ((cotx)^2+1)(cscx)^2(cotx)^5\,dx$$

$z=\cot x \qquad\qquad dz=-(cscx)^2dx$

$$\int \frac{(z^2+1)(cscx)^2z^5}{-(cscx)^2}dz$$

$$-\int (z^7+z^5)dz$$

$$-\left[\frac{z^8}{8}+\frac{z^6}{6}\right]+c$$

$$-\left[\frac{(cotx)^8}{8}+\frac{(cotx)^6}{6}\right]+c$$

في باقي الأمثلة عند فرض قيمة الz

فان السؤال قد انتهى والتكملة مجرد التأكد من الحل

لذلك فقط أثناء حل الدوسية نضع الفرض المناسب اختصارا

$$2)\int \frac{\cos x+\sin x}{\sin 2x}dx \longrightarrow \sin(\,2x\,)=2\sin x\cos x$$

$$\frac{1}{2}\int cscx\ dx+\frac{1}{2}\int secx\ dx$$

$$\frac{1}{2}\ln|\csc x-\cot x\,|+\frac{1}{2}\ln|\sec x+\tan x|+c$$

3) $\int (tanx)^2 + (tanx)^4 dx$

$\int (tanx)^2 (1 + (tanx)^2 dx$

$\int (tanx)^2 (secx)^2 dx$

$z = tanx$

$dz = (secx)^2 dx$

⋮

4) $\int (tanx)^3 secx\, dx$

$\int (tanx)^2 \sec x \tan x\, dx$

$\int ((secx)^2 - 1)\, secx\, tanx\, dx$

$z = secx$

$dz = secx\, tanx\, dx$

⋮

5) $\int \frac{dx}{cosx - 1} * \frac{cosx + 1}{cosx + 1}$

$\int \frac{cosx+1}{(cosx)^2-1} dx$

$\int \frac{cosx+1}{-(sinx)^2} dx \rightarrow \int (-cscx\, cotx - (cscx)^2) dx$

$cscx + cotx + c$

7.3 *Trigonometric Substitution*

$$a^2 - x^2 \rightarrow x = a\ sin\theta$$

$$a^2 + x^2 \rightarrow x = a\ tan\theta$$

$$x^2 - a^2 \rightarrow x = a\ sec\theta$$

هنا نختار الفرض المناسب للتخلص من مشاكل كالجذور والتكاملات الغريبة بالاستعانة بالمتطابقات

$Ex : Evaluate$

1) $\int \sqrt{4 - x^2}dx \rightarrow \sqrt{2^2 - x^2}$

$x = 2sin\theta \qquad dx = 2cos\theta d\theta$

$\int \sqrt{4 - 4(sin\theta)^2}\ 2cos\theta\ d\theta$

$\int \sqrt{4(1 - (sin\theta)^2)}\ .2cos\theta\ d\theta$

$\int 2\sqrt{(cos\theta)^2}\ .2cos\theta\ d\theta$

$\int 4(cos\theta)^2 d\theta \qquad note : \qquad cos2\theta = 2(cos\theta)^2 - 1$

$(cos\theta)^2 = \frac{1}{2}(cos2\theta - 1)$

$\int 4\ .\ \frac{1}{2}(cos2\theta - 1)\ d\theta$

$2\left[\frac{sin2\theta}{2} - \theta\right] + c$

$\theta = ??\qquad sin2\theta = ??$

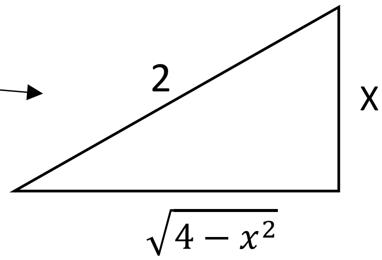
$sin2\theta - 2\theta + c$

$x = 2sin\theta$

$\theta = \sin^{-1}\frac{x}{2}$

$sin2\theta = 2sin\theta cos\theta$

$sin\theta = \frac{x}{2}$



$\rightarrow x.\frac{\sqrt{4-x^2}}{2} - 2\sin^{-1}\frac{x}{2} + c$

2) $\int \frac{dt}{t^2\sqrt{t^2-16}}$

$t = 4sec\theta$

$dt = 4sec\theta tan\theta\ d\theta$

$\int \frac{\cancel{4sec\theta}\, tan\theta\ d\theta}{16(\cancel{sec\theta})^2\,\cancel{4}\sqrt{(sec\theta)^2-1}}$

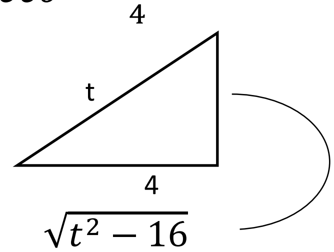
$\frac{1}{16}\int \frac{\cancel{tan\theta}\ d\theta}{sec\theta\ \cancel{tan\theta}}$

$\frac{1}{16}\int cos\theta\ d\theta \rightarrow \frac{1}{16}sin\theta + c$

$\frac{1}{16}\frac{\sqrt{t^2-16}}{t} + c$

$t = 4\ sec\theta$

$sec\theta = \frac{t}{4}$



3) $\int_0^2 \frac{dx}{\sqrt{4+x^2}}$

$x = 2tan\theta \qquad , x = 0 \rightarrow \theta = tan^{-1}(0) = 0$

$dx = 2sec^2\theta\ d\theta \qquad , x = 2 \rightarrow \theta = tan^{-1}(1) = \frac{\pi}{4}$

$\int_0^{\pi/4} \frac{2sec^2\theta}{2sec\theta}\ d\theta \qquad lnsec\theta + tan\theta \Big|_0^{\pi/4}$

$ln(\sqrt{2} + 1)$

4) $\int_0^{2/3} \sqrt{4 - 9x^2}\ dx = \int_0^{2/3} \sqrt{2^2 - (3x)^2}\ dx$

$x = 2sin\theta$

$3x = 2sin\theta$

$dx = \frac{2}{3} cos\theta\ d\theta$

$\int_0^{\pi/2} 2cos\theta \cdot \frac{2}{3}\ cos\theta\ d\theta$

$\int_0^{\pi/2} \frac{4}{3}\ cos^2\theta\ d\theta$

5) $\int_0^1 \sqrt{x - x^2}\ dx$ (هنا توجد مشكلة لذا نستخدم إكمال المربع)

$x - x^2$

$-(x^2 - x)$ نقسم معامل (س) على 2

ثم نربع

ثم نضيف ونطرح

$-\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right)$

$\frac{1}{4} - \left(x - \frac{1}{2}\right)^2$ ⟶ هنا نصبح جاهزين

$\Rightarrow \int_0^1 \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}\ dx$

$x - \frac{1}{2} = \frac{1}{2} sin\theta\ ,\ dx = \frac{1}{2} cos\theta\ d\theta$

$\int_{-\pi/2}^{\pi/2} \frac{1}{2} cos\theta \cdot \frac{1}{2} cos\theta\ d\theta$

$\int_{-\pi/2}^{\pi/2} \frac{1}{4} cos^2\theta\ \ d\theta \longrightarrow calculas1 \left(\int_{-r}^{r} even \to 2\int_0^r even\right)$

$\int_0^{\pi/2} \frac{1}{2} \cdot \frac{1}{2}\ (\cos 2\theta - 1)\ \ d\theta$

$\frac{1}{4}\int_0^{\pi/2}\ (\cos 2\theta - 1)\ \ d\theta$

$\frac{1}{4}\left(\frac{\sin 2\theta}{2} - \theta\right|_0^{\pi/2} = \frac{-\pi}{8}$

Extra problems:

1) $\int \frac{x^2}{(3+4x-4x^2)^{\frac{3}{2}}}\,dx$

→ 1) إكمال مربع

→ 2) $x - \frac{1}{2} = \sin\theta$

2) $\int x\sqrt{1-x^4}\;dx$

→ 1) $z = x^2$

→ 2) $z = \sin\theta$

3) $\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+sin^2 t}}$

→ 1) $z = sint$

→ 2) $z = \tan\theta$

7.4 *Partial fraction* ⟶ درجة المقام أكبر من درجة البسط

*قبل ما نبلش بالتكامل لازم نتمكن من موضوع ال *Decomposition*

Ex: Write down partial fraction decomposition for :

1) $\frac{2x-1}{(x^2-1)(x^2-4)}$

$= \frac{2x-1}{(x-1)(x+1)(x-2)(x+2)}$

$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$

Steps:

1) بحلل المقام

2) أجعل المقام بأبسط صورة ممكنة

3) أفصل كل مقام بكسر لوحده

2) $\frac{x^2+1}{(x^3+1)(x^3-x^2)}$

$= \frac{x^2+1}{(x+1)(x^2+x+1)(x^2)(x-1)}$

↓ حالة خاصة

$(x-0)^2$

$= \frac{A}{x+1} + \frac{Bx+c}{x^2+x+1} + \frac{D}{(x-0)^2} + \frac{E}{x-1} + \frac{F}{x}$

↓ مرة مع التربيع ومرة بدون والبسط دائما constant

المقام

- درجة أولى ⟶ A
- درجة ثانية لا يحلل ⟶ Ax+B
- درجة ثانية $(x-0)^2$ ⟶ مرة عادي A ، مرة تربيع B

3) $\frac{\boldsymbol{x^2+3}}{\boldsymbol{(x^3-1)(x^2-1)(x^3+x^2)}}$

$$\frac{x^2+3}{(x-1)(x^2-x+1)(x-1)(x+1)(x^2)(x+1)} \longrightarrow \frac{x^2+3}{(x-1)^2(x+1)^2\,(x^2)(x^2-x+1)}$$

$$\frac{A}{(x-1)}+\frac{B}{(x-1)}+\frac{C}{(x+1)}+\frac{D}{(x+1)^2}+\frac{E}{x^2}+\frac{F}{x}+\frac{(Gx+H)}{(x^2-x+1)}$$

<u>*Integration</u>

Evaluate:

1) $\int \frac{x+5}{x^2+5x+4}\,dx$

درجة المقام > درجة البسط

P.f

$$\frac{x+5}{(x+4)(x+1)}$$

$$\frac{A}{(x+4)}+\frac{B}{(x+1)}=\frac{(x+5)}{(x+4)(x+1)}$$

$$\frac{A(x+1)\;+\;B(x+4)}{(x+4)(x+1)}=\frac{x+5}{(x+4)(x+1)}$$

$$A(x+1)\;+B\;(x+4)=\;x+5$$

↓

to find the value of A/B we give Value to X that makes some terms equals zero

$x = -1$ / $x = -4$

At x=-1 → 3b=4 → b=$\frac{4}{3}$

At x=-4 → -3a=1 → a= $\frac{-1}{3}$

لو كنت فارض ال A مكان B
بطلع نفس الجواب وال 2 صح

$$\rightarrow \int \frac{\frac{-1}{3}}{x=4} + \frac{\frac{4}{3}}{x+1} dx$$

$$= \frac{-1}{3}\ ln|x+1| \ + \frac{4}{3}\ ln|x+1| + c$$

2) $\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\ dx$

$$\frac{(x-1)(x^2+1)}{(x-1)(x^2+1)} * \frac{A}{x-1} + \frac{B}{(x-1)^2} * \frac{(x^2+1)}{(x^2+1)} + \frac{Cx+D}{x^2+1} * \frac{(x-1)^2}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2(x^2+1)}$$

$$A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1) = x^2 - 2x - 1$$

x= 1 → 2B=-2 → B=-1

x= 0 → -A-1+D=-1 → A=D

x=-1 → -4A -2-4c+4D =2 → C=-1

x= 2 → 5A-5-2+D=-1 → A=1 , D=1

$$\int \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{1-x}{x^2+1} dx$$

$$\ln|x-1| - \int (x-1)^{-2} dx + \int \frac{1}{x^2+1} - \frac{x}{x^2+1} dx$$

$$In|x-1| + \frac{1}{x-1} + tan^{-1}(x) - \frac{1}{2} \ln|x^2+1| + c$$

3) $\int \frac{x^3+4x+3}{x^4+5x^2+4} dx$

$$\frac{(X^3+4x+3)}{(x^2+1)(x^2+4)} \Longrightarrow \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$\underline{(Ax+B)(x^2+4) + (Cx+D)(x^2+1)} = x^3+4x+3$$

<u>بفكهم</u>

Ax^3+ 4Ax + Bx^2+ 4B + Cx^3 + Cx + Dx^2 + D = x^3+0x^2+4x+3

بساوي المعاملات ببعض

A + C = 1 ⟶ 1

4A + C = 4 ⟶ 2

4B + D = 3 ⟶ 3

B + D = 0 ⟶ 4

1-2 ⟶ 3A = 3 ⟶ A=1 , C = 0

3-4 ⟶ 3B = 3 ⟶ B=1 , D=-1

$$\int \frac{x+1}{x^2+1} - \frac{1}{x^2+4}\,dx$$

$$\int \frac{x+1}{x^2+1}\,dx + \int \frac{-1}{x^2+4}\,dx$$

$$x = 2\ \tan\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{X}{x^2+1} + \frac{1}{x^2+1}\,dx + \int \frac{-2\,\sec^2\theta\,d\theta}{4\,\sec^2\theta}$$

$$\frac{1}{2}\,ln|x^2+1| + tan^{-1}x + \frac{-1}{2}\,\theta + c$$

$$\theta = \tan^{-1}\left(\frac{x}{2}\right)$$

*More problems:

1) $\int \frac{dx}{(x+1)^{3/2}(x+2)}$

$$Z = \sqrt{x-1}$$

$$dz = \frac{dx}{2\sqrt{x-1}}$$

$$dx = 2\sqrt{x-1}\ dz \longrightarrow \int \frac{2z\,dz}{z^{\frac{3}{2}}(z^2+1)}$$

$$\int \frac{2\,dz}{z^2(z^2+1)} \longrightarrow \text{P.f}$$

$$\frac{A}{z} + \frac{B}{z^2} + \frac{Cx+D}{z^2+1}$$

2) $\int \frac{dx}{e^x+1}$

$Z = e^x$

$dz = e^x dx$

$$\int \frac{dz}{z(z+1)} \longrightarrow p.f$$

3) $\int \frac{e^{2x}dx}{e^{2x}+3e^x+2}$

$Z = e^x$

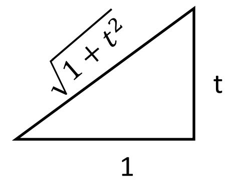
$dz = e^x dx$

$$\int \frac{zdz}{(z+1)(z+2)} \longrightarrow P.f$$

*Half angle substitution :

*الفكرة العامة احول تكامل فيه ($cosx/sinx$) ما بعرف اكامله لتكامل بنحل P.f

Method :



$$t = tan(\frac{x}{2})$$

$$sin(x) = 2sin(\frac{x}{2})\, cos(\frac{x}{2}) = 2\,\frac{t}{\sqrt{t^2+1}} \cdot \frac{1}{\sqrt{t^2+1}} = \frac{2t}{t^2+1}$$

$$cos(x) = cos^2(\frac{x}{2}) - sin^2(\frac{x}{2})$$

$$= \frac{1-t}{t^2+1}$$

So

$Sinx = \frac{2t}{t^2+1}$ $\qquad$ $cosx = \frac{1-t}{t^2+1}$

Examples :

Evaluate:

1) $\int \frac{dx}{1+sinx-cosx}$ ⟶ There is no method to solve this integration so half angle sub

$$t = tan(\frac{x}{2})$$

$\tan^{-1} t = \frac{x}{2}$ $\qquad$ $sinx = \frac{2t}{(t^2+1)}$

$x = 2\ tan^{-1} t \qquad cosx = \frac{1-t^2}{t^2+1}$

$dx = \frac{2dt}{1+t^2}$

$\rightarrow \int \frac{\frac{2dt}{(1+t^2)}}{(t^2+1)\frac{1}{t^2+1}+\frac{2t}{t^2+1}+\frac{t^2-1}{t^2+1}}$

$\rightarrow \int \frac{dt}{t(t+1)} \longrightarrow P.f$

$\frac{A}{t} + \frac{B}{t+1} = \frac{1}{t(t+1)}$

$A(t+1) + Bt = 1$

$t = 0 \longrightarrow A = 1$

$t = -1 \longrightarrow B = -1$

$\int (\frac{1}{t} + \frac{-1}{t+1})dt$

$ln|t| - ln|t+1| + c$

$ln\,(\frac{t}{t+1}) + c \qquad t = tan(\frac{x}{2})$

$ln\,(\frac{tan\left|\frac{x}{2}\right|}{tan\left|\frac{x}{2}\right|+1}) + c$

2) $\int \frac{dx}{3sinx-4cosx}$

$t = tan^{-1}(\frac{x}{2})$

$x = 2\,tan^{-1}\,t$

$dx = \frac{2dt}{t^2+1}$

$\int \frac{2dt}{(t^2+1)[\frac{6t}{t^2+1}+\frac{4t^2-4}{t^2+1}]}$

$\int \frac{2dt}{4t^2+6t-4}$

$\int \frac{dt}{2t^2+3t-2} \rightarrow \int \frac{dt}{(2t-1)(t+2)} \longrightarrow p.f$

7.8 Improper Integrals

هو مجرد تكامل عادي ولكن يوجد مشاكل في حدود التكامل حيث يمكن ان يولد مشكلة في اصفار المقام او المالانهاية فنستبدل القيمة ب const ثم ندخل lim

Ex1:

$\int_a^\infty f(x)dx$

$x = \infty$ لا يوجد قيمة لل

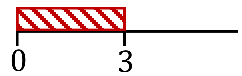
$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

Ex 2: $\int_0^3 \frac{dx}{x-3}$ at $x = 3$ صفر مقام

$x - 3 = 0$

$x = 3$

$\lim_{a \to 3^-} \int_0^a \frac{dx}{x-3}$



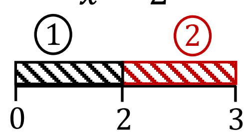
So $\lim_{a \to 3^+}$ or

$\lim_{a \to 3^-}$??

Ex 3: $\int_0^3 \frac{dx}{x-2}$ at $x = 2$ صفر مقام

$x - 2 = 0$

$x = 2$



$\Rightarrow \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2}$

$$\lim_{a \to 2^-} \int_0^a \frac{dx}{x-2} + \lim_{b \to 2^+} \int_b^3 \frac{dx}{x-2}$$

- Evaluate:

1) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\lim_{a\to 1^-} \int_0^a \frac{dx}{\sqrt{1-x^2}}$$

$$\lim_{a\to 1^-} \sin^{-1} x \Big|_0^a$$

$$\lim_{a\to 1^-} \sin^{-1} a - \sin^{-1} 0$$

$$\boxed{\frac{\pi}{2}}$$

2) $\int_0^1 r ln r\, dr$

$$\lim_{a\to 0^+} \int_a^1 r ln r\, dr$$

$$u = lnr \qquad dv = rdr$$

$$du = \frac{dr}{2} \qquad v = \frac{r^2}{2}$$

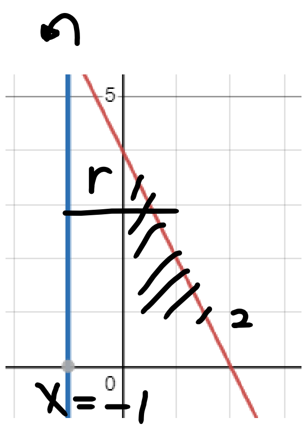
$$\lim_{a\to 0^+} \frac{r^2}{2} lnr \Big|_a^1 - \lim_{a\to 0^+} \int_a^1 \frac{r}{2} dr$$

$$\lim_{a\to 1^-} \left(\frac{-a^2}{2} lna - \frac{1}{4} + \frac{a^2}{4}\right)$$

$$\longrightarrow \quad \frac{r^2}{2} \cdot \frac{1}{2}$$

* Find the volume of the region bounded by:

y = 4-2X , X = 0, y =0 about X = -1 using Shells



$r = X - (-1) = X + 1$

$h = 4 - 2x$

$V = 2\pi \int \mathbf{20} \, (x+1)\,(4-2x)\,dx$

*Evaluate:

1) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\lim_{a\to 1^-} \int_0^a \frac{dx}{\sqrt{1-x^2}}$$

$$\lim_{a\to 1^-} \sin^{-1} x \,]_0^a$$

$$\lim_{a\to 1^-} \sin^{-1} a - \sin^{-1} 0 \;\rightarrow\; \frac{\pi}{2}$$

2) $\int_0^1 r\, lnr\, dr$

$$u = \ln x \qquad dv = r\, dr$$

$$du = \frac{dr}{r} \qquad v = \frac{r^2}{2}$$

$$\lim_{a\to 0^+} \frac{r^2}{2} \ln r \,]_a^1 - \lim_{a\to 0^+} \int_a^1 \frac{r^2}{2}\, dr$$

$$\lim_{a\to 0^+} \left(\frac{-9^2}{2}\, lna - \frac{1}{4} + \frac{9^2}{4}\right)$$

$$\lim_{a\to 0^+} \left(\frac{9^2}{4} - \frac{9^2\, lna}{2}\right) - \frac{1}{4}$$

$$-\frac{1}{2} \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a^2}} = \frac{\infty}{\infty}\ \text{L.R} \qquad -\frac{1}{4}$$

$$-\frac{1}{2} \lim_{a\to 0^+} \frac{\frac{1}{a}}{\frac{-2a}{a^4}} - \frac{1}{4}$$

$$-\frac{1}{2} \lim_{a\to 0^+} \frac{a^2}{-2} - \frac{1}{4}$$

$$= -\frac{1}{4}$$

3) $\int_0^{\infty} \frac{dz}{z^2+3z+2}$

$\lim_{a\to\infty} \int_0^a \frac{dz}{(z+2)(z+1)}$ ⟶ $\frac{A}{z+1} + \frac{B}{z+2}$

$\lim_{a\to\infty} \left[\int_0^a \frac{1}{(z+1)} dz + \int_0^a \frac{-1}{(z+2)} dz\right]$ ⟶ $A = 1, B = -1$

$\lim_{a\to\infty} (\ln|z+1|]_0^a - \ln|z+2|]_0^a)$

$\lim_{a\to\infty} \ln(a+1) - \ln(a+2) + \ln(2)$

$\lim_{a\to\infty} \ln\frac{a+1}{a+2} \quad +\ln(2)$

$0 + \ln(2) = \ln 2$

Try it

4) $\int_0^4 \frac{dx}{x^2-x-2}$ ⟶ $\int_0^2 \ldots \ldots.$, ⟶ $\int_2^4 \ldots \ldots.$

Try it

5) $\int_{-\infty}^{\infty} xe^{-x^2} dx$ ➔ ans. Zero

Extra Note:

$\int_0^1 \frac{1}{x^p} dx$ ⟶ find the value of P for which the integral converge

*converge: له جواب محدد وليس ∞\-∞

For P=1:

$\lim_{a\to 0^+} \int_a^1 \frac{1}{x} dx \to \ln|x|]_{0^+}^1 \to \ln(1) - \ln(0^+) \to \infty \to$ divergent

For P≠1:

$\lim_{a\to 0^+} \int_a^1 \frac{1}{x^p} dx \Rightarrow \lim_{a\to 0^+} \int_a^1 x^{-p} dx \Rightarrow \lim_{a\to 0^+} \frac{x^{-p+1}}{-p+1}]_a^1$

$\Rightarrow \frac{1}{1-p} \lim_{a\to 0^+} 1 - a^{1-p}$

P>1 ⟶ div.

P<1 ⟶ conv.

*رح توضح أكثر بالوحدة الثالثة

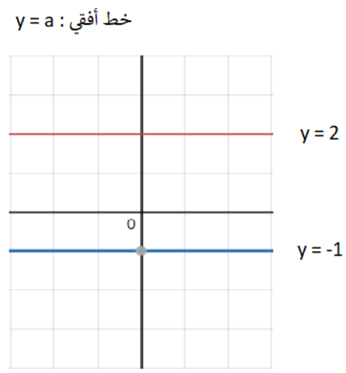
CHAPTER 6 :

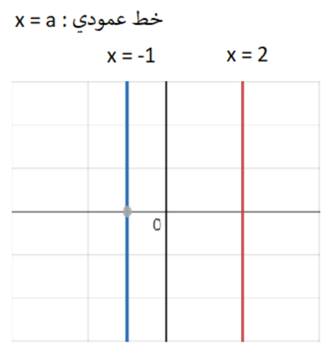
AREAS AND VOLUMES

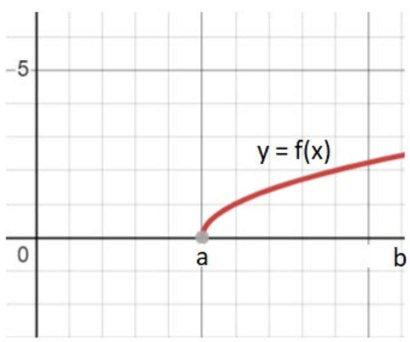
Chapter "6" Areas and volumes

6.1 : Area

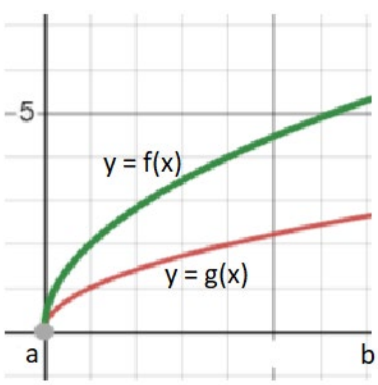
Area ➔ integration



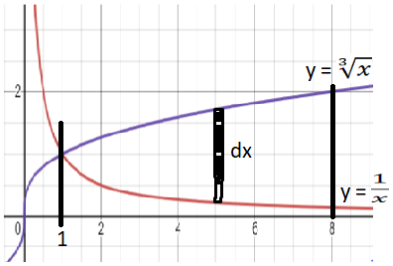




$$\text{Area} = \int_a^b f(x)\, dx$$

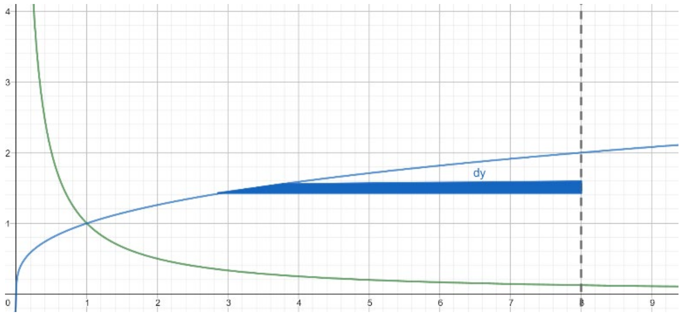


$$A = \int_a^b [\underbrace{f(x)}_{\text{العلوي}} - \underbrace{g(x)}_{\text{السفلي}}]\, dx$$



$$A = \int_1^8 \left(\sqrt[3]{x} - \frac{1}{x}\right) dx$$

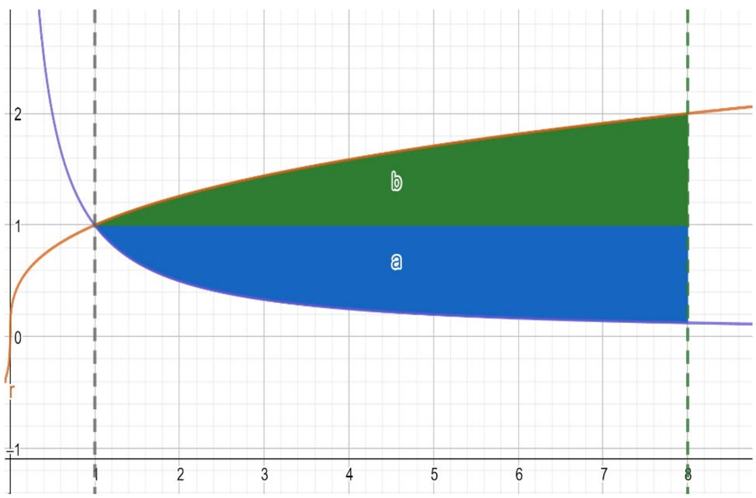
What if we took cross section of dy instead of dx ???



Length of dy can be calculated by

$8 - \sqrt[3]{x}$ $\qquad$ $8 - \frac{1}{x}$

2 Lengths → 2 parts of areas



We find the intersection between $x = 8\ and\ y = \frac{1}{x}$:

$$\boldsymbol{y = \frac{1}{8}}$$

$$Area = A_1 + A_2$$

$$Area = \int_{y=\frac{1}{8}}^{y=1} \left(8 - \frac{1}{x}\right) dy + \int_{y=1}^{y=2} (8 - \sqrt[3]{x})\, dy$$

هناك بعض التكاملات قد تحل ب
dy أو dx
ولكن مع الخبرة ستستخدم الطريقة الأقل حلًا

لما نكامل بالنسبة لdy لازم كل اشي يكون بدلالة y.

$$y = \frac{1}{x} \rightarrow x = \frac{1}{y}$$

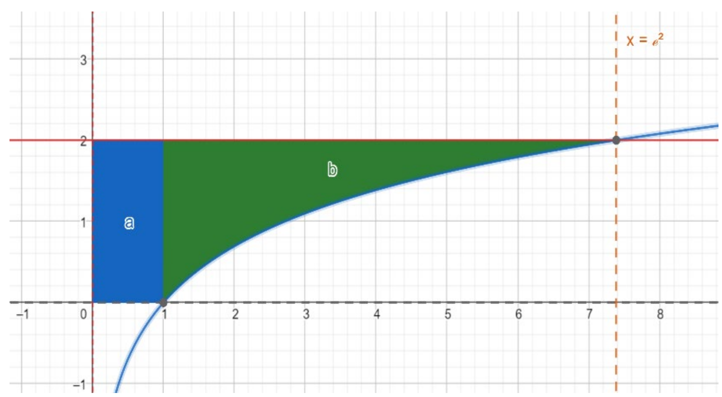
$$y = \sqrt[3]{x} \rightarrow x = y^3$$

$$Area = \int_{y=\frac{1}{8}}^{y=1} \left(8 - \frac{1}{y}\right) dy + \int_{y=1}^{y=2} (8 - y^3)\, dy$$

Find the area of the region bounded by :

$y = \ln(x)$, $y = 0$, $y = 2$, $x = 0$

Using dx :



نوجد نقاط التقاطع :

Point 1 : $\ln x = 0 \rightarrow x = 1$

Point 2 : $\ln x = e^x \rightarrow x = e^2$

$$A = a + b$$

$$A = \int_0^1 (2 - 0)\, dx + \int_1^{e^2} (2 - \ln(x))\, dx$$

$$A = 2 + 2(e^2 - 1) - \int_1^{e^2} \ln x\, dx$$

$$A = 2e^2 - x * \ln x \Big|_1^{e^2} + \int_1^{e^2} dx$$

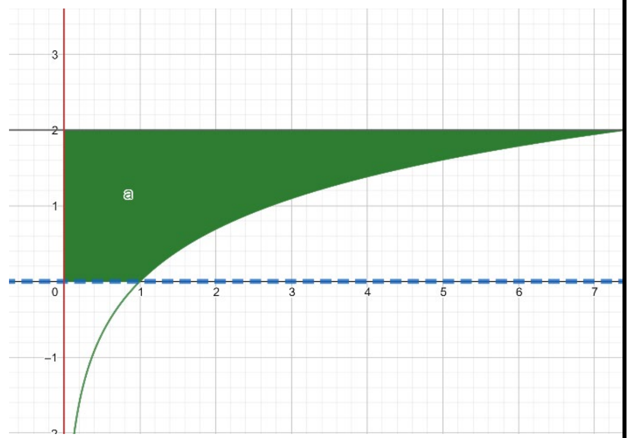
$$u = \ln x \qquad du = \frac{dx}{x}$$
$$dv = dx \qquad v = x$$

$$A = 2e^2 - 2e^2 + (e^2 - 1) = e^2 - 1$$

Using dy :

$$A = \int_{y=0}^{y=2} (e^y - 0)\ dy$$

$$A = e^y \Big|_0^2 = e^2 - e^0 = e^2 - 1$$



Both provides the same answer but dy is easier.

Find the area of the shaded region :

$$m = \frac{\Delta y}{\Delta x} \rightarrow \ m = -1$$

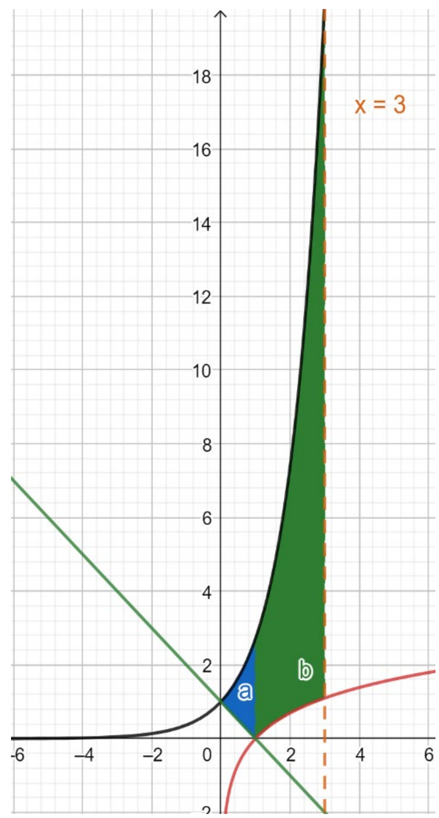
$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - 1)$$

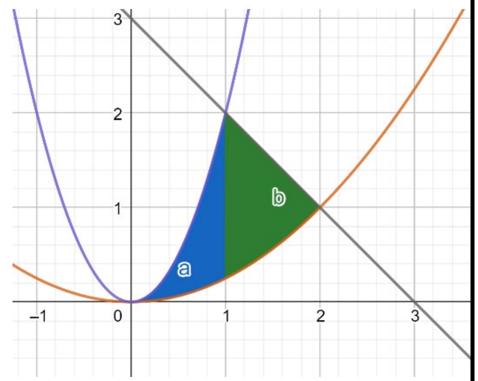
$$y = 1 - x$$

$$A = a + b$$

$$A = \int_0^1 [e^x - (1 - x)]\ dx + \int_1^3 (e^x - \ln(x))\, dx$$



Find the area between $y = \frac{1}{4}x^2, \quad y = 2x^2, \quad y = 3 - x$:



نجد نقاط التقاطع :

Point 1 : $3 - x = 2x^2 \rightarrow x = 1 \ \ or \ \ x = -\frac{3}{2}$

Point 2 : $3 - x = \frac{1}{4}x^2 \rightarrow x = 2 \ \ or \ \ x = -6$

$x = -\frac{3}{2}$, $x = -6$ → Rejected

$A = a + b$

$$A = \int_0^1 \left(2x^2 - \frac{1}{4}x^2\right) dx + \int_1^2 [(3 - x) - \left(\frac{1}{4}x^2\right)] \, dx$$

Find the area enclosed by $y = \sin^{-1} x, \quad y = \frac{\pi}{2}, \quad x = 0$:

نجد نقاط التقاطع :

Point 1 : $\sin^{-1} x = \frac{\pi}{2} \rightarrow x = 1$

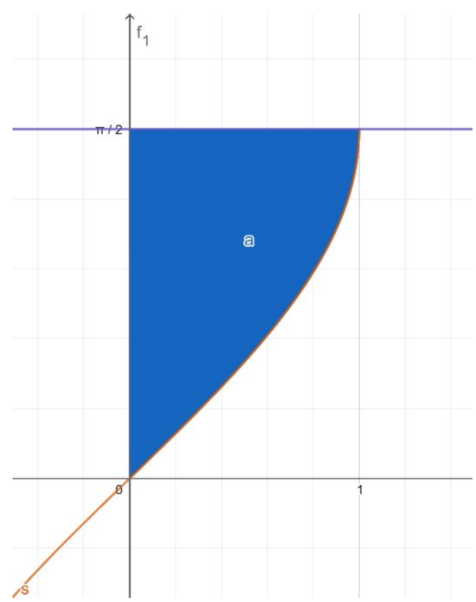
using dx :

$Area = \int_0^1 (\frac{\pi}{2} - \sin^{-1} x) \, dx \rightarrow$ تكاملها صعب

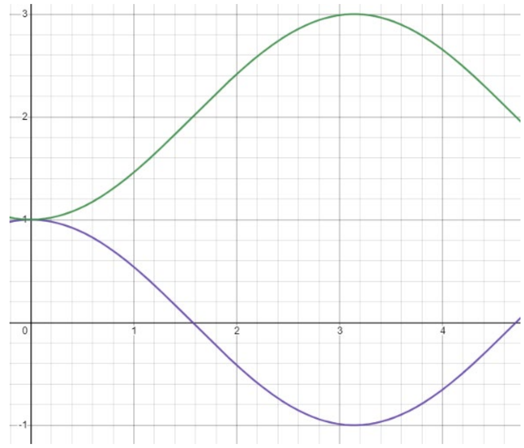
using dy :

$Area = \int_{y=0}^{y=\frac{\pi}{2}} \sin y \ dy \rightarrow \quad y = \sin^{-1} x$

$x = \sin y$



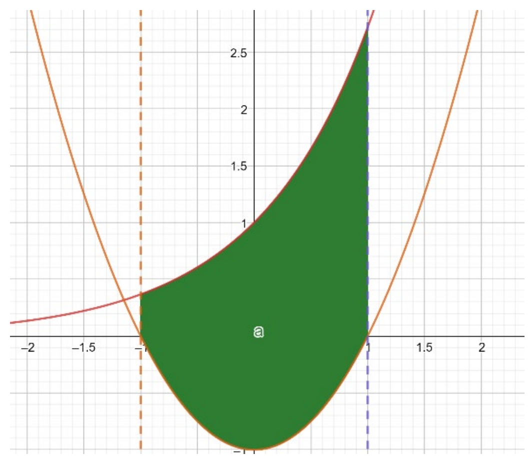
Find the area enclosed $y = \cos x$, $y = 2 - \cos x$, $0 \le x \le 2\pi$



$$Area = \int_0^{2\pi} (2 - \cos x - \cos x)\ dx$$

$$Area = \int_0^{2\pi} (2 - 2\cos x)\ dx$$

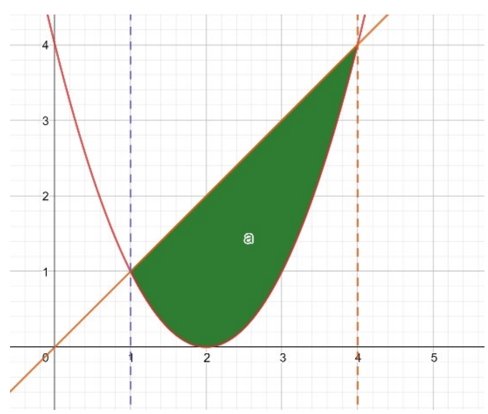
Find the area of the region bounded by :



1) $y = e^x,\ \ y = x^2 - 1,\ \ x = -1,\ \ x = 1$

$$Area = \int_{-1}^{1} [e^x - (x^2 - 1)]\ dx$$

2) $y = (x-2)^2, \quad y = x$



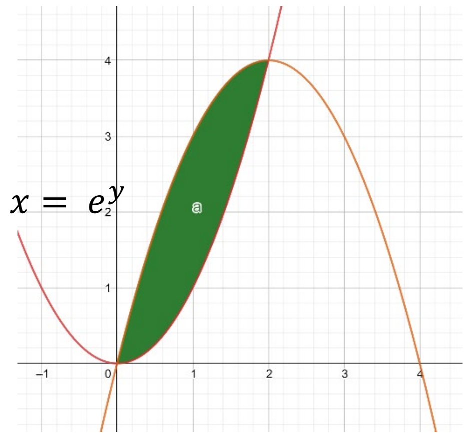
نجد نقاط التقاطع :

$(x-2)^2 = x \rightarrow \quad x = 1\ and\ x = 4$

$$Area = \int_1^4 [x - (x-2)^2]\ dx$$

3) $y = 4x - x^2, \quad y = x^2$

نجد نقاط التقاطع :



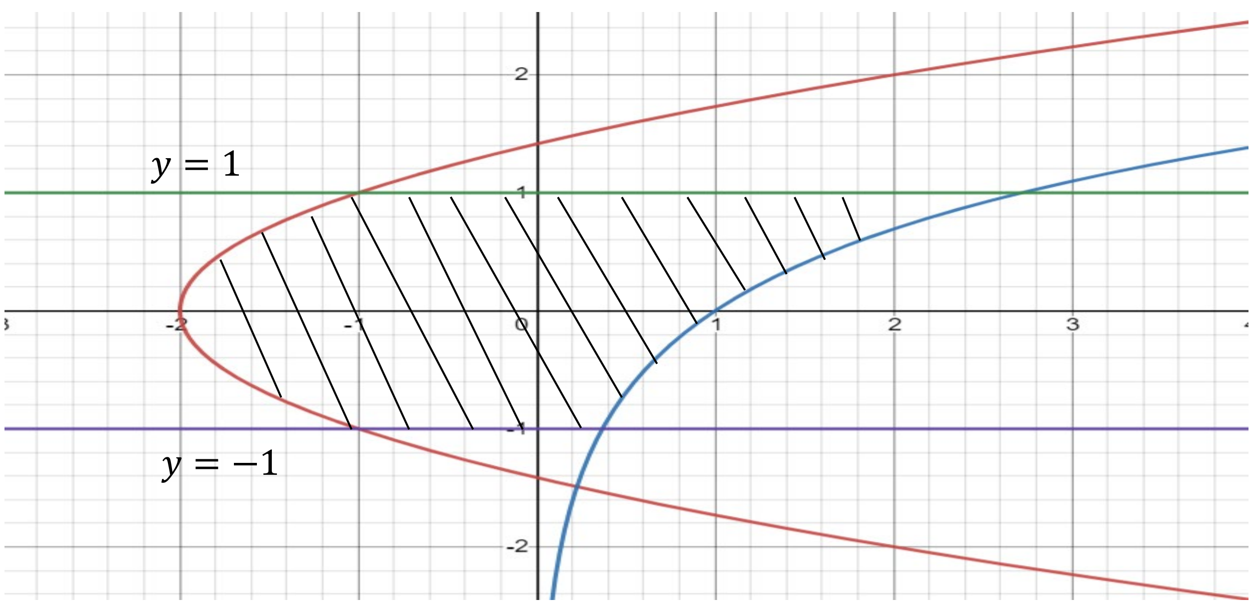
$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0 \rightarrow x = 0 \text{ and } x = 2$$

$$Area = \int_0^2 [4x - x^2 - x^2] \, dx$$

4) $x = y^2 - 2$



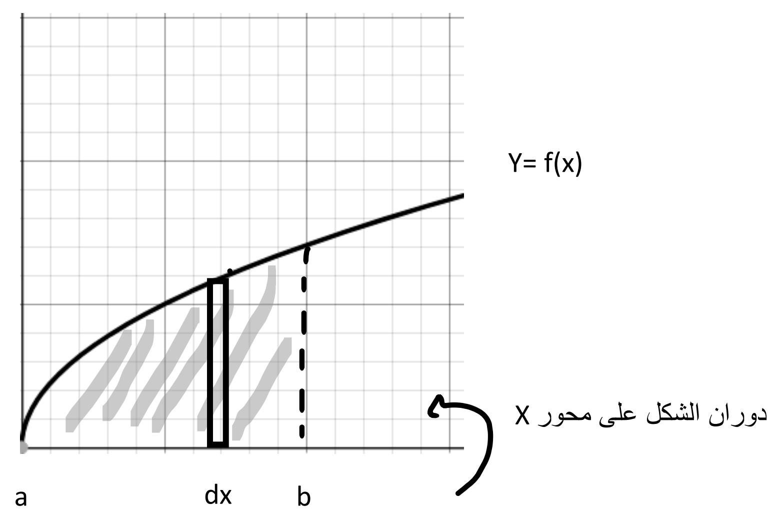
$$A = \int_{-1}^{1} [e^y - (y^2 - 2)] dy$$

$6.2\ \textit{Volume}$:

Volume (with revolution)

Disk	Washer	Cylindrical shells
(مقطع عامودي على محور الدوران)	(مقطع عامودي على محور الدوران)	(مقطع موازي لمحور الدوران)

1) *Disk*

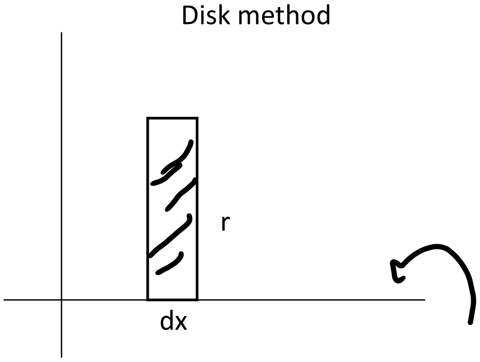


*Note:

$$* V = \int A\, x\, dx$$

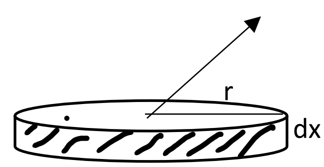
$$* V = \int A\, y\, dy$$

نلاحظ أن المساحة مغلقة لذلك نستخدم
Disk method



Cylinder

Volume(cylinder) = $\pi r^2 l$
$= \pi r^2 dx$



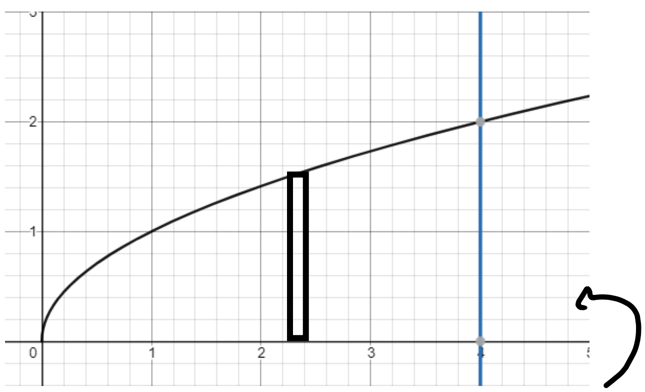
$$V = \pi \int_a^b r^2 \; dx$$

$$V = \pi \int_a^b f^2(x) \; dx$$

$Examples:$

$*$ $Find\ the\ volume\ of\ the\ solid\ generated\ by\ revolving\ the\ region\ enclosed\ by:$
$y = \sqrt{x}\,, x = 4\,, y = 0\ \ about:$

1) $X - axis$



About X-axis

*نأخذ مقطع عامودي

Disk نلاحظ أنه *

dx

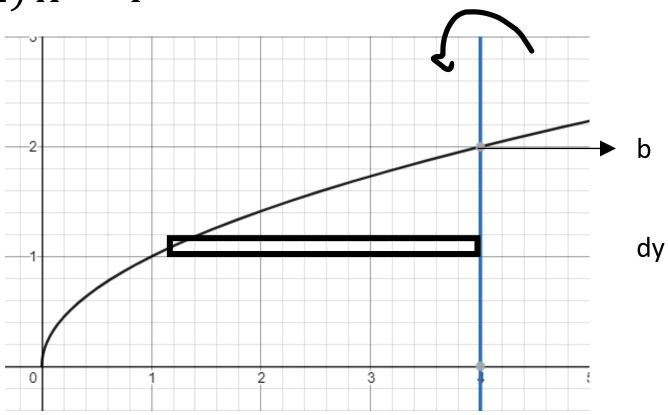
$$V = \pi \int_a^b r^2 \; dx$$

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 \; dx$$

$$= \pi \int_0^4 x \; dx$$

$$= \pi \,.\, \frac{x^2}{2}\Big|_0^4 = \frac{\pi}{2} \,.\, 16 = 8\pi$$

2) $X = 4$



نأخذ مقطع
*عامودي

نلاحظ أنه Disk

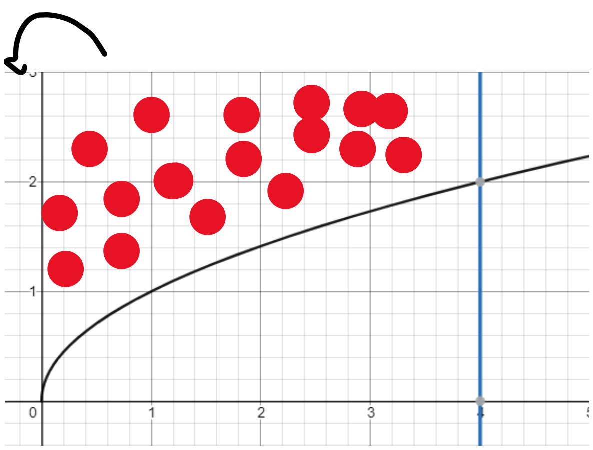
b: x=4, y=$\sqrt{x}$

y=2

$$V = \pi \int_a^b r^2 \ dy$$

$$V = \pi \int_0^2 (4 - y^2)^2 \ dy$$

3) $Y - axis$



نظرًا للفراغات الموجودة
(النقاط) فإن هذه الطريقة ليست

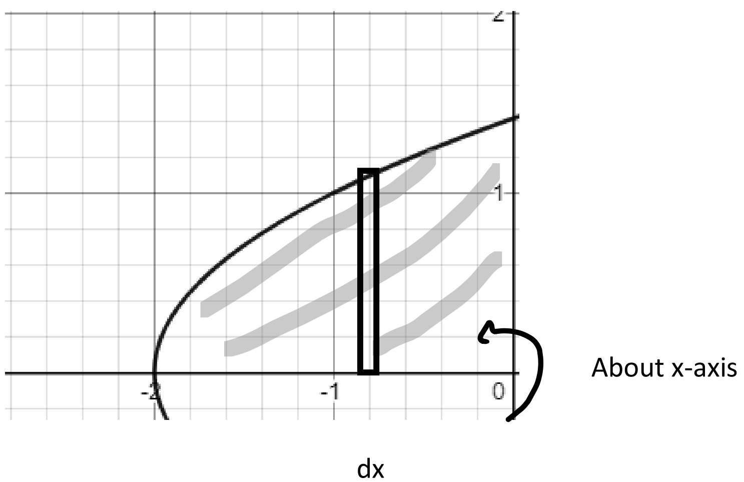
Disk

بل طريقة

Washer

وسنأخذها لاحقًا

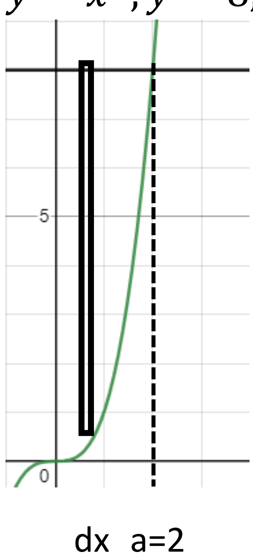
$* \, Find \; the \; volume :$



$$V = \pi \int_{-2}^{0} \left(\sqrt{x+2}\right)^2 \, dx$$

$$= \pi \int_{-2}^{0} (x+2) \, dx$$

$* \, Find \; the \; volume \; of \; the \; solid \; that \; is \; obtained \; from$:
$y = x^3, y = 8, x = 0 \;\; about \; y = 8$:



لا يوجد فراغات بين
المساحة ومحور الدوران

⟶ Disk

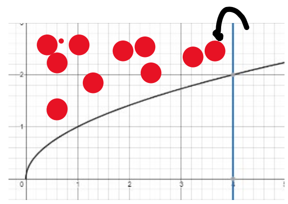
$$V = \pi \int_{0}^{2} (8 - x^3)^2 \, dx$$

.

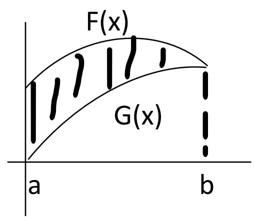
.

$$\boxed{\frac{576\,\pi}{7}}$$

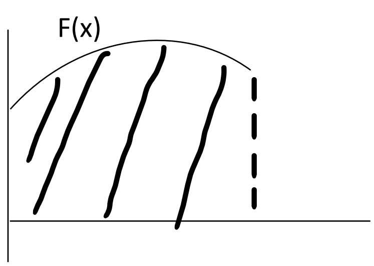
2) $Washer$ → يوجد فراغ بين المساحة
المحصورة ومحور الدوران



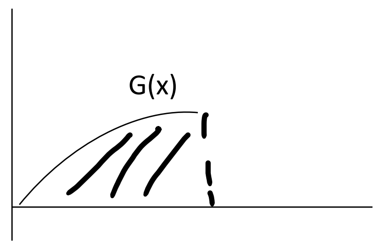
About y -axis



=



-



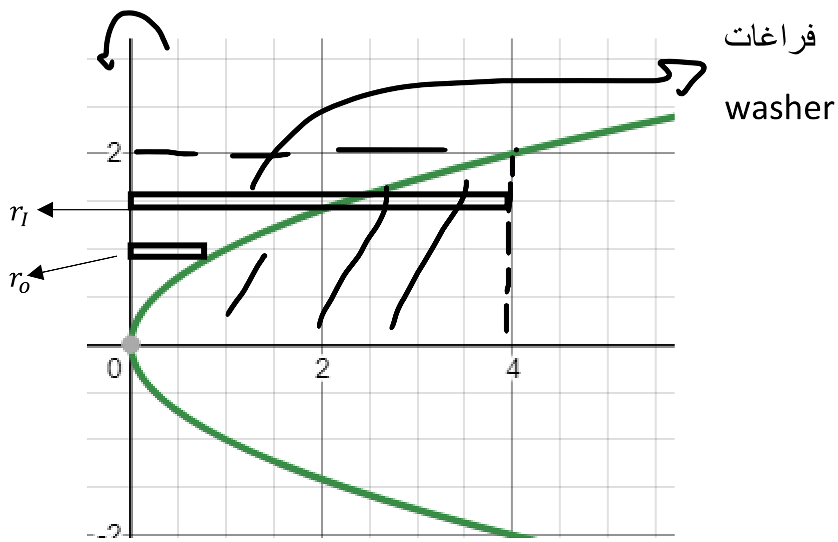
Disk - Disk

$$V = \pi \int_a^b ({r_0}^2 - {r_I}^2)\ dx$$

$$V = \pi \int_a^b (f^2(x) - g^2(x))\ dx$$

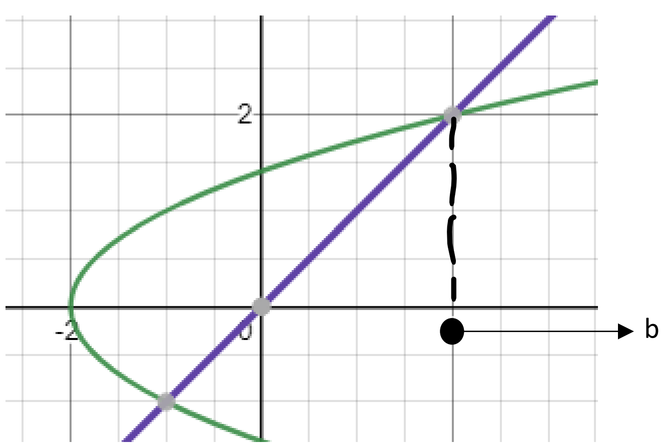
r_o: $outer\ r$
r_I: $Inner\ r$

$Example : find\ volume\ about\ y - axis$



$$V = \pi \int_0^2 (4^2 - (y^2 - 0^2)^2)\ dy$$

$* find\ the\ volume :$



b : y=y

$x = x^2 - 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1$ ✗

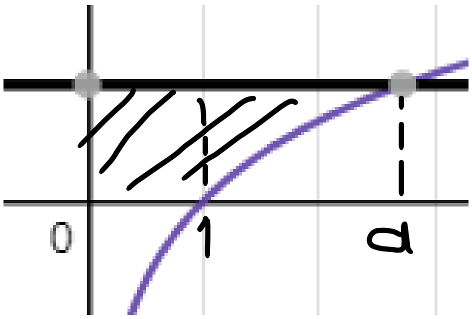
$x = 2$ ✓

$r_0 = \sqrt{x+2}$

$r_I = x$

$$V = V_{1(disk)} + V_{2(washer)}$$
$$= \pi \int_{-2}^{0} \sqrt{x+2}^{\,2}\ dx + \pi \int_0^2 \left[\left(\sqrt{x+2}\right)^2 - x^2\right] dx$$

$* \ find\ the\ volume\ about : 1)\ x-axis\ \ 2) y-axis$



$a: \ln x = 1$
$x = e$

$y = \ln x$

1) $X-axis\ (Washer + disk)$

$$V = \pi \int_0^1 1^2\ dx + \pi \int_1^e [(1)^2 - (\ln x)^2]\, dx$$

$y = \ln x$
$y = e^x$

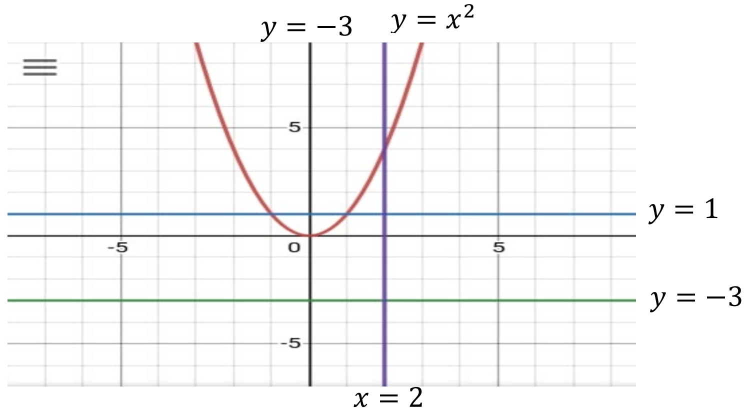
2) $y-axis\ (disk)$

$$V = \pi \int_0^1 (e^y)^2\ dy$$

مهم *Find the volume:

Region: $y=x^2$, $y=1$, $x=2$

About $y= -3$



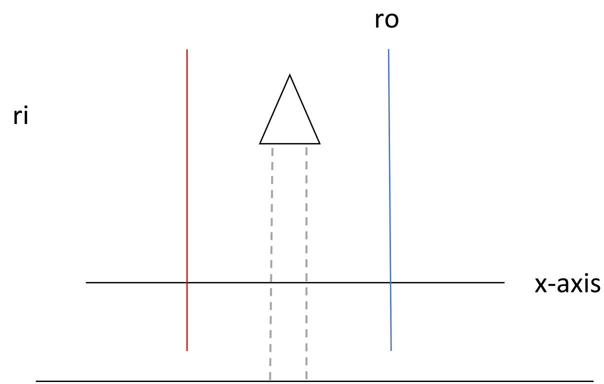
Washer :

$R0= x^2 -(-3)$

$R0=x^2+3$

$Ri=1-(-3)$

$Ri=4$



$V=\pi\int_1^2 [(x^2+3)^2-(4)^2]dx$

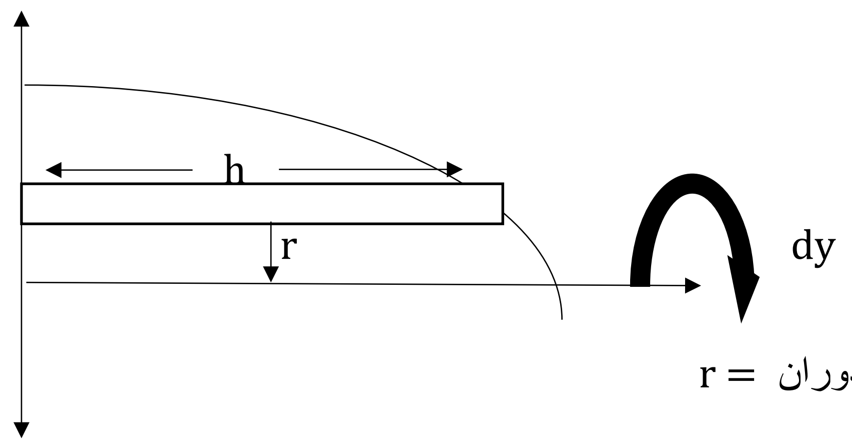
A→ y=y

$x^2=1$

x=1 yes

x=-1 no

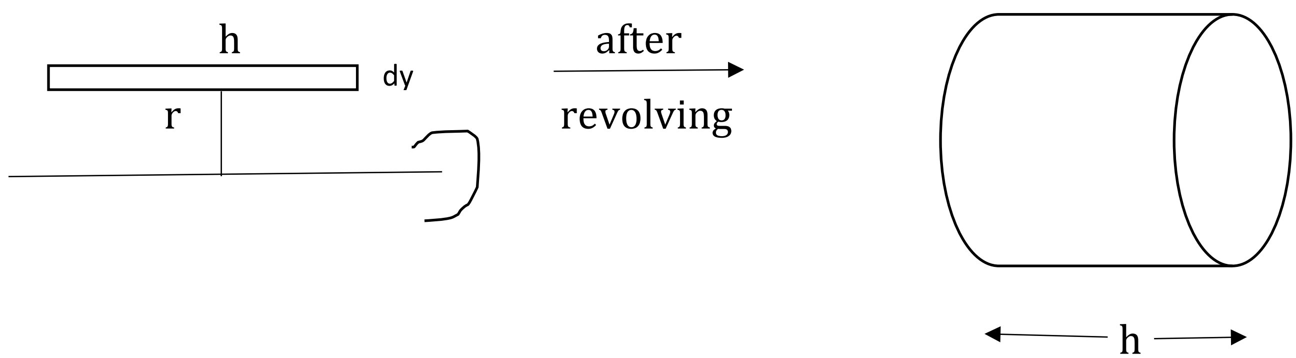
6.3 Cylindrical Shells (مقطع موازي لمحورالدوران)

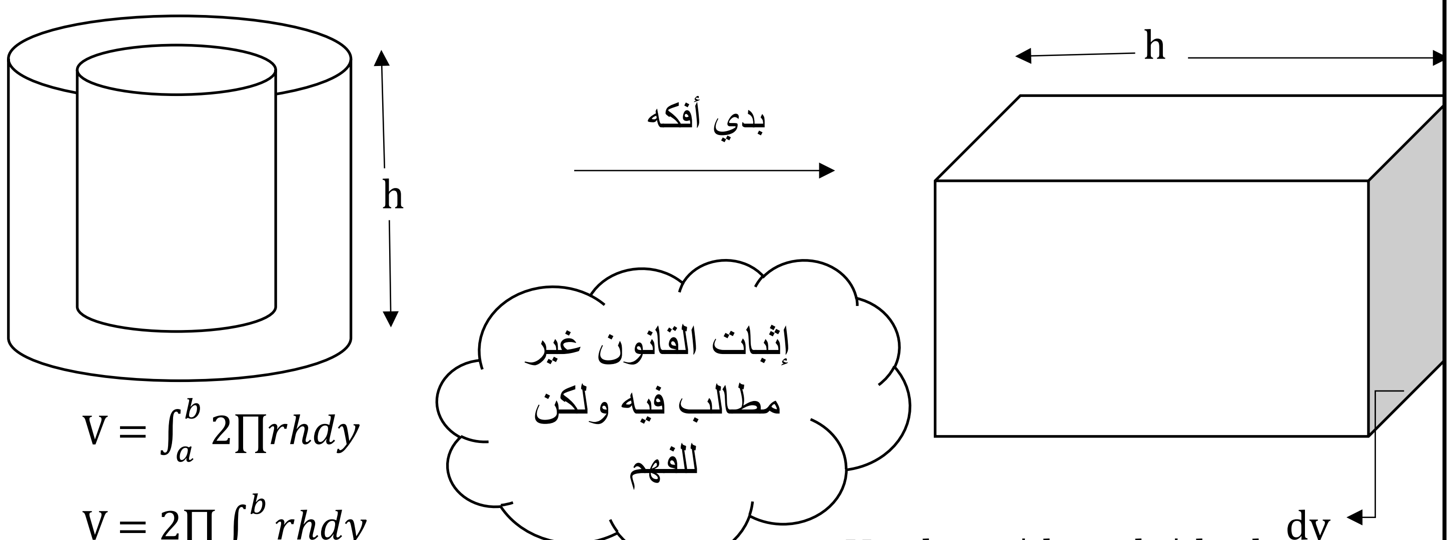


نأخذ مقطع موازي

r = المسافة بين المقطع ومحور الدوران

h = طول المقطع





إثبات القانون غير مطالب فيه ولكن للفهم

$V = \int_a^b 2\prod rhdy$

$V = 2\prod \int_a^b rhdy$

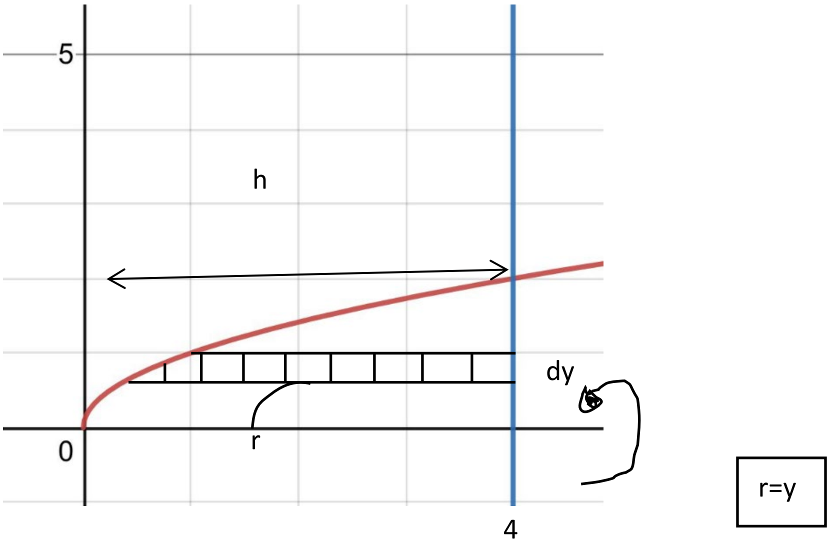
V = base * length * hight

لازم أحدد المقطع (الموازي)

اعرف قيمة h/r

Ex: find the volume (using shells)

1)

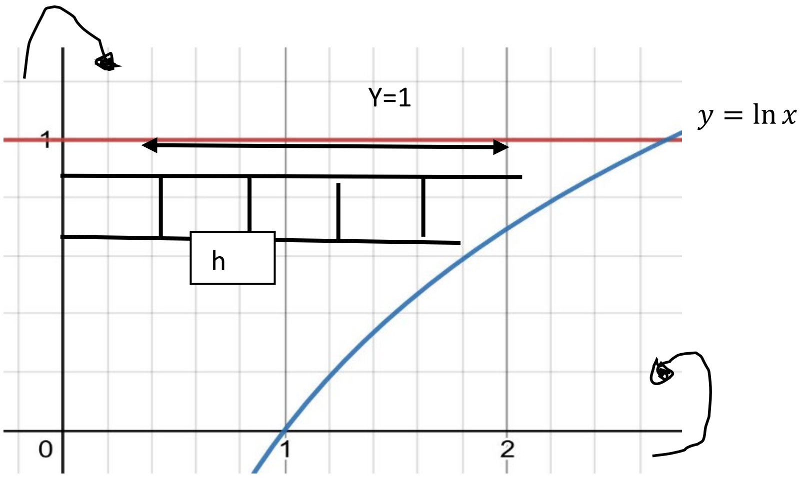


$V= 2\pi \int_{y=9}^{y=6} rh\, dhy$

$= 2\pi \int_{0}^{24} y\,(4- \sqrt{x})dy$

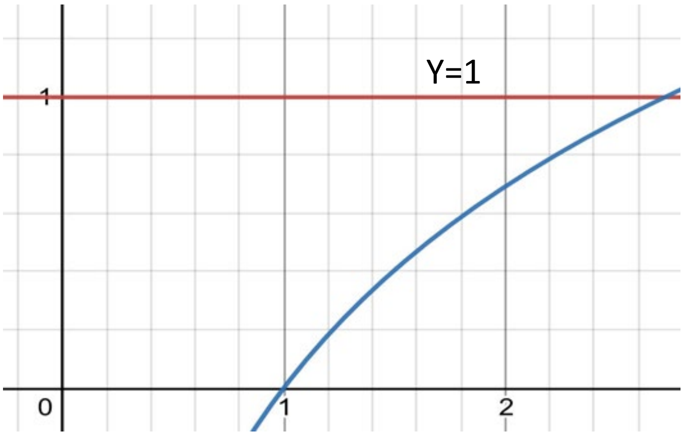
$=2\pi \int_{0}^{4} y\,(4-y^2)dy$

2)



$V= 2\pi \int_{0}^{1} y\,(e^{y})\, dy$

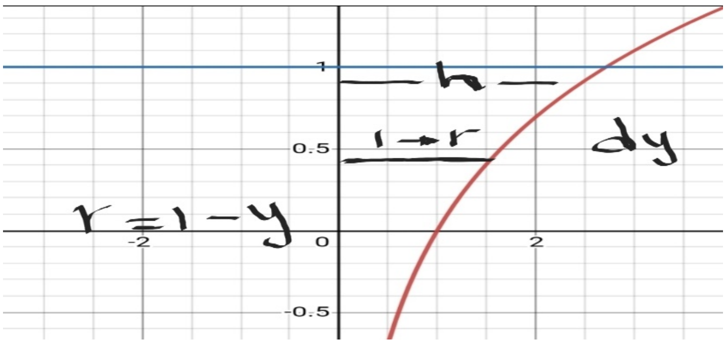
3)



$y = \ln x$

$$About\ y = 1$$

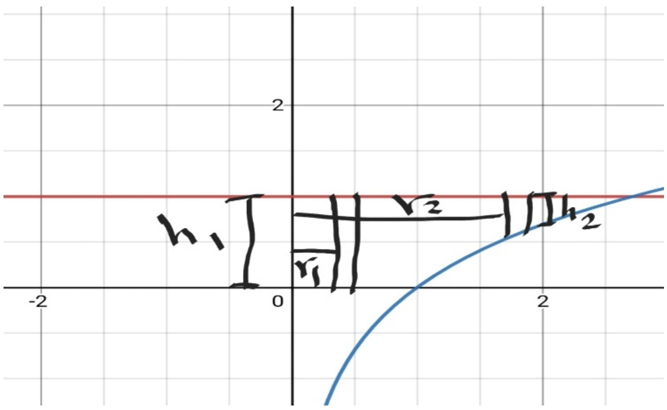
Find the volume using shells



$r = 1 - y$

$$v = 2\pi \int_0^1 (1 - y) e^y \, dy$$

1)



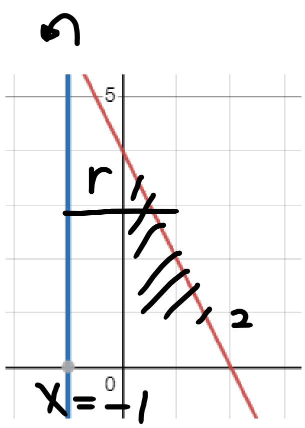
Here (h) maybe

2 lengths

$$V = 2\pi \int_0^1 x\,(1)dx + 2\pi \int_1^e x\,(1 - \ln x)\,dx$$

* find the volume of the region bounded by:

y = 4-2X , X = 0, y =0 about X = -1 using Shells



$r = X - (-1) = X + 1$

$h = 4 - 2x$

$V = 2\pi \int_{0}^{2} (x + 1)(4 - 2x)\, dx$

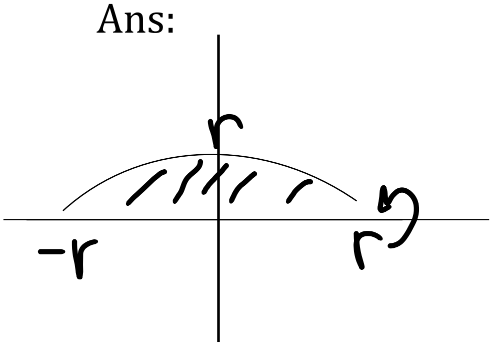
* Example

أسئلة توضيحية وتفهيمية
لموضوع ال (Volume by rotation)
بإمكانك عمل Skip لهم ☺

* Show that the volume of sphere (كرة) is given by:

V =3/4 π r3

Ans:



half circle
$y = \sqrt{r^2 - x^2}$
rotate it around x-axis the
Sphere is formed.

disk :

$$V = \pi \int_{-r}^{r} (\sqrt{\boldsymbol{r^2 - x^2}})^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx \longrightarrow \int_{-r}^{r} even = 2 \int_0^r f(x)$$

❖ $= 2\pi \int_0^r (r^2 - x^2) \, dx \qquad 2\pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_0^r$

$$= 2\pi \left(\frac{3r^3}{3} - \frac{r^3}{3} \right) = 2\pi \left(\frac{2r^3}{3} \right) = \frac{4}{3} \pi r^3$$

Volume (without rotation) → Slicing method

V = $\int A$

Cross - section perpendicular to x - axis

→ V = $\int_{x_1}^{x_2} A(x)dx$

Cross - section perpendicular to y - axis

→ V = $\int_{y_1}^{y_2} A(y)dy$

A(x) , A(y)

- **Square**
  A = x^2
- **circle**
  A = πr^2
- **Triangle**
  - مثلث قائم الزاوية
    A = $\frac{1}{2}$A . B
  - أي مثلث
    A = $\frac{1}{2}$A . B . sin θ
    → θ : Angle between A & B

➔ Note

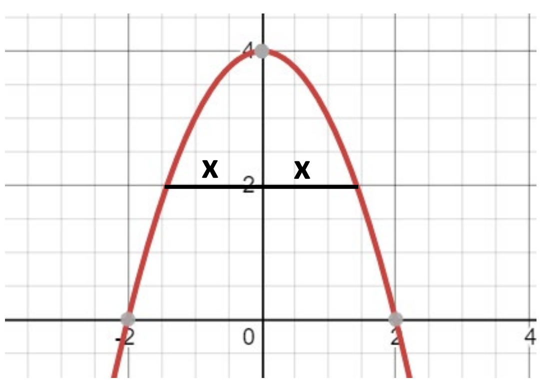
(1) Equilateral triangle → مثلث متساوي الأضلاع
(2) Right angle triangle → مثلث قائم الزاوية
(3) Isosceles triangle → مثلث متساوي الساقين

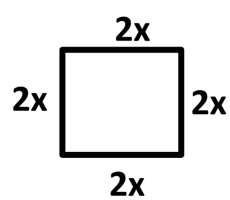
❖ **Steps :**

1- نرسم الاقتران
2- برسم *cross section* داخل المنطقة المظللة
3- نجد طول *cross section*
4- نجد مساحة *cross section*

Examples:

- **Find the volume of the solid whose base is $y = 4 - x^2$ and x-axis and the cross section is perpendicular to y-axis square.**





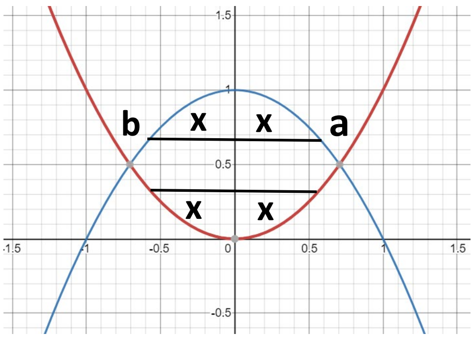
$$A = (2x)^2 = 4x^2$$

$$A = 4(4 - y)$$

$$V = \int_0^4 A(y)\,dy$$

$$\boxed{V = \int_0^4 4(4 - y)\,dy}$$

- Find the volume of $y = x^2$, $y = 1 - x^2$ solid whose base perpendicular to y-axis and cross section is square.



2 diff. lengths → بجزئ

$A = (2x)^2 = 4x^2$

$y = y$

$x^2 = 1 - x^2$

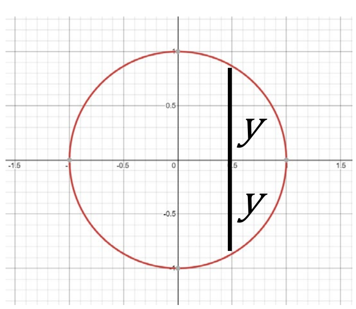
$2x^2 = 1$

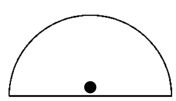
$x^2 = \frac{1}{2} \rightarrow x = \frac{-1}{\sqrt{2}} \rightarrow b \quad // \quad x = \frac{1}{\sqrt{2}} \rightarrow a$

$V = \int_{y=0}^{y=\frac{1}{2}} 4x^2 \, dy + \int_{y=\frac{1}{2}}^{y=1} 4x^2 \, dy$

$y \qquad\qquad 1 - y$

- Find the volume of the solid whose base is circle $x^2 + y^2 = 1$ and cross sections is perpendicular to x-axis is :
  a) Semi – circle

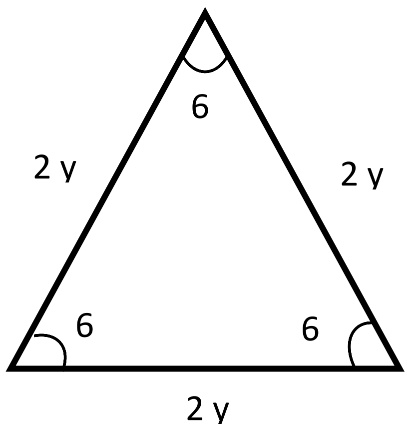




$A = \frac{1}{2} \pi y^2 = \frac{1}{2} \pi (1 - x^2)$

$V = \int_{-1}^{1} \frac{1}{2} \pi (1 - x^2) \, dx$

b) Equilateral triaxial



$$A = \frac{1}{2} . A . B . \sin\theta$$

$$= \frac{1}{2} . 2y . 2y . \sin 60$$

$$= \frac{1}{2} . 2y^2 . \frac{\sqrt{3}}{2}$$

$$V = \int_{-1}^{1} \sqrt{3}\,(1 - \boldsymbol{x^2})\, dx$$

CHAPTER 8 :

APPLICATION OF INTEGRATION

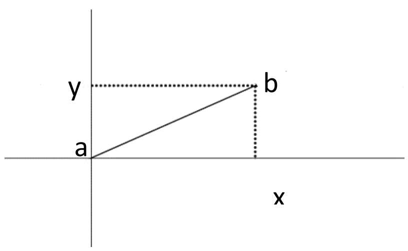
ch 8 : application of integration

عندي تطبيقين عن التكامل :

1) Arc length

2) Surface Area

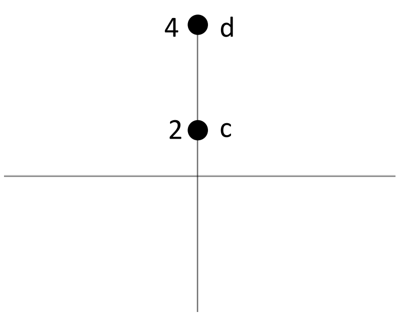
8.1 : Arc length



ab → بقدر اعرف طوله ؟

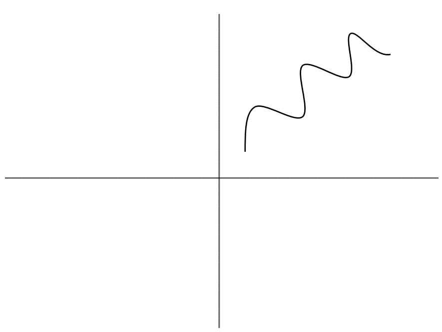
$$ab^2 = x^2 + y^2$$

$$ab = \sqrt{x^2 + y^2}$$



cd → بقدر اعرف طوله ؟

cd = 4 – 2 = 2



هون في هاي الحالة

صعب أقيسه سواء

بقانون او مسطرة

او تقدير , طيب شو الحل ؟

هون بلجأ لل Arc length

Arc length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

بدلالة x

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

بدلالة y

الخطوات :

1) بشتق الإقتران / العلاقة

2) بربع الطرفين

3) بزيد 1 على الطرفين

4) بحط الجذر و بكامل

Note : لا يوجد داعي للرسم

Ex : find arc length for :

1) $y = x^{\frac{3}{2}}$, $1 \leq x \leq 4$

حدود التكامل

*واضح هون إنه الاقتران بدلالة x
يعني من المنطق اشتق بالنسبة x
$\left(\frac{dy}{dx}\right)$

$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$$

ربع الطرفين

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

زيد 1

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{9}{4} x + 1$$

باخد الجذر و بكامل

$$L = \int_{x=1}^{x=4} \sqrt{\frac{9}{4} x + 1}$$

$$L = \int_1^4 \left(\frac{9}{4} x + 1\right)^{\frac{1}{2}} \frac{dy}{dx}$$

بالعادة بطلبوا الشكل النهائي للتكامل دون حسابه (Setup integer)

$$L = \int_1^4 \left(\frac{9}{4} x + 1\right)^{\frac{1}{2}} dx \longrightarrow \left. \frac{\left(\frac{9}{4} x + 1\right)^{\frac{3}{2}}}{\frac{2}{3} * \frac{9}{4}} \right|_1^4$$

*Note :

اذا كانت المشتقة $\frac{dy}{dx}$

حدود التكامل تكون من $x = a , x = 6$

اذا كانت المشتقة $\frac{dx}{dy}$

حدود التكامل تكون من $y = a , y = 6$

2) $y = \ln(\sec x)$, $x \in [0, \frac{\pi}{3}]$

$$\frac{dy}{dx} = y` = \frac{\sec x \tan x}{\sec x}$$

$$y` = \tan x$$

ربع

$$(y`)^2 = \tan^2 x$$

زيد 1

$$(y`)^2 + 1 = \tan^2 x + 1$$ → remember : $\sec^2 x = \tan^2 x + 1$

جذر + تكامل

$$(y`)^2 + 1 = \sec^2 x$$ → $|sec|$, but $0 \to \frac{\pi}{3}$ ⊕

$$L = \int_0^{\frac{\pi}{3}} \sec x \; dx$$ → $\ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{3}}$

$$L = \ln|2 + \sqrt{3}| - \ln|1 + 0|$$

$$L = \ln(2 + \sqrt{3})$$

3) $y^2 = x$, $P_1(0,0)$

$P_0(1,1)$

*طيب هون شو بدي اخد حدود تكامل ؟ كيف أشتق ؟

*العلاقة مبينة إنه أشتق بالنسبة ل y أفضل

و كانه علاقة بدلالة y ← $\frac{dx}{dy}$ ← الحدود باخذها من $y = a , y = 6$

$$\frac{dx}{dy} = 2y$$

$$\left(\frac{dx}{dy}\right)^2 = 4y^2$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = 4y^2 + 1$$

$$L = \int_{y=0}^{y=1} \sqrt{4y^2 + 1}\, dy$$

في حال بدي اكمل حل → $let \quad 2y = \tan x$

4) $xy = 1$, $P_1(1,1)$

$P_0(2,\frac{1}{2})$

*هون قدامي خيارين :

1) $y = \frac{1}{x}$, $2 \geq x \geq 1$, $\frac{dy}{dx}$

or

2) $x = \frac{1}{y}$, $1 \geq y \geq \frac{1}{2}$, $\frac{dx}{dy}$

H.W

Ans choice 1 : $L = \int_1^2 \sqrt{\frac{1}{x^4} + 1}\ dx$

choice 2 : $L = \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{y^4} + 1}\ dy$

مهم جدا

5) $x = \frac{y^4}{8} + \frac{1}{4y^2}$ $1 \leq y \leq 2$

$$\frac{dx}{dy} = \frac{4y^3}{8} + \frac{-2y}{4y^4}$$

$$\frac{dx}{dy} = \frac{y^3}{2} - \frac{1}{2y^3}$$

يتبع

$\frac{dx}{dy} = x^1 \qquad = \frac{y^3}{2} - \frac{1}{2y^3} \qquad , \ 1 \le y \le 2$

$$(x^1)^2 = \underbrace{\frac{y^6}{4}}_{\text{مربع ثاني}} - 2\underbrace{\left(\frac{y^3}{2}\right)}_{\text{الاول}}\underbrace{\left(\frac{1}{2y^3}\right)}_{\text{الثاني}} + \underbrace{\frac{1}{4y^6}}_{\text{مربع ثاني}}$$

$$(x^1)^2 = \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6}$$

$$1 + (x^1)^2 = \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6} + 1$$

في هذه الحالات دائما تكون الجهة اليمين مربع كامل يعني

$$a^2 + b + c^2 = (a + bc)^2$$

$$L = \int_1^2 \sqrt{\left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2}\ dy$$

$$= \int_1^2 (\frac{y^3}{2} + \frac{1}{2y^3}\ dy$$

Past Paper :

Find Arc length for :

$(y-1)^3 = x^2 \qquad if \qquad 1 \le y \le 5$

يا بحلها $\frac{dx}{dy}$ و بكمل او بطلع حدود x عن طريق تعويض y

$if \qquad y = 1 \longrightarrow x = 0$

$y = 5 \longrightarrow x = 8$

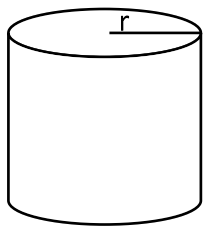
Choice 1 $\longrightarrow \frac{dx}{dy} \longrightarrow \int_1^5$

Choice 2 $\longrightarrow \frac{dy}{dx} \longrightarrow \int_0^8$

Ans choice 1 $\longrightarrow$ ☺ بستنى منكم إجابة

choice 2 $\longrightarrow L = \int_0^8 \sqrt{1 + \frac{4}{9x^{\frac{2}{3}}}}$

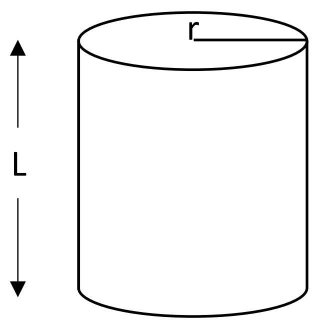
8.2 : Surface Area :



Surface Area ⟶ مساحة السطح

(كإني بدي ادهنه للجسم يعني مساحة الأوجه الخارجية)

S.A for cylinder (الأسطوانة) ⟶ محيط القاعدة * الارتفاع

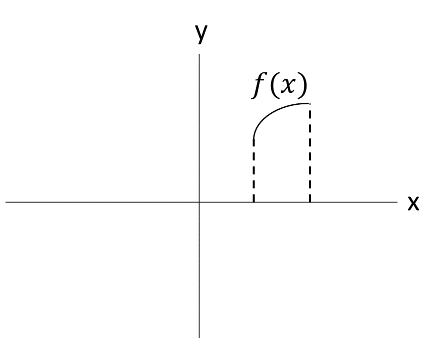


$$S.A = 2\pi r * L$$

$But, let\ f(x)\ is\ function\ shown\ in\ the\ stretch\ and$

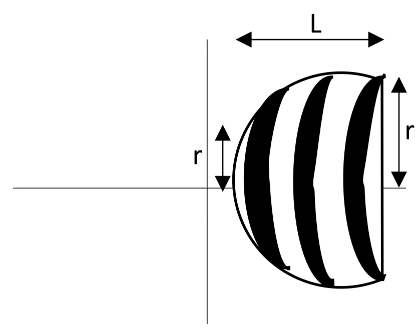
$we\ want\ to\ rotate\ it\ around\ x - axis$

$find\ surface\ area\ of\ the\ resultant\ shape :$



*بلاحظ إنه هون عندي ال r متغيرة مش ثابتة نتيجة تغيرها بالنسبة لكل دائرة فدائما r بتكون رمز متغير مش ثابت في هاي الحالة

*بلاحظ انه الارتفاع L هو فعليا ال Arc length كما تعلمنا



- S.A

بدلالة X

$$2\pi \int_a^b r \sqrt{1 + (\frac{dy}{dx})^2} dx$$

بدلالة Y

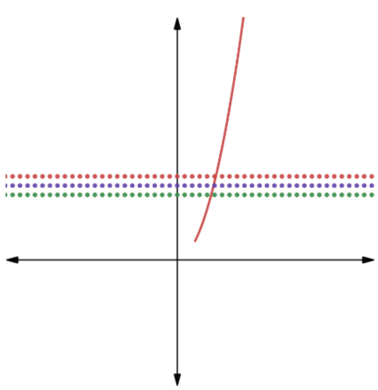
$$2\pi \int_c^d r \sqrt{1 + (\frac{dx}{dy})^2} dy$$

و كإنها $2\pi r * L$ و لكن عشان عندي اكثر من دائرة فبجمعهم بالتكامل

Note: مهمة

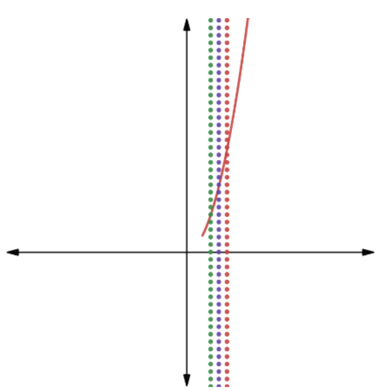
r : is the distance between the curve and the revolution axis.

Example of " r " with selected general cases:



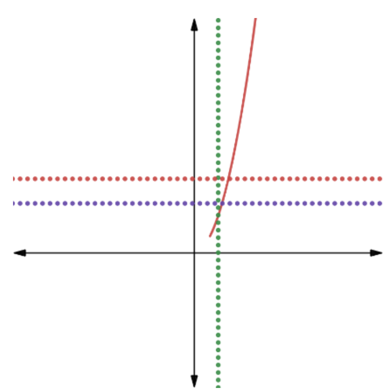
If the y axis (x=0) is the rotational axis then:

r= f(y)-0=x



If the x axis (y=0) is the rotational axis then:

r= f(x)-0=y



If the x=a axis is the rotational axis then:

r= x-a

How to find r ?

- الرسم
- r , $const$, متغيرة (حسب)
- $if\ r\ //\ x-axis\ , r = x$
- $if\ r\ //\ y-axis\ , r = y$

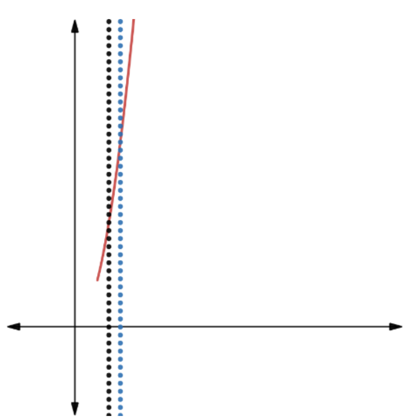
Examples on finding r :

- Find r if a curve described by $y = x^2, x \in \{2,9\}$ and rotates about :

  1) x-axis   2) y-axis   3) x = -3   4) y = -3   5) x = 2

ANS,

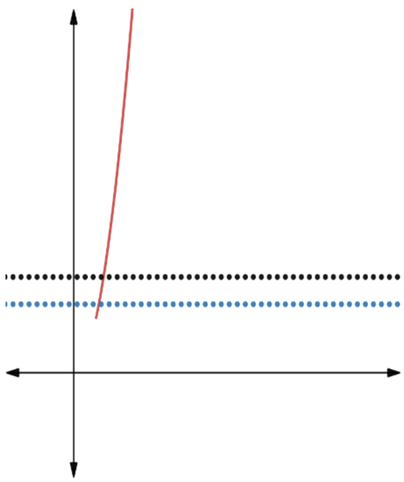
1)



$r = y$

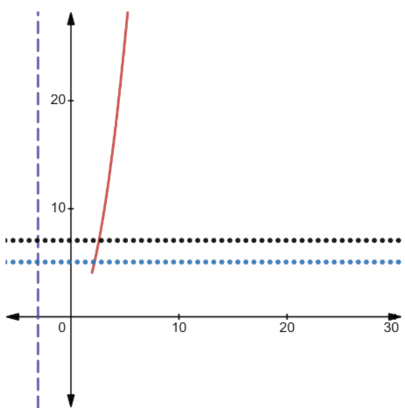
$r = x^2$

2)



$r = x$

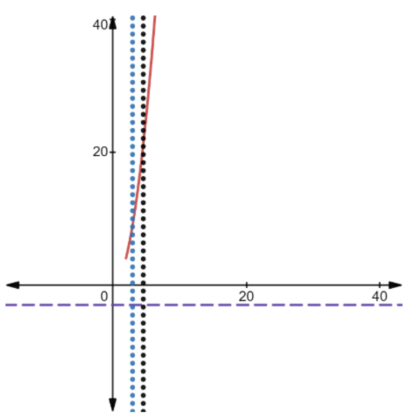
3)



$r = x - -3$

$r = x + 3$

4)



$r = y - -3$

$r = y + 3$

$r = x^2 + 3$

ملاحظة:

في حال لم تستطيع رسم الاقتران يمكنك من معرفة محور الدوران تشكيل معادلتين:

*اذا المحور موازي محور السينات:

$r = f(x) +$ ازاحة

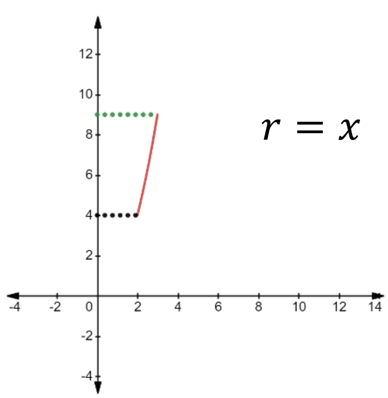
*اذا المحور موازي لمحور الصادات:

$r = f(y) +$ ازاحة

Example: find the surface area for the following

1) $y = x^2; 2 \leq x \leq 3; about\ the\ y-axis$

Finding r as first step:

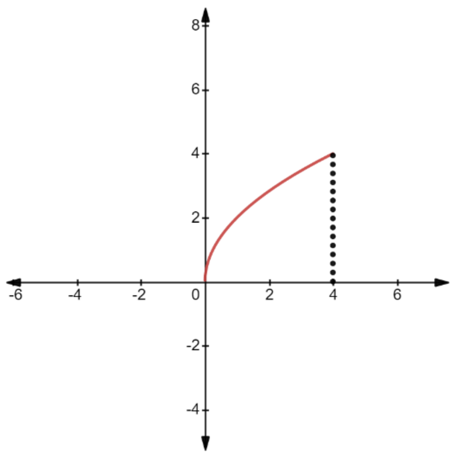


Finding the arc length:

- Our "r" is set with respect to x so we use the S.A wit respect to x formula:
- $L = S.A = 2\pi \int_a^b r\sqrt{1 + (\frac{dy}{dx})^2}dx$
- a = 2 and b = 3 as they are our curve terminals
- $\acute{y}^2 = (\frac{dy}{dx})^2 = (\frac{dx^2}{dx})^2 = 4x^2$
- $L = 2\pi \int_2^3 x\sqrt{1 + (4x^2)^2}dx\ , setup\ integral$

2) $y = 2\sqrt{x}; 0 \leq x \leq 4; about\ the\ x - axis$

Finding r as first step:



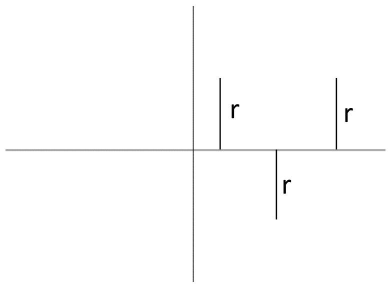
Finding the arc length:

- Our “r” is set with respect to x so we use the S.A with respect to x formula:
- $\text{L} = \text{S.A} = 2\pi \int_a^b r \sqrt{1 + (\frac{dy}{dx})^2} dx$
- $\text{a} = 2$ and $\text{b} = 3$ as they are our curve terminals
- $\acute{y}^2 = (\frac{dy}{dx})^2 = (\frac{d2\sqrt{x}}{dx})^2 = (\frac{1}{\sqrt{x}})^2 = \frac{1}{x}$
- $\boldsymbol{L = 2\pi \int_0^4 2\sqrt{x} \sqrt{1 + \frac{1}{x}} dx}$

$$\boldsymbol{= 2\pi \int_0^4 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx}$$

$$\boldsymbol{= 2\pi \int_0^4 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 2\pi \int_0^4 2\sqrt{x+1} dx}$$

3) $y = \sqrt{1+4x}; 1 \le x \le 5; about\ the\ x - axis$



ملاحظة (اذا ما عرفت ترسم الرسمة):

لان محور الالتفاف تبعنا مستقيم او اقتران ثابت فنصف القطر سيبقي موازي لمحور الصادات بهاي الحاله مهما كانت الرسمه

$r = y = \sqrt{1+4x}$

Finding the arc length

- Our "r" is set with respect to x so we use the S.A with respect to x formula:
- $\mathrm{L} = \mathrm{S.A} = 2\pi \int_a^b r\sqrt{1 + (\frac{dy}{dx})^2}dx$
- $\mathrm{a} = 2$ and $\mathrm{b} = 3$ as they are our curve terminals
- $\acute{y}^2 = (\frac{dy}{dx})^2 = (\frac{d\sqrt{1+4x}}{dx})^2 = (\frac{4}{2\sqrt{1+4x}})^2 = \frac{4}{1+4x}$
- $\boldsymbol{L = 2\pi \int_1^5 \sqrt{1+4x}\sqrt{\frac{4}{1+4x} + 1}dx \ = 4\pi \int_1^5 \sqrt{5+4x}dx}$

H.W

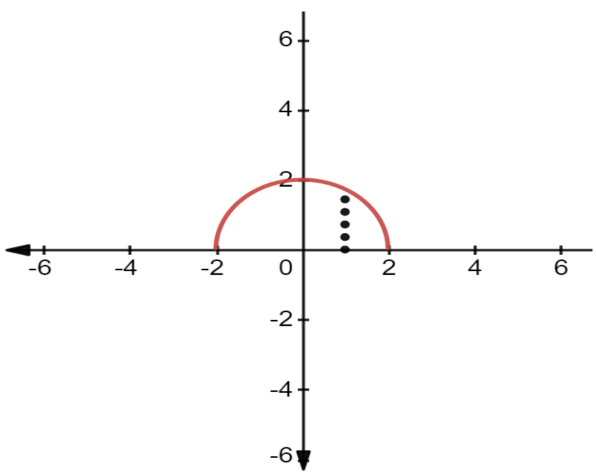
4) $y = \ln(x); 1 \le x \le e; about\ the\ y - axis$

Ans,

- $L = S.A = 2\pi \int_1^e x\sqrt{1+\frac{1}{x^2}}dx = 2\pi \int_1^e \sqrt{x^2+1}dx$

5) $y = \sqrt{4-x^2}; -2 \le x \le 2; about\ the\ x - axis$

يحل هذا السؤال بطريقتين :



طريقة كالك 1 (للمهندسين)

- نص دائرة $y = \sqrt{4-x^2}$
- لما ألفه رح يتشكل عندي كرة كاملة
- $v = \frac{4}{3}\pi r^3$
- $S.A = 4\pi r^2$
- $S.A = 4\pi(2)^2$
- $S.A = 16\pi$

طريقة كالكولاس 2 :

- $r = y$
- $r = \sqrt{4-x^2}$
- $y` = \frac{2x}{2\sqrt{4-x^2}}$
- $(y`)^2 + 1 = \frac{x^2}{4-x^2} + 1$
- $S.A = 2\pi \int_{-2}^{2} \sqrt{4-x^2} * \sqrt{\frac{x^2}{4-x^2}+1}\, dx$

$= 2\pi \int_{-2}^{2} \sqrt{4-x^2} * \frac{\sqrt{4}}{\sqrt{4-x^2}}\, dx$

$= 2\pi (4)(2)$

$S.A = 16\pi$