Calculus II

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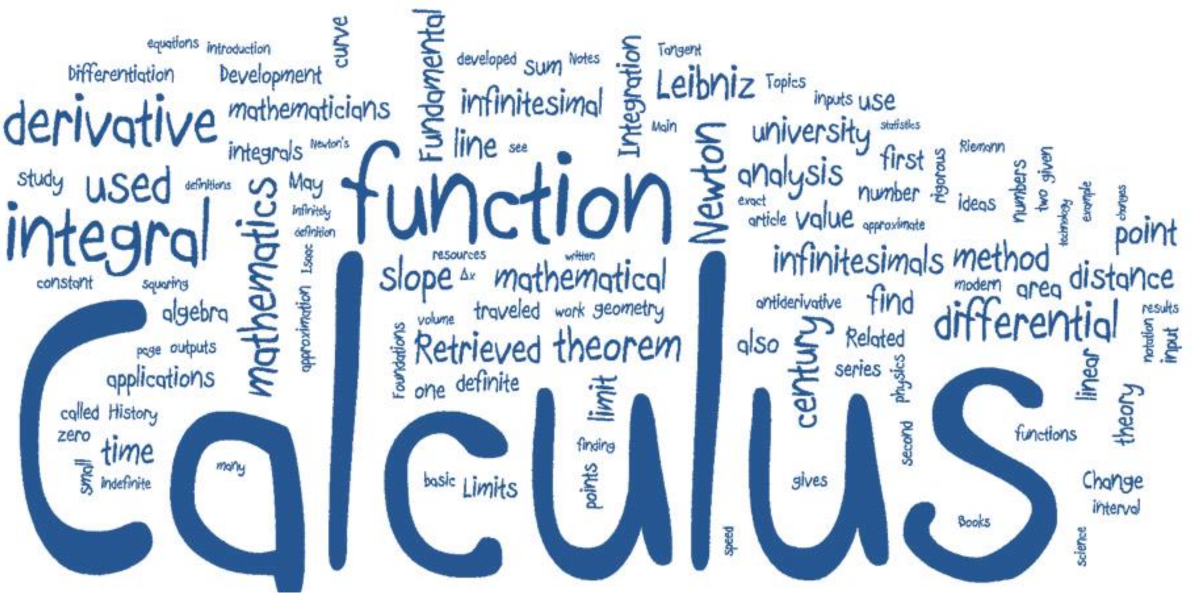
إعداد الطالبة: سجى البدور



Integration by Part

Ex: $\int \tan^{-1}x \, dx$ $\quad u = \tan^{-1}x \quad dv = 1\,dx$

$du = \frac{1}{1+x^2}dx \quad v = x$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}dx = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$

Ex: $\int_0^1 \tan^{-1}x \, dx = \left[x\tan^{-1}x - \frac{1}{2}(\ln(1+x^2))\right]_0^1$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$

Find $\int \sin^n x \, dx$ ~~=~~

sol: $\int \sin^{n-1}x \, \sin x \, dx$ $\quad u = \sin^{n-1}x \quad dv = \sin x$

$du = (n-1)\sin^{n-2}x \cos x \, dx \quad v = -\cos x$

$= -\sin x \cos x + (n-1)\int \sin^{n-2}x \cos^2 x \, dx$

$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x (1-\sin^2 x)\, dx$ $\quad \boxed{\sin^2 x + \cos^2 x = 1}$

$\int \sin^n x \, dx = -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \, dx - (n-1)\int \sin^n x \, dx$

$n \int \sin^n x \, dx = \frac{-\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \, dx}{n}$

$\int \sin^n x \, dx = \frac{-\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \, dx}{n}$

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Find $\int \cos^n x \, dx$

$u = \cos^{n-1} x \qquad dv = \cos x$

$du = -(n-1)\cos^{n-2} x \sin x \qquad v = \sin x$

Sol : $\int \cos^{n-1} x \cos x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$

$\frac{n}{n} \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx$

$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx$

Ex : $\int \cos^6 (3x) \, dx$

[1] Find $\int \cos^n (3x) \, dx$

$u = \cos^{n-1}(3x) \qquad dv = \cos(3x)$

$du = 3(n-1) \cos^{n-2}(3x) \sin(3x) \qquad v = \frac{\sin(3x)}{3}$

$= \frac{1}{3} \cos^{n-1}(3x) \sin(3x) - 3(n-1) \int \cos^{n-2}(-3x) \sin^2(3x) \, dx$

$(1 - \cos^2(3x))$

$= \frac{1}{3} \cos^{n-1}(3x) \sin(3x) - (n-1) \int \cos^{n-2}(3x) \, dx - (n-1) \int \cos^n (3x) \, dx$

$\int \cos^n (3x) \, dx = \frac{1}{3n} \cos^{n-1}(3x) \sin(3x) + \frac{n-1}{n} \int \cos^{n-2}(3x) \, dx$

[2]

2 $\int \cos^6(3x)\,dx$ نعوضها بآخر خطوة

$$= \frac{1}{18}\cos^5(3x)\sin(3x) + \frac{5}{6}\int \cos^4(3x)\,dx$$

$$= \frac{1}{18}\cos^5(3x)\sin(3x) + \frac{5}{6}\left[\frac{1}{12}\cos^3(3x)\sin(3x) + \frac{3}{4}\int \cos^2(3x)\,dx\right]$$

$$= \frac{1}{18}\cos^5(3x)\sin(3x) + \frac{5}{72}\cos^3(3x)\sin(3x) + \frac{15}{24}\left[\frac{1}{6}\cos(3x)\sin(3x) + \frac{1}{2}\int 1\,dx\right] + C$$

Ex(53): P.(477) $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{(n-2)} x\,dx \quad (n \neq 1)$

Sol: $\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$

$$= \int \tan^{n-2}(\sec^2 x - 1)\,dx = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^{n-2}\,du - \int \tan x^{n-2}\,dx$$

$$= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

Ex (54): [redacted] $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$

Prove

Sol: $\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$dv = \sec^2 x\,dx$

$u = \sec^{(n-2)} x \longrightarrow v = \tan x$

$du = (n-2)\sec^{(n-3)} x \tan x \sec x$

$= \sec x \tan x - (n-2)\int \sec^{(n-2)} x \tan x\,dx$

$= \sec^{(n-2)} x \tan x - (n-2)\int \sec^{(n-2)} x (\sec^2 x - 1)\,dx$

$= \sec^{(n-2)} x \tan x - (n-2)\int \sec^n x\,dx - (n-2)\int \sec^{n-2} x\,dx$

$\int \sec^n x\,dx = \dfrac{\tan x \sec^{(n-2)} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{(n-2)} x\,dx$

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Ex (51) A) show that $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$
P. 477 B) Find $\int (\ln x)^3 dx$

sol:

A) $\int (\ln x)^n dx =$

$u = (\ln x)^n \qquad dv = dx$

$du = n(\ln x)^{n-1} \frac{1}{x} dx \qquad v = x$

$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$ ##

B) $\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx$

$= x(\ln x)^3 - 3x(\ln x)^2 - 6\int \ln x \, dx$

$= x(\ln x)^3 - 3x(\ln x)^2 - 6x(\ln x) + \int 6 \, dx$

$= x(\ln x)^3 - 3x(\ln x)^2 - 6x(\ln x) + 6x + C$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \qquad v = x$

H.W

Ex (50) P. 477 Prove

1) $\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots (2n)} \frac{\pi}{2}$

& $\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2.4.6 \cdots 2n}{3.5.7 \cdots (2n+1)}$

Ex : 39 P. 477

$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) d\theta$

$u = \theta^2$

$du = 2\theta dx$

$dx = \frac{du}{2\theta}$

$\int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos u \frac{du}{2\theta} = \int_{\frac{\pi}{2}}^{\pi} \frac{u}{2} \cos u \, du$

$w = \frac{u}{2} \qquad dv = \cos u$

$dw = \frac{1}{2} du \qquad v = \sin u$

$= \frac{u}{2} \sin u - \int \frac{1}{2} \sin u$ يتبع

39 P.472

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\cos u\, du = \frac{u}{2}\sin u - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \sin u\, du$$

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\cos u\, du = \frac{u}{2}\sin u\Big|_{\frac{\pi}{2}}^{\pi} + \frac{1}{2}(\cos u)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{2}\sin\pi - \frac{\pi}{4}\sin\frac{\pi}{2} + \frac{1}{2}\cos\pi - \frac{1}{2}\cos\frac{\pi}{2}$$

$$= 0 - \frac{\pi}{4} - \frac{1}{2} - 0 = \boxed{-\frac{\pi}{4} - \frac{1}{2}}$$

$\boxed{H.W} \Rightarrow$ Ex (41) $\int x\ln(x+1)\,dx$.

Ex(42) $\int \frac{\arcsin(\ln x)}{x}\,dx$

Ex (40) $\Rightarrow \int_0^{\pi} e^{\cos t}\sin(2t)\,dt$.

* $\int \sin^m x \cos^n x\, dx$

A] If n is odd $(n = 2K+1)$

$$\int \sin^m x \cos^{2K+1} x\, dx = \int \sin^m x\,(\cos^2 x)^K \cos x\, dx$$

$$= \int \sin^m x\,(1-\sin^2 x)^K \cos x\, dx$$

let $u = \sin x$

B] If m is odd $(m = 2K+1)$

$$\int \sin^{(2K+1)} x \cos^n x\, dx = \int (\sin^2 x)^K \cos^n x \sin x\, dx$$

$$= \int (1-\cos^2 x)^K \cos^n x \sin x\, dx$$ let $u = \cos x$

③ If n & m are even

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

Ex: $\int \sin^5 x \cos^2 x \, dx$

$$= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx$$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$$= -\int (1-u^2)^2 u^2 \, du$$

$$= -\int (u^2 - 2u^4 + u^6) \, du = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\left(\frac{\cos^3 x}{3} - 2\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}\right) + C$$

Ex: $\int \sin^7 x \cos^3 x \, dx$

$$= \int \sin^7 x \cos^2 x \cos x \, dx = \int \sin^7 x (1-\sin^2 x) \cos x \, dx$$

$$= \int u^7 (1-u^2) \, du = \int u^7 - u^9 \, du$$

$u = \sin x$

$du = \cos x \, dx$

$dx = \frac{du}{\cos x}$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + C = \boxed{\frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + C}$$

Ex 48 : $\int \frac{dx}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1}$
P. 485

$= \int \frac{\cos x + 1}{-\sin^2 x} = -\int \left(\frac{\cos x}{\sin x} * \frac{1}{\sin x} + \frac{1}{\sin^2 x} \right) dx$

$= - \int (\cot x \csc x + \csc^2 x) dx$

$= - \csc x + \cot x + c$

Ex 49 : $\int x \tan^2 x \, dx$
P. 485

$u = x \qquad dv = \tan^2 x \; (\sec^2 x - 1)$

$du = 1 dx \qquad v = \tan x - x$

$= x \tan x - x^2 - \int (\tan x - x) \, dx$

$= x \tan x - x^2 - \ln|\sec x| + \frac{x^2}{2} + c$

$= \boxed{x \tan x - \frac{x^2}{2} - \ln|\sec x| + c}$

Ex 45 :

$\int_0^{\pi/6} \sqrt{1 + \cos 2x} \, dx = \int_0^{\pi/6} \sqrt{\cancel{1} + 2\cos^2 x - \cancel{1}} \, dx$

$= \sqrt{2} \int_0^{\pi/6} |\cos x| \, dx = \sqrt{2} \int_0^{\pi/6} \cos x \, dx$

$= \sin x \Big|_0^{\pi/6} = \boxed{\frac{1}{2}}$

P. 484 / P. 485

7.3

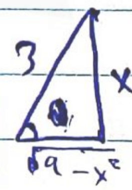
Trigonometric substitution

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$	$\sin^2 x + \cos^2 x = 1$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$	$\tan^2 x + 1 = \sec^2 x$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $\left(0 \le \theta \le \frac{\pi}{2} \text{ or } \pi \le \theta \le \frac{3\pi}{2}\right)$	$\tan^2 x + 1 = \sec^2 x$

Ex: $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$

Sol: let $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$

$dx = 3\cos\theta\,d\theta$



$$\Rightarrow \int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} * 3\cos\theta\,d\theta$$

$$= \int \frac{\cancel{3}\cdot 3\sqrt{1 - \sin^2\theta}}{\cancel{9}\sin^2\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

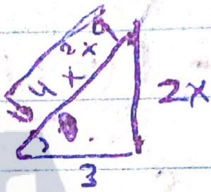
$$= \boxed{-\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

Ex: $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$

$2x = 3\tan\theta \qquad x = \frac{3}{2}\tan\theta$

$\theta = \tan^{-1}\left(\frac{2x}{3}\right)$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$



$$\int_0^{3\sqrt{3}/2} \frac{\frac{27}{8}\tan^3\theta \cdot \frac{3}{2}\sec^2\theta\, d\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}$$

$$= \left(\frac{3}{2}\right)^4 \frac{1}{9^{\frac{3}{2}}} \int \frac{\tan^3\theta\,\sec^2\theta\, d\theta}{(\sec^2\theta)^{\frac{3}{2}}} = \left(\frac{3}{2}\right)^4 \cdot \frac{1}{(3)^3} \int \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^3\theta}\cdot\cos\theta\, d\theta = \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= \frac{3}{16}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\sin\theta\, d\theta = \frac{3}{16}\int \frac{1-\cos^2\theta}{\cos^2\theta}\cdot\sin\theta\, d\theta$$

$$= \frac{-3}{16}\int \frac{1-u^2}{u^2}\, du = \frac{-3}{16}\int u^{-2} - 1\, du$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\frac{-3}{16}\left(\frac{u^{-1}}{-1} - u\right) = \left(\frac{3}{16u} + \frac{3u}{16}\right) \cancel{\frac{3}{16\cos\theta}} \longrightarrow$$

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cont.

Ex: $\left(\frac{3}{16u} + \frac{3u}{16}\right) = \left(\frac{3}{16\cos\theta} + \frac{3\cos\theta}{16}\right)$

$= \cancel{\frac{3.3}{16\sqrt{}}} \quad \frac{-3}{16}\left(-\frac{\sqrt{4x^2+9}}{3} - \frac{3}{\sqrt{4x^2+9}}\right)\Big|_0^{\frac{3}{2}}$

$= \boxed{\frac{3}{32}}$

Ex: $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ complete square

Sol: $3-2x-x^2$

$-(x^2+2x-3)$

$-\left(x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2-3\right) \leftarrow \left(\frac{x}{2}\right)^2$

$-\left((x^2+2x+1)-4\right)$

$-\left((x+1)^2-4\right) = 4-(x+1)^2$

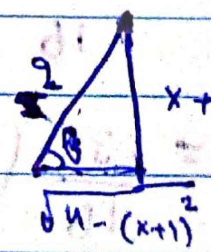
$\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx = \quad x+1 = 2\sin\theta$

$x = 2\sin\theta - 1$

$dx = 2\cos\theta\,d\theta$

$= \int \frac{(2\sin\theta-1)\cdot 2\cos\theta\,d\theta}{\sqrt{4-4\sin^2\theta}} \qquad \theta = \sin^{-1}\left(\frac{x+1}{2}\right)$

$= \frac{2}{2}\int \frac{(2\sin\theta-1)\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = -\int (2\sin\theta-1)\,d\theta$



$= -\frac{2\sqrt{4-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$

Ex 23 : $\int \frac{dx}{\sqrt{x^2+2x+5}}$

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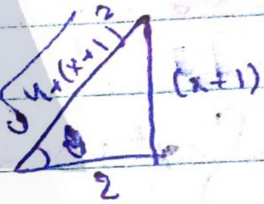
Sol: x^2+2x+5

$(x^2+2x+1)(-1+5) = (x+1)^2+4$

$\int \frac{dx}{\sqrt{4+(x+1)^2}}$ $\quad x+1 = 2\tan\theta \quad x = 2\tan\theta - 1$

$\theta = \tan^{-1}\frac{(x+1)}{2} \quad dx = 2\sec^2\theta \, d\theta$



$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}} = \frac{2}{2}\int \frac{\sec^2\theta \, d\theta}{\sqrt{\sec^2\theta}}$

$= \int \sec\theta \, d\theta = \ln|\tan\theta + \sec\theta| + c$

$= \boxed{\ln\left|\frac{(x+1)}{2} + \frac{\sqrt{4+(x+1)^2}}{2}\right| + c}$

H.W

Ex: $\int \frac{x+1}{\sqrt{x^2+2x+5}} dx$ $\quad$ 1 Method $\quad x+1 = 2\tan\theta$

2 Method $\quad u = x^2+2x+5$

(7.4)

- integration of rational functions by Partial fractains

التكامل باستخدام الكسور الجزئية

Polyn / Poly ÷ Just

· Rational func. $\Rightarrow \dfrac{P(x)}{Q(x)}$ where P, Q are Polys.

إذا درجة البسط ≥ مقام
يجب انه نقوم بالقسمة الطويلة

مهم

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C}$$

Ex: $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r|l} & x^2+x+2 \\ x-1 & x^3+x \\ & -x^3+x^2 \\ \hline & x^2+x \\ & -x^2+x \\ \hline & 2x \\ & -2x+2 \\ \hline & 2 \end{array}$$

$$\int x^2+x+2\,dx + \int \frac{2}{x-1}\,dx$$

$$= \boxed{\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C}$$

Partial fractions.

$\dfrac{P(x)}{Q(x)}$ where deg $(P(x)) <$ deg $(Q(x))$

* Case I

If $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots\cdots(a_nx+b_n)$

then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

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Ex: write the form of the Partial fraction decompostion of $\dfrac{x^2+2x-1}{2x^3+3x^2-2x}$

$$= \frac{x^2+2x-1}{x(2x^2+3x-2)} = \frac{x^2+2x-1}{x(2x-1)(x+2)}$$

$$= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$= \frac{A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)}{x(2x-1)(x+2)} = \frac{x^2+2x-1}{x(2x-1)(x+2)}$$

$\boxed{x = 0, \frac{1}{2}, -2}$

ax^2+bx+c

- two real roots: $b^2-4ac>0$
- one real root: $b^2-4ac=0$
- No real Root: $b^2-4ac<0$

$ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Now to find the constents A, B, C

$$\frac{C(a_1x+b_1)(a_2x+b_2) + A(a_2x+b_2)(a_3x+b_3) + B(a_1x+b_1)(a_3x+b_3)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)}$$

$$A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) = x^2+2x-1$$

let

$x=0 \Rightarrow -2A+0+0=-1 \quad \boxed{A=\frac{1}{2}}$

$x=\frac{1}{2} \Rightarrow \frac{5}{4}B = \frac{1}{4} \Rightarrow \boxed{B=\frac{1}{5}}$

$x=-2 \Rightarrow 10C = -1 \Rightarrow \boxed{C = \frac{-1}{10}}$

$$= \frac{\frac{1}{2}}{x} + \frac{\frac{1}{5}}{2x-1} - \frac{\frac{1}{10}}{x+2}$$

Ex: find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

نفس السؤال

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{1}{2x} + \frac{1}{5(3x-1)} - \frac{1}{10(x+2)}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\ln|3x-1| - \frac{1}{10}\ln|x+2| + C$$

- Case 2

Q(x) is a product of linear factors, some of which are repeated.

$$\frac{1}{(ax+b)^r} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_r}{a_rx+b_r}$$

Ex: $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

Ex: $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & -x^4+x^3+x^2-x \\ \hline & x^3-x^2+3x+1 \\ & -x^3+x^2+x-1 \\ \hline & 4x \end{array}$$

$$= \int x+1\,dx + \int \frac{4x}{x^3-x^2-x+1}\,dx$$

$$= \frac{x^2}{2} + x + \int \frac{4x}{(x-1)(x^2-1)}\,dx$$

$$= \frac{x^2}{2} + x + \int \frac{4x}{(x-1)(x-1)(x+1)}\,dx$$

Ex 8 $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$

$A(x^2-1) + B(x+1) + C(x-1)^2 = 4x$

Method 1

$x = 1 \Rightarrow 2B = 4 \Rightarrow \boxed{B = 2}$

$x = -1 \Rightarrow 4C = -4 \quad \boxed{C = -1}$

$x = 0 \Rightarrow -A + B + C = 0 \qquad -A + 2 - 1 = 0 \quad \boxed{A = 1}$

Method 2

$x^2 \Rightarrow A + C = 0 \qquad \boxed{A = -C}$

$x^1 \Rightarrow B - 2C = 4 \qquad \boxed{B = 4 + 2C}$

$x^0 \Rightarrow -A + B + C = 0$

$C + 4 + 2C + C = 0 \qquad 4C = 4 \qquad \boxed{C = -1}$

$\therefore \boxed{A = 1}$

$\boxed{B = 2}$

$= \int \left[(x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)} \right] dx$

$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{(x-1)} - \ln|x+1| + C$ #

تكملة 2

- Case 3: $Q(x)$ contains irreducible quadratic factors (with no repeted factors).

$ax^2 + bx + c \Rightarrow b^2 - 4ac < 0$

$\Rightarrow \dfrac{Ax + B}{ax^2 + bx + c}$

Ex: $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx \Rightarrow \frac{2x^2 - x + 4}{x\,(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 4)}$

$A(x^2 + 4) + x(Bx + C) = 2x^2 - x + 4$

$x^2 \Rightarrow 2 = A + B \quad \boxed{1 = B}$

$x \Rightarrow \boxed{-1 = C}$

$4 \Rightarrow 4 = 4A \quad \boxed{A = 1}$

$\displaystyle\int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$

$= \ln|x| + \frac{1}{2}\ln|x^2 + 4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$\boxed{\dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C.} \Rightarrow$ مهم

Ex: $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int 1 + \frac{x-1}{4x^2-4x+3}\,dx$ → irreducible

$$\begin{array}{r|l} 4x^2-4x+3 & \overline{\;4x^2-3x+2\;}^{\;1} \\ & -4x^2 \pm 4x \mp 3 \\ \hline & \quad x-1 \end{array}$$

complete square

$4x^2-4x+3 = 4\left(x^2-x+\frac{3}{4}\right)$

$= 4\left(x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{3}{4}\right) = 4\left(\left(x^2-x+\frac{1}{4}\right)+\frac{1}{2}\right)$

$= 4\left(\left(x-\frac{1}{2}\right)^2+\frac{1}{2}\right) = 4\left(x-\frac{1}{2}\right)^2+2$

$= \left(2\left(x-\frac{1}{2}\right)\right)^2+2 = (2x-1)^2+2$

$u = 2x-1$

$du = 2\,dx$

$= x + \int \frac{\frac{u}{2}+\frac{1}{2}-1}{u^2+2}\,\frac{du}{2}$

$= x + \frac{1}{8}\int \frac{u}{u^2+2}\,du - \frac{1}{4}\int \frac{1}{u^2+2}\,du$

$= x + \frac{1}{8}\ln\left|u^2+2\right| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$

$= x + \frac{1}{8}\ln\left|(2x-1)^2+2\right| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$

- Case (4)

$Q(x)$ contains repeted irreducible quadratic Factors

~~A_1x+B_1~~ $Q(x)$ has the factor $(ax^2+bx+c)^r$ $r>1$

$$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Ex: write out the form of Partial fraction (Do not find the constants)

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{(x^2+x+1)}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

Ex: $\displaystyle\int\frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

$$A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$$

$$=A(x^2+1)^2+(Bx+C)(x^3+x)+(Dx^2+Ex)=1-x+2x^2-x^3$$

x^4 : $A+B=0$ $\boxed{A=-B}$

x^3 : $\boxed{C=-1}$

x^2 : $2A+B+D=2$

x : $C+E=-1$

x^0 = $\boxed{A=1}$ $\boxed{E=0}$

$\boxed{B=-1}$

$2-1+D=2$

$\boxed{D=1}$

$$=\int\left(\frac{1}{x}-\frac{x+1}{(x^2+1)}+\frac{x}{(x^2+1)^2}\right)dx$$

$$\ln|x|-\frac{1}{2}\int\frac{2x}{x^2+1}\,dx-\int\frac{1}{x^2+1}\,dx+\int\frac{x}{(x^2+1)^2}\,dx$$

$$=\ln|x|-\frac{1}{2}\ln|x^2+1|-\tan^{-1}(x)-\frac{1}{2(x^2+1)}+C$$

Ex: $\int \frac{\sqrt{x+4}}{x}\,dx$

$u = \sqrt{x+4}$

$u^2 = x+4$

$2u\,du = dx$

$$= \int \frac{u}{u^2-4}\cdot 2u\,du = 2\int \frac{u^2}{u^2-4}\,du$$

$$\begin{array}{r} 1 \\ u^2-4\,\overline{)\,u^2} \\ \underline{-u^2+4} \\ 4 \end{array}$$

$$= 2\int 1 + \frac{4}{u^2-4}\,du \rightarrow \text{partial fraction}$$

$$= \boxed{2\sqrt{x+4} + 2\ln\left|\sqrt{x+4} - 2\right| - 2\ln\left|\sqrt{x+4}+2\right| + C}$$

Ex 59 let $z = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$

Prove

(1) $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+z^2}}$

(2) $\sin\left(\frac{x}{2}\right) = \frac{z}{\sqrt{1+z^2}}$

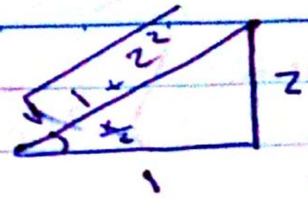
(3) $\cos(x) = \frac{1-z^2}{1+z^2}$

(4) $\sin(x) = \frac{2z}{1+z^2}$

(5) $dx = \frac{2}{1+z^2}\,dz$

Proof

$$z = \tan\left(\frac{x}{2}\right)$$



(1) $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+z^2}}$

(2) $\sin\left(\frac{x}{2}\right) = \frac{z}{\sqrt{1+z^2}}$

$\sin(2a) = 2\sin a \cos a$

$\cos(2a) = \cos^2 a - \sin^2 a$

(3) $\cos(x) = \cos\left(2\left(\frac{x}{2}\right)\right)$

$= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

$= \frac{1}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-z^2}{1+z^2}$

(4) $\sin(x) = \sin\left(2\left(\frac{x}{2}\right)\right)$

$= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$= 2\left(\frac{z}{\sqrt{1+z^2}}\right)\left(\frac{1}{\sqrt{1+z^2}}\right) = \frac{2z}{1+z^2}$

(5) $dx = \frac{2}{1+z^2}\, dz$

$dz = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$

$2\,dz = \left(\frac{\sqrt{1+z^2}}{1}\right)^2 dx$

$\Rightarrow \boxed{dx = \frac{2}{1+z^2}\, dz}$

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Ex : 61
P. 502 $\int_{0}^{\pi} \frac{1}{3\sin x - 4\cos x}\, dx$

Sol : $z = \tan\left(\frac{x}{2}\right)$

$$\Rightarrow \int \frac{1}{3\left(\frac{2z}{1+z^2}\right) - 4\left(\frac{1-z^2}{1+z^2}\right)} \cdot \frac{2}{1+z^2}\, dz$$

$$= \int \frac{1+z^2}{6z - 4 + 4z^2} \cdot \frac{2}{1+z^2}\, dz = \int \frac{dz}{2z^2 + 3z - 2}$$

$$= \int \frac{dz}{(2z-1)(z+2)} \Rightarrow \frac{1}{(2z-1)(z+2)} = \frac{A}{2z-1} + \frac{B}{(z+2)}$$

$$A(z+2) + B(2z-1) = 1$$

$z = -2 \Rightarrow -5B = 1$ $\boxed{B = \frac{-1}{5}}$

$z = \frac{1}{2} \Rightarrow \frac{5}{2}A = 1$ $\boxed{A = \frac{2}{5}}$

$$\Rightarrow \int \frac{\frac{2}{5}}{2z-1}\, dz - \int \frac{\frac{1}{5}}{z+2}\, dz = \frac{1}{5}\ln|2z-1| - \frac{1}{5}\ln|z+2| + c$$

$$= \frac{1}{5}\ln\left|\frac{2z-1}{z+2}\right| + c = \boxed{\frac{1}{5}\ln\left|\frac{2\tan\left(\frac{x}{2}\right) - 1}{\tan\left(\frac{x}{2}\right) + 2}\right| + c}$$

Ex 60 : $\int \frac{1}{1-\cos x}\,dx$

* Method 1

$\int \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} = \int \frac{1+\cos x}{\sin^2 x}\,dx = \int \csc^2 x + \cot x \csc x \, dx$

$= \boxed{-\cot x - \csc x + c}$

* Method 2

$z = \tan\left(\frac{x}{2}\right)$ ..... ونكمل عليه

(7.5)

strategy for integration

Ex : $\int \frac{\tan^3 x}{\cos^3 x}\,dx$

Method 1

$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx$ $u = \sec x$ $du = \sec x \tan x \, dx$

⋮

Method 2 $\int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{\sin^2 x \sin x}{\cos^6 x}\,dx$ $u = \cos x$ $du = \sin x \, dx$

$= \int \frac{1-\cos^2 x}{\cos^6 x}\,dx = \int \frac{1-u^2}{u^6}\,du = \int u^{-6} - u^{-4}\,du$

$= \frac{u^{-5}}{-5} + \frac{u^{-3}}{3} + c = \frac{-1}{5} \cdot \frac{1}{\cos^5 x} + \frac{1}{3} \cdot \frac{1}{\cos^3 x} + c$

Ex: $\int e^{\sqrt{x}}\,dx$ $\qquad u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\int e^{u}\,u\,du \qquad w = u \qquad dv = e^{u}$

$dw = 1\,du \qquad v = e^{u}$

$2ue^{u} - 2e^{u} + c = \boxed{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

Ex: $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$

$$\begin{array}{r} x^2+3x+19 \\ x^3-3x^2-10x \,\big)\, x^5 + 1 \\ -x^5 + 3x^4 + 10x^3 \\ \hline 3x^4 + 10x^3 + 1 \\ -3x^4 + 9x^3 + 30x^2 \\ \hline 19x^3 + 30x^2 + 1 \\ 19x^3 - \end{array}$$

partial fraction

Ex: $\int \frac{dx}{x\sqrt{\ln x}}$

Sol: $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

Ex 22 P.507 $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$ $\qquad u = \ln x \qquad du = \frac{1}{x}\,dx$

$\int \frac{u}{\sqrt{1+u^2}}\,du \qquad z = u^2$ or $z = 1 + u^2$.

or $u = \tan\theta \rightarrow$ اجدع

Ex 62 : $\int \frac{dx}{1+\cos^2 x}$ $\quad z = \tan\left(\frac{x}{2}\right)$

التكامل المعتل

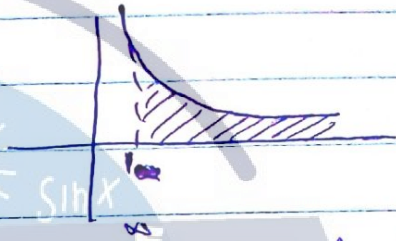
7.8 Improper Integrals.

* خاص بالتكامل المحدود

- Type I infinite intervals

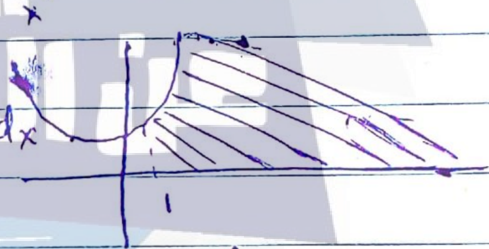
(a) $\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^{t} f(x)\,dx$

$\frac{1}{x^2}$



(b) $\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_t^{b} f(x)\,dx$ $\quad \int_1^{\infty} \frac{1}{x^2}\,dx$

(c) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$ $\quad x^2$



$= \lim_{z \to -\infty} \int_z^{a} f(x)\,dx + \lim_{t \to \infty} \int_a^{t} f(x)\,dx$ $\quad \int_1^{\infty} x^2\,dx$

If the limits in (a)(b)(c) are exist then the integrals is called converge. otherwise are called diverge.

Ex: $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_1^{t} x^{-2}\,dx = \lim_{t \to \infty} -x^{-1} \Big|_1^{t}$

$= \lim_{t \to \infty} \left(-\frac{1}{t} + 1\right) = \boxed{1}$ $\boxed{\text{Converge}}$

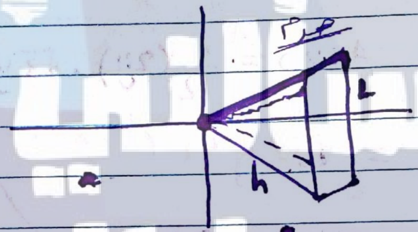
Ex: find the Volume of a pyramid whose base is a squre with side L and heights is h.
(use integration)

Sol: $A(x) = (2y)^2$

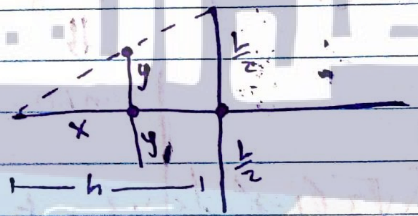
$= 4y^2$

$= 4\left(\dfrac{L^2x^2}{4h^2}\right)$

$= \boxed{\dfrac{L^2x^2}{h^2}}$

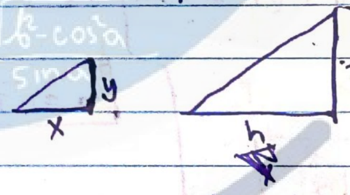


$$V = \int_0^h A(x)\,dx = \frac{L^2}{h^2}\int_0^h x^2\,dx$$



$$= \frac{L^2}{h^2}\left(\frac{x^3}{3}\Big|_0^h\right) = \frac{L^2}{3h^2}\left(h^3 - 0\right)$$

$$= \frac{L^2}{3\cancel{h^2}}\cancel{h^3} = \boxed{\frac{1}{3}L^2h}$$



تشابه مثلثات

$$\frac{\frac{L}{2}}{y} = \frac{h}{x}$$

$$2yh = Lx$$

$$y = \frac{Lx}{2h}$$

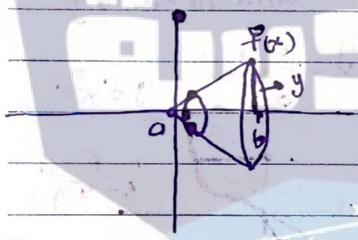
* Rotation

1. The volume of a solid obtain by rotating $y = f(x)$ about x-axis from $x=a$ to $x=b$ is

$$V = \pi \int_a^b (f(x))^2 \, dx$$

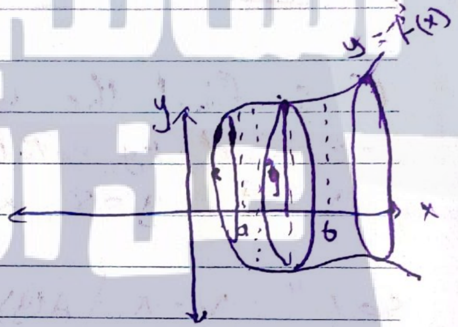
2. Rotating $x = g(y)$ about y-axis from $y=c$, $y=d$ is

$$V = \pi \int_c^d (g(y))^2 \, dy$$



$$A(y) = \pi y^2$$

$$= \pi (f(x))^2$$

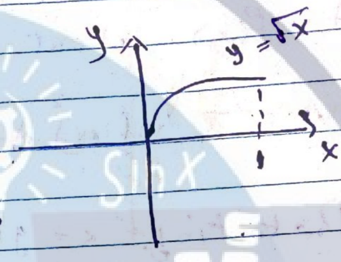


$$A(x) = \pi y^2 = \pi (f(x))^2$$

$$V = \int_a^b A(x) \, dx = \pi \int_a^b (f(x))^2 \, dx$$

ch(6)

Ex 8: Find the volume of the solid obtained by rotating about x-axis, $y = \sqrt{x}$ from 0 to 1.

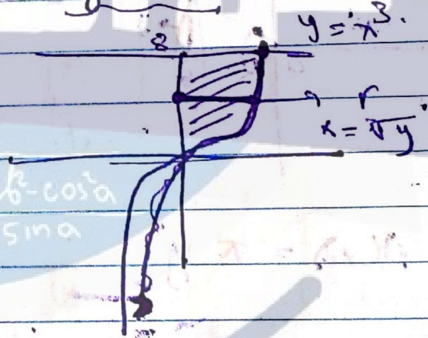


Sol: $V = \pi \int_0^1 (f(x))^2 \, dx$

$= \pi \int_0^1 x \, dx = \frac{\pi}{2} (x^2) \Big|_0^1$

$= \boxed{\frac{\pi}{2}}$

Ex: find the volume of the solid ~~of~~ obtained by rotating the region bounded by $y = x^3$, $y = 8$, $x = 0$ about y-axis.



Sol: $V = \pi \int A(y) \, dy$

$= \pi \int (g(y))^2 \, dy$

$= \int_0^8 \pi y^{\frac{2}{3}} \, dy$

$= \boxed{\frac{96\pi}{5}}$

$A(y) = \pi x^2$

$A(y) = \pi (\sqrt[3]{y})^2$

$= \pi y^{\frac{2}{3}}$

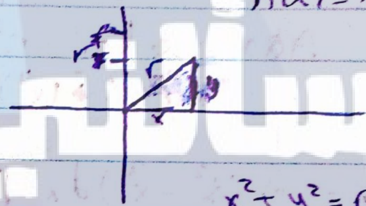
Ex (49): Find the volume of a cap of a sphere
P. 447 with radius r and hieght h.

$$V = \int_{-r}^{r} A(x)\,dx$$

$$= \int_{-r}^{r} \pi (r^2 - x^2)\,dx = \frac{4}{3}\pi r^3$$

$A(x) = \pi y^2$

$$= \int_{r-h}^{r} A(x)\,dx$$



$x^2 + y^2 = r^2$

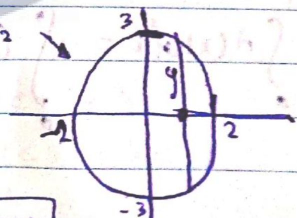
$\boxed{y^2 = r^2 - x^2}$

$$= \pi \int_{r-h}^{r} (r^2 - x^2)\,dx = \pi h^2 \left(r - \frac{h}{3}\right)$$

Ex (55): Find the volume of the solid S
P: 447 where base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$.
Cross-sections perpendicular to the x-axis are
متساوي الساقين isosceles right tringles with hypotenons in the base.

$9x^2 + 4y^2 - 36 = 0$

$y^2 = 9 - \frac{9}{4}x^2$



sol: $A(x) = \frac{1}{2} L^2$

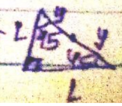
$A(x) = y^2$

$$V = \int_{-2}^{2} A(x)\,dx = 2\int_{0}^{2} \left(9 - \frac{9}{4}x^2\right)dx = \boxed{24}$$

$L^2 + L^2 = 4y^2$

$L^2 = 2y^2$

$= 2L^2 = 4y^2$



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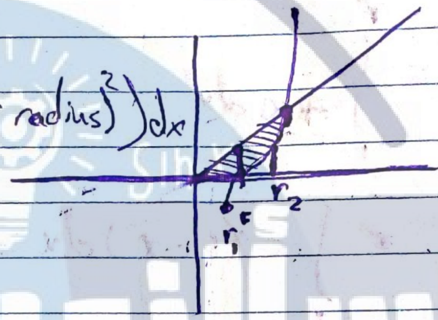
Ex: Find the volume of the region R enclosed by the curves $y = x$ and $y = x^2$ is rotatal about the x-axis.



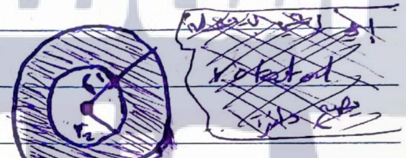
Sol:

$$V = \pi \int_a^b \left((\text{outer radius})^2 - (\text{inner radius})^2 \right) dx$$

$\Rightarrow x^2 - x = 0 \Rightarrow x = 0, 1$

$$A(x) = \pi (r_1^2 - r_2^2)$$

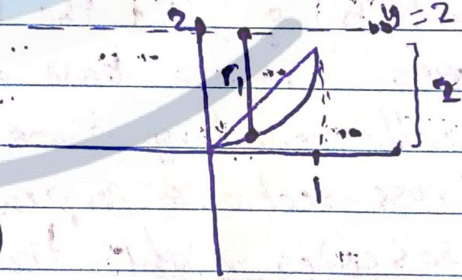
$$= \pi ((x^2) - (x^4))$$



$$V = \pi \int_0^1 (x^2 - x^4)\, dx = \boxed{\frac{2\pi}{15}}$$

Ex: Find the volume of region R enclosed by the curves $y = x$ and $y = x^2$ is rotatal about the line $y = 2$.

Sol:



$x = x^2$

$x^2 - x = 0 \Rightarrow \boxed{x = 0, 1}$

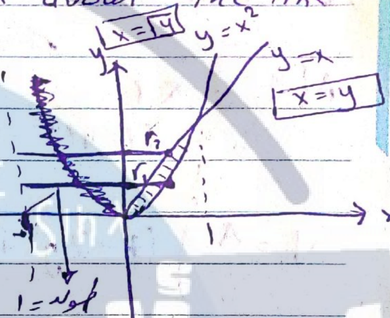
$$A(x) = \pi (r_1^2 - r_2^2)$$

$$= \pi \left((2 - x^2)^2 - (2 - x)^2 \right)$$

$$V = \int_0^1 A(x)\, dx = \pi \int_0^1 \left((2 - x^2)^2 - (2 - x)^2 \right) dx = \boxed{\frac{8\pi}{15}}$$

6.2

Ex: Find the volume of the solid obtained by rotating the region enclosed by $y = x$, $y = x^2$ about the line $x = -1$



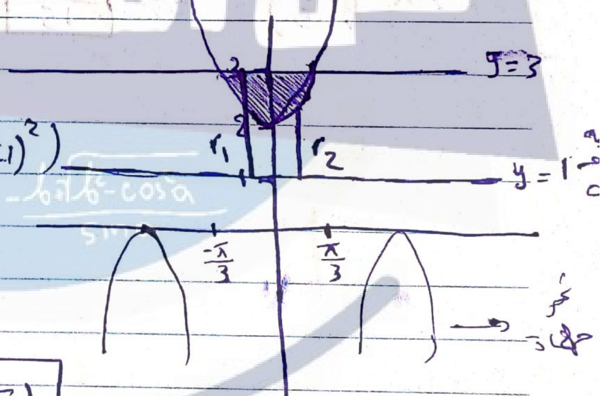
$$A(y) = \pi(r_1^2 - r_2^2)$$

$$A(y) = \pi\left((\sqrt{y} + 1)^2 - (y+1)^2\right)$$

y axis

$$V = \pi \int_0^1 \left((\sqrt{y}+1)^2 - (y+1)^2\right) dy = \boxed{\frac{\pi}{2}}$$

Ex: Find the volume of the solid obtained by rotating the region enclosed by $y = 1 + \sec x$, $y = 3$ about the line $y = 1$



Sol: $A(x) = \pi(r_1^2 - r_2^2)$

$$= \pi\left((3-1)^2 - (1+\sec x - 1)^2\right)$$

$$A(x) = \pi(4 - \sec^2 x)$$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4 - \sec^2 x)\,dx = \boxed{2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)}$$

$1 + \sec x = 3$

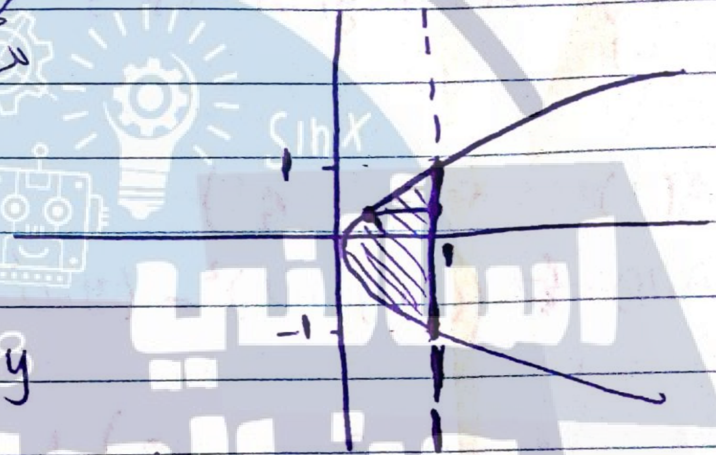
$\sec x = 2$

$\frac{1}{\cos x} = 2$

$\cos x = \frac{1}{2}$

$\boxed{x = \pm\frac{\pi}{3}}$

Ex: Find the volume of the solid obtained by rotating the region enclosed by $x=y^2$, $x=1$ about the line $x=1$



$$A(y) = \pi r^2$$
$$= \pi (1-y^2)^2$$
$$V = \pi \int_{-1}^{1} (1-2y^2+y^4)\,dy$$
$$= 2\pi \left(y - \frac{2}{3} y^3 + \frac{y^5}{5} \right) \Big|_0^1$$
$$= 2\pi \left(\frac{15\times1}{15} - \frac{5\times2}{5\times3} + \frac{3\times1}{3\times5} \right) = 2\pi \left(\frac{15-10+3}{15} \right)$$
$$= 2\pi \left(\frac{8}{15} \right) = \boxed{\frac{16\pi}{15}}$$

6.3 ⇒ Volums by cylidrical shells

★ rotating $y=f(x)$ about y-axis.

$$V = \int_a^b 2\pi x f(x)\, dx \quad \leftrightarrow \quad \text{مساحة إسطوانة}$$

محيط القاعدة × الارتفاع

$2\pi(r) = x \qquad f(x)$

★ rotating $x=g(y)$ about x-axis.

$$V = \int_c^d 2\pi y\, g(y)\, dy$$

Ex: Find the volume of the solid obtained by rotating about y-axis the region bounded by $y = 2x^2 - x^3$ and $y=0$;

{~~cylindrical shell~~}

sol: by cylindrical shells.

$$2x^2 - x^3 = 0$$

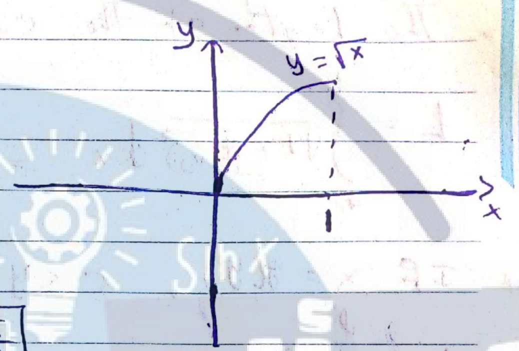
$$x^2(2-x) = 0$$

$$x = 0, 2$$

$$V = \int_0^2 2\pi x(2x^2 - x^3)\, dx$$

$$= \boxed{\frac{16\pi}{5}}$$

Ex: Find the Volume of the solid obtained by rotating about x-axis the region Under the curve $y = \sqrt{x}$ from $x=0$ to $x=1$.

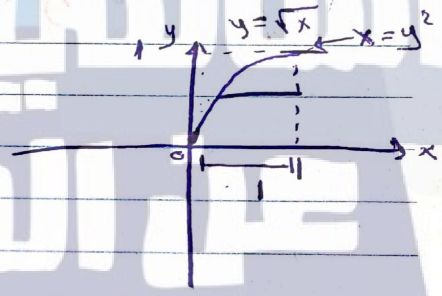


Method 1 7.2

$$A(x) = \pi(\sqrt{x})^2$$
$$= \pi x$$

$$V = \pi \int_0^1 x \, dx = \frac{\pi x^2}{2} = \boxed{\frac{\pi}{2}}$$

Method 2 cylindrical shells

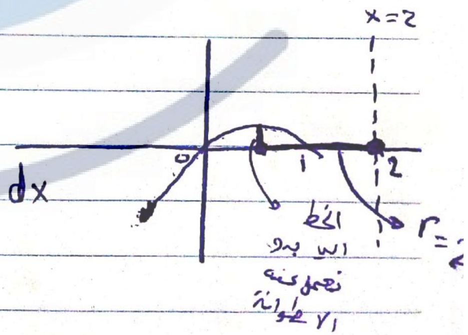


$$V = \int_0^1 2\pi y (1 - y^2) \, dy$$
$$= \boxed{\frac{\pi}{2}}$$

Ex: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Method 1 → 8 تنط

Method 2



$$V = \int_0^1 2\pi (2 - x)(x - x^2) \, dx$$

$$\boxed{V = \frac{\pi}{2}}$$

ch(8)

(8.1) → Arc Length

* The length of the curve $y = f(x)$, $a \le x \le b$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

* If $x = g(y)$, $c \le y \le d$

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

ch(8)

$S(y) = \int_a^B \sqrt{1+(g'(t))^2}\, dt \qquad S(x) = \int_a^x \sqrt{1+f'(t))^2}\, dt$

$y = f(x) \qquad x = g(y) \quad ds = \sqrt{1+(f'(x))^2}\, dx$

$ds = \sqrt{1+(g'(y))^2}\, dy$

8.2 Area of a surface of revolution.

Rotation about x-axis.	Rotation about y-axis
$S = \int 2\pi y\, ds$	$S = \int 2\pi x\, ds$ (Arc length of function)
$= \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$	$= \int_a^b 2\pi x \sqrt{1+(f'(x))^2}\, dx$
or $\int_c^d 2\pi y \sqrt{1+(g'(y))^2}\, dy$	$= \int_c^d 2\pi\, g(y) \sqrt{1+(g'(y))^2}\, dy$

Ex: The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ in an arc of the circle $x^2+y^2=4$ Find the area of a surface obtained by rotating this arc about the x-axis.

Sol:

$y = \sqrt{4-x^2}$

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4-x^2} = \frac{4}{4-x^2}$

$S = \int 2\pi y\, ds$

$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \sqrt{\frac{4}{4-x^2}}\, dx$

$= 2\pi \int_{-1}^{1} 2\, dx = 4\pi x \Big|_{-1}^{1} = \boxed{8\pi}$

Ex: The arc of $y = x^2$ from $(1,1)$ to $(2,4)$ is rotating about y-axis. Find the area of this surface.

Sol: Method 1

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x^2$$

$$u = 1 + 4x^2$$

$$du = 8x\,dx$$

$$S = \int 2\pi x\, ds$$

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}\,dx$$

$$= \frac{2\pi}{8}\int_1^2 \sqrt{u}\,du = \frac{\pi}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right)$$

$$= \frac{\pi}{6}\sqrt{u^3} = \frac{\pi}{6}\left[\sqrt{(1+4x^2)^3}\right]_1^2$$

$$= \boxed{\frac{\pi}{6}\left(\sqrt{125} - \sqrt{(17)^3}\right)}$$

Method 2

$$y = x^2$$

$$x = \pm\sqrt{y} \qquad 1 \le y \le 4$$

$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4y}$$

$$S = \int 2\pi x\, ds$$

$$= 2\pi \int_1^4 \sqrt{y}\sqrt{\frac{4y+1}{4y}}\,dy$$

$$= \frac{2\pi}{2}\int_1^4 \sqrt{4y+1}\,dy$$

$\vdots$

$$= \boxed{\frac{\pi}{6}\left(\sqrt{125} + \sqrt{(17)^3}\right)}$$

ch(8) <u>8.2</u>

Sunday
10/Mar/2019

Ex: Find the area of the surface generated by rotating $y = e^x$ $0 \le x \le 1$, about the x-axis.

sol:

$$S = \int 2\pi \, y \, ds \qquad y = e^x$$

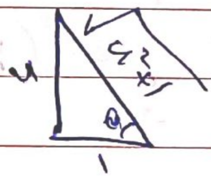
$$\frac{dy}{dx} = e^x$$

$$= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} \, dx \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + e^{2x}$$

$$= 2\pi \int_1^e \sqrt{1 + u^2} \, du \qquad u = e^x, \quad du = e^x dx$$

$$1 + u = \tan\theta \Rightarrow u = \tan\theta - 1$$

$$= 2\pi \int_1^e \sec^2\theta \, d\theta \, \cancel{\sec^2\theta} \qquad du = \sec^2\theta \, d\theta$$



$$= \cancel{2}\pi \, \frac{1}{\cancel{2}} \left[\sec\theta + \ln|\sec\theta + \tan\theta|\right]$$

$$= \pi \left[\sqrt{1 + u^2} + \ln\left[\sqrt{1 + u^2} + u\right]\right]_1^e$$

$$\cancel{= \pi\left(\left[\sqrt{1+e^2} + \ln\left[\sqrt{1+e^2} + e\right]\right] - \ldots\right)}$$

$$= \pi\left[\left(\sqrt{2} - \sqrt{1+e^2}\right) + \left(\ln[\sqrt{2} + 1] - \ln\sqrt{1+e^2} + e\right)\right]$$

10/Mar/2019

Ex 5 : set up integral for the area of the surface
P. 555 obtained by rotating $x = y + y^3$, $0 \le y \le 1$ about.

[1] x - axis.

$S = \int 2\pi y \, ds$

$x = y + y^3$

$\frac{dx}{dy} = 1 + 3y^2$

$= 2\pi \int_0^1 y \sqrt{2 + 6y^2 + 9y^4} \, dy$

$1 + \left(\frac{dx}{dy}\right)^2 = \cancel{1 + 9y^4}$
$2 + 6y^2 + 9y^4$

[2] y - axis

$S = \int_0^1 2\pi x \, ds$

$= \int_0^1 2\pi (y + y^3) \sqrt{2 + 6y^2 + 9y^4} \, dy$

بعوضي من عند رقم 1 لـ ∞ شو ما بدي

ch (11)

Sequence :

Ex : show that the seq. $\left\{ \frac{3}{n+5} \right\}$ is decreasing

Method $\underline{1}$: ~~decr~~ decr. $a_n > a_{n+1}$ to prove

$n+5 < (n+1)+5 \quad , \quad n \geq 1$

$\left(\frac{1}{n+5} > \frac{1}{(n+1)+5} \right) 3$

$\frac{3}{n+5} > \frac{3}{(n+1)+5} \qquad \frac{3}{n+5} > \frac{3}{n+6}$

$\boxed{a_n > a_{n+1}}$

Method $\underline{2}$ by using derivative.

$f(n) = \frac{3}{n+5}$

اذا $f(x)$ اكل من بعض
يكون decreasing

$f'(n) = \frac{-3}{(n+5)^2} < 0$

Ex : $\left\{ \frac{n}{n^2+1} \right\}$ decresing or increasing ?

Sol : $f(n) = \frac{n}{n^2+1} \Rightarrow f'(n) = \frac{(n^2+1) - n(2n)}{(n^2+1)^2} = \frac{-n^2+1}{(n^2+1)^2}$

$= \frac{-(n^2-1)}{(n^2+1)^2}$

$\therefore \left\{ \frac{n}{n^2+1} \right\}$ decreasing

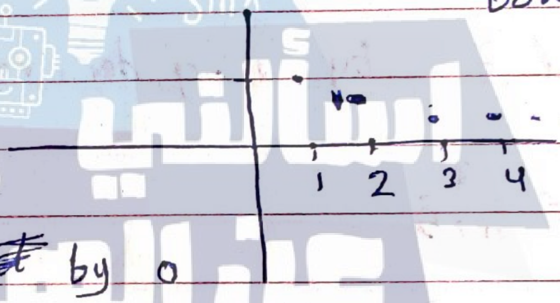
Def : 1) If a seq. is increasing or decreasing then it is <u>monoton</u>.

2) If there exist m such that $a_n \leq m$ for all $n \geq 1$ <u>then</u> $\Rightarrow$ $\{a_n\}$ is bounded above.

3) If there exist M such that $a_n \geq M$ for all $n \geq 1 \Rightarrow \{a_n\}$ is bounded below.

4) If $\{a_n\}$ is bounded above and below $\Rightarrow \{a_n\}$ is bounded.

Ex : $\left\{\frac{1}{n}\right\}$



$\left\{\frac{1}{n}\right\}$ is bounded below ~~but~~ by 0
& bounded above by 1

$\Rightarrow \left\{\frac{1}{n}\right\}$ is bounded.

\# Thm : Every bounded monoton sequance is converge.

Ex: The seq. $\left\{\frac{1}{n}\right\}$ is decreasing and bounded.

$\Rightarrow \left\{\frac{1}{n}\right\}$ is converge

Ch 11

11.1

Ex: ~~a_1~~ let the recurrence seq. $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$, $n = 1, 2, \ldots$

show that the seq. is conv., the find $\lim_{n \to \infty} a_n$

Sol: 1) to prove a_n is bounded ~~show that~~

2) " " " a_n is increasing

then by thm the seq. is conv.

Since the seq. conv. ⇒ conv. to uniqe limit L,

$$a_{n+1} = \frac{1}{2}(a_n + 6)$$

$$\lim_{n \to \infty} a_{n+1} = \frac{1}{2}\left(\lim_{n \to \infty} a_n + 6\right)$$

*Ex: the seq. $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing an bounded find limit:

Sol: inc + bdd ⇒ conv. to L

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \left(3 - \frac{1}{a_n}\right)$$

$$L = 3 - \frac{1}{L} \Rightarrow L = \frac{3L - 1}{L}$$

$$L^2 - 3L + 1 = 0, \quad L = \frac{3 \pm \sqrt{9 - 4}}{2} \Rightarrow L = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$L = \frac{3}{2} + \frac{\sqrt{5}}{2}$ (cause increasing) بنوخذ الموجب لأنها increasing

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11.2

$$a_0 + a_1 + a_2 + \dots = \sum_{K=0}^{\infty} a_K$$

infinite series.

Def.: partial sum.

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{K=1}^{n} a_n$ Partial sum

we have the seq. $\{S_n\}$

if the seq. is conv. => the series conv.

if the seq. is div. => the series div.

Ex: show that the seq. $\sum_{K=1}^{\infty} \frac{1}{K(K+1)}$ is conv.

sol

$$\frac{A}{K} + \frac{B}{K+1} = \frac{1}{K(K+1)}$$

$$\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = \sum_{K=1}^{\infty} \frac{1}{K} - \frac{1}{K+1}$$

$A(K+1) + BK = 1$

$K=0$. $\boxed{A=1}$

$K=-1$ $-B = 1$ $\boxed{B=-1}$

$S_1 = 1 - \frac{1}{2}$

$S_2 = a_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3})$

$= 1 - \frac{1}{3}$

$S_3 = a_1 + a_2 + a_3 = (1 - \frac{1}{2}) + (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{3})$

$= 1 - \frac{1}{4}$

$S_n = 1 - \frac{1}{n}$

$\Rightarrow \lim_{n \to \infty} S_n = 1$ conv.

$\Rightarrow$ series $\sum \frac{1}{K(K+1)} \xrightarrow{\text{conv.}}$

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نوع (2) $\sum a r^{n}$

r^{n+1}

$r^{2n-1} = (r^2)^n \cdot r^{-1}$

$|r|<1 \Rightarrow$ serize ~~ser~~ conV. $\Rightarrow \sum = \frac{\text{first term}}{1-r}$

$|r| \geq 1 \Rightarrow$ Div.

نوع (1) # $\sum_{k=1}^{\infty} a_k$ / $s_n = a_1 + a_2 + \cdots + a_3$

$\lim_{n \to \infty} s_n = L$ exist $\Rightarrow \sum a_k$ conV.

$\sum \left(\frac{1}{n} - \frac{1}{n+1} \right)$

Div. Test (D.V)

If $\lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum a_n$ div.

* Note : If $\lim_{n \to \infty} a_n = 0 \Rightarrow$ test fail.

Thm : If $\sum a_n$ & $\sum b_n$ ConV. then :

(1) $\sum c\, a_n$ ConV.

(2) $\sum (a_n \pm b_n)$ ConV.

ConV, Div $\qquad$ Div, Div لا يمكن

Div المجموع تكون $\qquad$ ConV or Div تحديد إذا المجموع

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Integral Test

let f cont, Positive, decreasing func. on $[1,\infty)$, $a_n = f(n)$ then.

(1) IF $\int_1^\infty f(n)\,dn$ Conv. then $\Rightarrow$ $\sum_{n=1}^\infty a_n$ Conv.

(2) If $\int_1^\infty f(n)\,dn$ Div $\Rightarrow$ $\sum_{n=1}^\infty a_n$ Div.

- Ex: For what values of P is the series $\sum_{n=1}^\infty \frac{1}{n^P}$ conv. or div ?

Sol: $f(n) = \frac{1}{n^P}$ cont, Positive, decreasing

by Integral Test (I.T)

إذا ال series تبدأ بـ أي قيمة نبدأ التكامل بالقيمة التي بدأ بها ال series

لكن بشرط يكون cons, positive, decreasing func

نحدد بالاشتقاق

$P \neq 1$

$$\int_1^\infty \frac{1}{n^P}\,dn = \lim_{t\to\infty}\int_1^t n^{-P}\,dn = \lim_{t\to\infty} \left. \frac{n^{-P+1}}{-P+1} \right|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-P}\left[\frac{1}{n^{P-1}}\right]_1^t = \lim_{t\to\infty} \frac{1}{1-P}\left[\frac{1}{t^{P-1}} - 1\right]$$

2 Case

(1) $P-1>0 \Rightarrow P>1$

$$\int_1^\infty \frac{1}{n^P}\,dn = \frac{1}{1-P}[0-1] = \frac{1}{P-1} \quad \text{Conv}$$

(2) $P-1<0$ $\int_1^\infty \frac{1}{n^P}\,dn = \frac{1}{1-P}[\infty - 1] = \infty$ Div

(3) $P-1=0$ $\boxed{P=1}$ $\int_1^\infty \frac{1}{n}\,dn = \lim_{t\to\infty}[\ln n]_1^t = \lim_{t\to\infty}[\ln t - 1] = \infty - 1 = \infty$ Div

(3)

P- Series

$\sum \frac{1}{n^p}$ 2 case

Thm

$$\text{}= \begin{cases} \frac{1}{P-1}, & (\text{conv}) \text{ if } P>1 \\ \text{Div}, & \text{if } P \le 1 \end{cases}$$

Ex: (1) $\sum \frac{1}{n}$ div (P-Series) $P=1$

(2) $\sum \sqrt[3]{n^5} = \sum \frac{1}{n^{-5/3}}$ div (P-Series $P=\frac{-5}{3}<1$)

(3) $\sum \frac{1}{n^{10}}$ conv. (P- series $P=10>1$)

Ex: Test the following series.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Sol: by (I.T)

$\frac{1}{n^2+1}$ cont, Positive, decreasing

$$\int_1^\infty \frac{1}{n^2+1}\,dn = \lim_{t\to\infty} \int_1^t \frac{1}{n^2+1}\,dn = \lim_{t\to\infty} \tan^{-1}(n)\Big|_1^t$$

$$= \lim_{t\to\infty} [\tan^{-1}t - \tan^{-1}(1)] = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}} \quad \boxed{\text{Conv}}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv

(2) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

by

Sol: (I.T)

$f(n) = \frac{\ln n}{n}$ cont, Positive, decreasing

$\int_1^{\infty} \frac{\ln n}{n}\, dn = \lim_{t \to \infty} \int_1^{t} \frac{\ln n}{n}\, dn$

$= \int z\, dz = \frac{z^2}{2}\Big]_1^t$

$\lim_{t \to \infty} \frac{(\ln t)^2}{2} - 0$

$= \infty$ div.

$f'(n) = \frac{1 - \ln n}{n^2}$

$1 - \ln n = 0$

$\ln n = 1$

$n = e$

$+ + + \quad e \quad - - - \quad \infty$

decreasing

$z = \ln n \Rightarrow dz = \frac{dn}{n}$

$\int \frac{\ln n}{n}\, dn = \int z\, dz$

$= \frac{z^2}{2}$

$\frac{\ln^2 n}{2}$